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PARETO SUBDIFFERENTIAL CALCULUS FOR CONVEX SET-VALUED MAPPINGS AND APPLICATIONS TO SET OPTIMIZATION

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Abstract. This paper provides an extension of a recent work by El Maghri and Laghdir, dealing with the subdifferential calculus for convex vector mappings. The purpose of this paper is to study the Pareto subdifferential (weak and proper) for convex set-valued mappings defined via Pareto efficiency from a point of view of characterizations and calculus rules. We develop calculus rules of the Pareto subdifferentials for the sum and/or the composition of two convex set-valued mappings. The obtained formulas are original and hold under the weak conditions of the connectedness or Attouch–Brézis and the regular subdifferentiability. Some applications to a general set-valued optimization problem are given to illustrate our main results.
Keywords. Pareto subdifferential; Pareto efficiency; Regular subdifferentiability; Set-optimization; Set-valued convex mappings.

1. INTRODUCTION

This paper is motivated by the results presented by El Maghri and Laghdir [1]. The authors discussed the calculus rules of the sum and composition of the subdifferential for vector valued mappings defined in the Pareto sense and their applications to vector optimization. By using the concept of the regular subdifferentiability, they obtained under Moreau-Rockafellar or Attouch-Brézis condition the following formula expressing the calculus rules of the Pareto subdifferential for the sum of two convex single vector mappings $f_1, f_2 : X \to Y \cup \{+\infty_Y\}$

$$\partial^{\sigma}(f_1+f_2)(\bar{x}) = \partial^{\sigma} f_1(\bar{x}) + \partial^{s} f_2(\bar{x}), \tag{1.1}$$

where $\sigma \in \{w, p\}$ stands for weak or proper Pareto concept, $\partial^{\sigma} f_1(\bar{x})$ is the weak or proper Pareto subdifferential at $\bar{x}$, $\partial^{s} f_2(\bar{x})$ is the strong subdifferential at $\bar{x}$, X is a locally convex space for Moreau-Rockafellar condition and is Banach space for Attouch-Brézis condition, Y is a partially ordered vector space, $+\infty_Y$ is an abstract maximal element of the space Y.

In a more general framework, a similar result was proved in the case $\sigma = w$, by Lin [2] for convex set-valued mappings, that is, if $F_i : X \rightrightarrows Y$, $i = 1,2$, are convex set-valued mappings, then, under the connectedness assumption,

$$\partial^{w}(F_1+F_2)(\bar{x}, \bar{u}_1+\bar{u}_2) \subset \partial^{w} F_1(\bar{x}, \bar{u}_1) + \partial^{w} F_2(\bar{x}, \bar{u}_2), \tag{1.2}$$

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where $\bar{u}_i \in F_i(\bar{x})$ and $\partial^w F_i(\bar{x}, \bar{u}_i)$ stands for the weak subdifferential at $(\bar{x}, \bar{u}_i)$ (see Definition (2.5)). Let us emphasize that the reverse inclusion is not true. A counter example in [2] presented that the equality in (1.2) does not hold. In [3], Taa proved that, under the weaker assumption that $\mathbb{R}_+[\mathrm{dom}F_1 - \mathrm{dom}F_2]$ is a closed vector subspace of X, inclusion (1.2) holds.

In this work, motivated by the related results presented in [1, 3, 4, 5, 6], our main objective is to attempt to prove that the equality (1.1) holds for convex set-valued mappings. In fact, the presence of the strong subdifferential enables us to establish the desired equality. To achieve the aim of this work, we deal with an extension of the lower semicontinuity adapted to set-valued mappings, namely, the star epi-closedness, and we establish a characterization of the Pareto subdifferential by a scalarization process. We suitably extend the concept of regular subdifferentiability due to Raffin [7] in the setting of set-valued mappings which play a crucial role in establishing our main results. This concept enables us to find the gap between the strong and Pareto subdifferential.

By using the normal cone intersection formula for two convex subsets and applying a scalarization process, we obtain calculus rules for the weak and proper subdifferential of the sum and the composition of two convex and star epi-closed set-valued mappings taking values in a partially ordered topological linear space, under a qualification condition and the concept of regular subdifferentiability introduced. The approach that we will use for computing the Pareto subdifferential of the composed set-valued mappings is to transform it as the Pareto subdifferential of the sum of two convex set-valued mappings. The scalarization process is based essentially by stating at first, the sum for the class of convex set-valued mappings taking values in $\mathbb{R}$. As a consequence, when the mappings are single-valued, we recapture a result of [1] expressing the Pareto subdifferential calculus for convex vector mappings. As the applications of these results, we establish the σ-efficient optimality conditions in terms of the Lagrange–Kuhn–Tucker multipliers and the vector normal cone of a general constrained convex set-valued mathematical programming problem.

The paper is structured as follows. In Section 2, we give some preliminaries and introduce a kind of lower semicontinuity adapted for set-valued mappings. We describe the different notions of Pareto subdifferential underlining their preliminary connections with the efficient sets. In Section 3, we present the scalarization principle characterizing scalarly the Pareto subdifferential, and we extend the concept of regular subdifferentiability to set-valued mappings. Section 4 is the central part of this paper. It is divided into two subsections. Subsection 4.1 is devoted to state the sum calculus rule for two convex set-valued mappings under a constraint qualification. In Subsection 4.2, we establish a chain rule for convex set-valued mappings. In Section 5, by virtue of these calculus rules, we study necessary and sufficient optimality conditions of a constrained set-valued optimization problem.

2. PRELIMINARIES

In this section, we provide some basic notations, definitions and theorems. Throughout this paper, let X and Y be two real locally convex topological vector spaces with Y separated and $Y_+ \subset Y$ be a proper ($Y_+ \neq \emptyset$, $Y_+ \neq Y$), pointed ($Y_+ \cap -Y_+ = \{0_Y\}$) and closed convex cone with nonempty topological interior. The convex cone Y_+ induces in Y the following preorder relations

$$y_1 \leq_{Y_+} y_2 \Longleftrightarrow y_2 - y_1 \in Y_+,$$

$$y_1 <_{Y_+} y_2 \Longleftrightarrow y_2 - y_1 \in \operatorname{int} Y_+,$$

and

$$y_1 \lneq_{Y_+} y_2 \Longleftrightarrow y_2 - y_1 \in Y_+ \backslash \{0_Y\},$$

where y_1 and y_2 are two elements of Y and $\operatorname{int} Y_+$ stands for the topological interior of Y_+. The topological dual spaces of X and Y are denoted by X^* and Y^*, respectively, paired in duality by $\langle .\,, .\rangle$. The polar cone Y_+^* of Y_+ is the set of $y^* \in Y^*$ such that $y^*(Y_+) \subseteq \mathbb{R}_+$, while the strict polar cone Y_+^{s*} is the set of $y^* \in Y^*$ such that $y^*(Y_+ \backslash \{0_Y\}) \subseteq \mathbb{R}_+ \backslash \{0\}$. Let C be a nonempty convex subset of Y. The normal cone of C at $\bar{y} \in C$ is denoted by

$$N_C(\bar{y}) := \{y^* \in Y^* : \langle y^*, y - \bar{y}\rangle \leq 0, \quad \forall y \in C\}.$$

In what follows, we denote by $L(X,Y)$ the set of all continuous linear operators from X into Y. Given two nonempty subsets $A, B \subset Y$ and $\alpha \in \mathbb{R}$, we write $A + B := \{a + b : (a,b) \in A \times B\}$ and $\alpha A := \{\alpha a : a \in A\}$. We set $A + \emptyset = \emptyset + A = \emptyset$ and $\alpha \emptyset = \emptyset$. Let $F : X \rightrightarrows Y$ be a set-valued mapping, the effective domain, graph, and image are defined respectively by

$$\operatorname{dom} F := \{x \in X : F(x) \neq \emptyset\},$$

$$\operatorname{gr} F \quad := \{(x,y) \in X \times Y : y \in F(x)\},$$

and

$$\operatorname{Im} F \quad := \bigcup_{x \in X} F(x).$$

If we define the set-valued mapping $F + Y_+$ from X into Y by $(F + Y_+)(x) = F(x) + Y_+$ for any $x \in X$, then the subset

$$\operatorname{epi} F := \operatorname{gr}(F + Y_+) = \{(x,y) \in X \times Y : y \in F(x) + Y_+\}$$

is called the epigraph of F.

Definition 2.1. [8] The set-valued mapping F is said to be

(a) Y_+-convex if its epigraph is a convex subset of $X \times Y$.
(b) Proper if its effective domain $\operatorname{dom} F \neq \emptyset$.

Let $G : Y \rightrightarrows Z$, where Z is a real locally convex topological vector space equipped with a nonempty convex cone Z_+. The composite set-valued mapping $G \circ F : X \rightrightarrows Z$ is defined by

$$(G \circ F)(x) := \begin{cases} G(F(x)) = \bigsqcup_{y \in F(x)} G(y), & \text{if } x \in \operatorname{dom} F, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We have $\operatorname{dom}(G \circ F) = F^{-1}(\operatorname{dom} G) \cap \operatorname{dom} F$, where $F^{-1}(\operatorname{dom} G) := \{x \in X : F(x) \cap \operatorname{dom} G \neq \emptyset\}$. Recall that the set-valued mapping $G : Y \rightrightarrows Z$ is said to be (Y_+, Z_+)-nondecreasing if $G(y_2) \subseteq G(y_1) + Z_+$ for all $(y_1, y_2) \in Y \times Y$ satisfying $y_1 \leq_{Y_+} y_2$. If G is (Y_+, Z_+)-nondecreasing, Z_+-convex, and F is Y_+-convex, then $G \circ F$ is Z_+-convex (see [3]).

The set-valued indicator mapping $R_S^v : X \rightrightarrows Y$ is defined for the nonempty subset $S \subseteq X$ by

$$R_S^v(x) := \begin{cases} \{0_Y\}, & \text{if } x \in S, \\ \emptyset, & \text{elsewhere.} \end{cases}$$

Let us recall the concept of connectedness [9] and the concept of continuity [10] of a set-valued mapping.

Definition 2.2. Let $F : X \rightrightarrows Y$ be a set-valued mapping.

(a) F is said to be connected at $x_0 \in X$ if there exists a mapping $h : X \to Y$ continuous at x_0 and $h(v) \in F(v)$ for all v in some neighborhood of x_0 and h is continuous at x_0.
(b) F is said to be upper-semicontinuous at x_0 if, for any open subset $V \supseteq F(x_0)$, there exists a neighborhood U of x_0, such that $F(U) \subseteq V$.
(c) F is said to be lower-semicontinuous at x_0 if, for any open subset V such that $V \cap F(x_0) \neq \emptyset$, there exists a neighborhood U of x_0, such that $F(U) \cap V \neq \emptyset$.
(d) F is said to be Y_+-upper-semicontinuous at x_0 if, for any open subset $V \supseteq F(x_0)$, there exists a neighborhood U of x_0, such that $F(U) \subseteq V + Y_+$.

We say that F is continuous at x_0 if it is upper-semicontinuous and lower-semicontinuous at x_0, and we say that F is continuous (resp. Y_+-upper-semicontinuous) on X if it is continuous (resp. Y_+-upper-semicontinuous) at each point $x \in X$.

In the sequel, we use two extensions of the lower semicontinuity adapted to set-valued mappings, namely the Y_+-epi-closedness and the star Y_+-epi-closedness.

Definition 2.3. Let $F : X \rightrightarrows Y$ be a set-valued mapping.

(a) F is said to be Y_+-epi-closed if its epigraph is closed in the product topology on $X \times Y$.
(b) F is said to be star Y_+-epi-closed if, for any $y^* \in Y_+^*$, the real set-valued mapping $y^* \circ F$ is $\mathbb{R}_+$-epi-closed.

Let us note that the Y_+-epi-closedness of a set-valued mapping does not imply in general that this set-valued mapping is star Y_+-epi-closed.

Example 2.1. Let $Y = \mathbb{R}^2$ and $Y_+ = \mathbb{R}^2_+$. Define $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ as

$$F(x) = \begin{cases} \{(a,b) \in \mathbb{R}^2 : \ a \geq \frac{1}{x}, \quad b \geq x\}, & \text{if } x > 0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then, F is Y_+-epi-closed, but F is not star Y_+-epi-closed. Indeed, for $y^* = (0,1) \in Y_+^* = \mathbb{R}^2_+$ one has

$$(y^* \circ F)(x) = \begin{cases} \{b \in \mathbb{R} : b \geq x\}, & \text{if } x > 0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

It is easy to see that, for any $n \in \mathbb{N}^*$, $(\frac{1}{n+1}, \frac{1}{n}) \in \operatorname{epi}(y^* \circ F)$ and $(\frac{1}{n+1}, \frac{1}{n}) \to (0,0)$, but $(0,0) \notin \operatorname{epi}(y^* \circ F)$. This shows that F is not star Y_+-epi-closed.

Proposition 2.1. *Let $F : X \rightrightarrows Y$ be a set-valued mapping with compact values (i.e., $F(x)$ is compact for all $x \in X$). If F is Y_+-upper-semicontinuous, then F is star Y_+-epi-closed.*

Proof. Let $y^* \in Y_+^*$ and define the following function on X by $\varphi(x) := \inf\limits_{y \in F(x)} \langle y^*, y \rangle$ for all $x \in X$. As F is compact values, it follows that, for any $x \in X$, there exists $z \in F(x)$ such that $\varphi(x) = y^*(z)$. Hence, we easily obtain that $\operatorname{epi}(y^* \circ F) = \operatorname{epi} \varphi$. Now, we check that φ is lower semicontinuous on X. Let $x_0 \in X$ and $\varepsilon > 0$. Since y^* is continuous at the origin of Y, there exists an open neighborhood V of the origin in Y such that $y^*(V) \subset [-\varepsilon, \varepsilon]$. By taking the open set $F(x_0) + V$, which contains $F(x_0)$, due to $0_Y \in V$, it follows from the Y_+-upper-semicontinuity of F at x_0 that there exists a neighborhood U of x_0 such that $F(x) \subseteq F(x_0) + V + Y_+$, $\forall x \in U$, which yields

$$\inf_{y \in F(x_0)} \langle y^*, y \rangle + \inf_{y \in V} \langle y^*, y \rangle + \inf_{y \in Y_+} \langle y^*, y \rangle \leq \inf_{y \in F(x)} \langle y^*, y \rangle, \quad \forall x \in U.$$

Consequently $\varphi(x_0) - \varepsilon \leq \varphi(x)$, $\forall x \in U$. Thus φ is lower semicontinuous at x_0. This completes the proof of the proposition. □

Let us recall several variants of the concept of efficient points of a subset $A \subset Y$ and some properties.

Definition 2.4. [10] Let $A \subset Y$ be a nonempty subset. $\bar{y} \in A$ is said to be

(a) Strongly efficient point of the subset A if $A \subseteq \bar{y} + Y_+$.
(b) Pareto or efficient point of the subset A if $(\bar{y} - Y_+) \cap A = \{\bar{y}\}$.
(c) Weak Pareto or weakly efficient point of the subset A if $(\bar{y} - \operatorname{int} Y_+) \cap A = \emptyset$.
(d) Proper Pareto or (Henig) properly efficient point of the subset A, if there exists a proper convex cone $\hat{Y}_+ \subset Y$ such that $Y_+ \backslash \{0_Y\} \subseteq \operatorname{int} \hat{Y}_+$ and $(\bar{y} - \hat{Y}_+) \cap A \subseteq \bar{y} + l(\hat{Y}_+)$, where $l(\hat{Y}_+) = \hat{Y}_+ \cap -\hat{Y}_+$ stands for the lineality of $\hat{Y}_+$.

The efficient set, strongly, properly, and weakly efficient sets will be denoted, respectively, by $E_e(A, Y_+)$, $E_s(A, Y_+)$, $E_p(A, Y_+)$, and $E_w(A, Y_+)$, respectively. We denote by $E_\sigma(A, Y_+)$ the set of efficient points depending on the choice of $\sigma \in \{s, p, e, w\}$. By using the well-known property $\operatorname{int} Y_+ \subset Y_+ \backslash \{0_Y\}$, we obtain $E_p(A, Y_+) \subseteq E_e(A, Y_+) \subseteq E_w(A, Y_+)$. We unify the notation of the polar cones by putting

$$Y_+^{*\sigma} = \begin{cases} Y_+^* \backslash \{0\}, & \text{if } \sigma = w\ , \\ Y_+^{s*}, & \text{if } \sigma = p\ . \end{cases}$$

The following theorem is useful in the sequel

Theorem 2.1. [10] *Let A be a subset of Y such that $A + Y_+$ is convex. Then*

$$E_s(A, Y_+) = \bigcap_{y^* \in Y_+^* \backslash \{0\}} \arg\min_{y \in A} \langle y^*, y \rangle,$$

and

$$E_\sigma(A, Y_+) = \bigcup_{y^* \in Y_+^{*\sigma}} \arg\min_{y \in A} \langle y^*, y \rangle.$$

Consider now the following set-valued optimization problem

$$(\mathrm{P}) \quad \begin{cases} \text{Min F(x)}, \\ x \in S, \end{cases}$$

where $F : X \supseteq S \rightrightarrows Y$ is a set-valued mapping. There are several types of solutions for (P). A pair $(\bar{x}, \bar{y}) \in X \times Y$ is said to be

(a) Strongly efficient solution of (P) if $(\bar{x}, \bar{y}) \in (S \times Y) \cap \operatorname{gr} F$ and $\bar{y} \in E_s(F(S), Y_+)$.
(b) Pareto or efficient solution of (P) if $(\bar{x}, \bar{y}) \in (S \times Y) \cap \operatorname{gr} F$ and $\bar{y} \in E_e(F(S), Y_+)$.
(c) Weak Pareto or weakly efficient solution of (P) if $(\bar{x}, \bar{y}) \in (S \times Y) \cap \operatorname{gr} F$ and $\bar{y} \in E_w(F(S), Y_+)$.
(d) Proper Pareto or (Henig) properly efficient solution of (P) if $(\bar{x}, \bar{y}) \in (S \times Y) \cap \operatorname{gr} F$ and $\bar{y} \in E_p(F(S), Y_+)$.

The efficient set, strongly, properly and weakly efficient sets for (P) will be denoted respectively, by $K_e(F(S),Y_+)$, $K_s(F(S),Y_+)$, $K_p(F(S),Y_+)$ and $K_w(F(S),Y_+)$, respectively. To unify the presentation, we denote by $K_\sigma(F(S),Y_+)$ the set of σ-efficient points depending on the choice of $\sigma \in \{s,e,w,p\}$. Note that $K_p(F(S),Y_+) \subseteq K_e(F(S),Y_+) \subseteq K_w(F(S),Y_+)$. The notion of the subdifferential defined in Pareto sense is important for dealing with vector set-valued optimization problems. The different efficient sets defined in the above enable us to introduce the concept of σ-subdifferential for a set-valued mapping.

Definition 2.5. Let $F : X \rightrightarrows Y$ be a set-valued mapping and $(\bar{x},\bar{y}) \in \mathrm{gr}F$. The σ-subdifferential of F at $(\bar{x},\bar{y})$ with $\sigma \in \{s,p,e,w\}$ is defined as

$$\partial^\sigma F(\bar{x},\bar{y}) := \{T \in L(X,Y) : \quad (\bar{x},\bar{y}-T(\bar{x})) \in K_\sigma((F-T)(X),Y_+)\}.$$

This definition is justified by the importance of the following immediate property

$$(\bar{x},\bar{y}) \in K_\sigma(F(X),Y_+) \quad \iff \quad 0 \in \partial^\sigma F(\bar{x},\bar{y}). \tag{2.1}$$

Another equivalent formulation of the definition of $\sigma-$subdifferential ($\sigma \in \{s,e,w,p\}$) of a Y_+-convex set-valued mapping $F : X \rightrightarrows Y$ at $(\bar{x},\bar{y}) \in \mathrm{gr}\,F$ is given by

$$\begin{array}{rcl}
\partial^s F(\bar{x},\bar{y}) & := & \{T \in L(X,Y) : \quad \forall x \in X,\ \forall y \in F(x), \quad T(x-\bar{x}) \leq_{Y_+} y-\bar{y}\},\\
\partial^e F(\bar{x},\bar{y}) & := & \{T \in L(X,Y) : \quad \nexists x \in X,\ \exists y \in F(x),\ \ y-\bar{y} \lneq_{Y_+} T(x-\bar{x})\},\\
\partial^w F(\bar{x},\bar{y}) & := & \{T \in L(X,Y) : \quad \nexists x \in X,\ \exists y \in F(x),\ \ y-\bar{y} <_{Y_+} T(x-\bar{x}),\\
\partial^p F(\bar{x},\bar{y}) & := & \{T \in L(X,Y) : \quad \exists \hat{Y}_+ \subset Y \text{ a proper convex cone such that}\\
 & & Y_+\backslash\{0\} \subseteq \mathrm{int}\hat{Y}_+, \nexists x \in X,\ \exists y \in F(x),\ \ y-\bar{y} \lneq_{\hat{Y}_+} T(x-\bar{x})\}.
\end{array}$$

By convention, we take $\partial^\sigma F(\bar{x},\bar{y}) = \emptyset$ if $(\bar{x},\bar{y}) \notin \mathrm{gr}\,F$, and we say that F is σ-subdifferentiable at $(\bar{x},\bar{y})$ with $\sigma \in \{s,e,p,w\}$ if $\partial^\sigma F(\bar{x},\bar{y}) \neq \emptyset$. Moreover, one can easily presents that $\partial^p F(\bar{x},\bar{y}) \subseteq \partial^e F(\bar{x},\bar{y}) \subseteq \partial^w F(\bar{x},\bar{y})$, and $\partial^s F(\bar{x},\bar{y}) \subseteq \partial^e F(\bar{x},\bar{y})$. In scalar case ($Y = \mathbb{R}$ and $Y_+ = \mathbb{R}_+$), all these subdifferentials coincide with the classical subdifferential denoted by ∂F and defined by

$$\partial F(\bar{x},\bar{y}) := \{x^* \in X^* : \langle x^*, x-\bar{x}\rangle \leq y-\bar{y}, \quad \forall (x,y) \in \mathrm{gr}\,F\}.$$

3. σ-Subdifferentiability for Set-Valued Mappings via Scalarization

Let $F : X \rightrightarrows Y$ be a set-valued mapping and $y^* \in Y_+^*$. Following the definition of the composite set-valued mapping, the real set-valued mapping $y^* \circ F : X \rightrightarrows \mathbb{R}$ is defined by

$$(y^* \circ F)(x) = \begin{cases} \bigcup_{y \in F(x)} \langle y^*, y\rangle, & \text{if } x \in \mathrm{dom}\,F,\\ \emptyset, & \text{if } x \notin \mathrm{dom}\,F. \end{cases}$$

Note that, for all $y^* \in Y_+^*$,

- $\mathrm{dom}(y^* \circ F) = \mathrm{dom}\,F$.
- F is Y_+-convex $\Rightarrow y^* \circ F$ is $\mathbb{R}_+$-convex.
- F is connected $\Rightarrow y^* \circ F$ is connected.

The next result characterizes scalarly the σ-subdifferential for $\sigma \in \{s,p,w\}$.

Theorem 3.1. *Let $F : X \rightrightarrows Y$ be a Y_+-convex set-valued mapping. Then*

$$\partial^s F(\bar{x},\bar{y}) = \bigcap_{y^* \in Y_+^* \backslash \{0\}} \{T \in L(X,Y) : y^* \circ T \in \partial(y^* \circ F)(\bar{x},\langle y^*,\bar{y}\rangle)\}, \tag{3.1}$$

and, for $\sigma \in \{p,w\}$,

$$\partial^{\sigma} F(\bar{x},\bar{y}) = \bigcup_{y^* \in Y_+^{*\sigma}} \{T \in L(X,Y) : y^* \circ T \in \partial(y^* \circ F)(\bar{x}, \langle y^*, \bar{y}\rangle)\}. \tag{3.2}$$

Proof. For (3.1), we have

$$\begin{aligned} T \in \partial^s F(\bar{x},\bar{y}) &\iff \bar{y} - T(\bar{x}) \in E_s(\mathrm{Im}(F-T), Y_+), \\ &\iff \forall y^* \in Y_+^* \backslash \{0\},\ \bar{y} - T(\bar{x}) \in \operatorname*{arg\,min}_{z \in \mathrm{Im}(F-T)} \langle y^*, z\rangle, \\ &\iff \forall y^* \in Y_+^* \backslash \{0\},\ \forall z \in \mathrm{Im}(F-T),\ \langle y^*, \bar{y} - T(\bar{x})\rangle \le \langle y^*, z\rangle, \\ &\iff \forall y^* \in Y_+^* \backslash \{0\},\ \forall x \in \mathrm{dom}\, F, y \in F(x),\ \langle y^*, \bar{y} - T(\bar{x})\rangle \le \langle y^*, y - T(x)\rangle, \\ &\iff \forall y^* \in Y_+^* \backslash \{0\},\ \forall (x,y) \in \mathrm{gr}\, F,\ \langle y^* \circ T, x - \bar{x}\rangle \le \langle y^*, y - \bar{y}\rangle, \\ &\iff \forall y^* \in Y_+^* \backslash \{0\},\ \forall (x,\alpha) \in \mathrm{gr}(y^* \circ F),\ \langle y^* \circ T, x - \bar{x}\rangle \le \alpha - \langle y^*, \bar{y}\rangle, \\ &\iff \forall y^* \in Y_+^* \backslash \{0\},\ y^* \circ T \in \partial(y^* \circ F)(\bar{x}, \langle y^*, \bar{y}\rangle). \end{aligned}$$

For (3.2), we have

$$\begin{aligned} T \in \partial^{\sigma} F(\bar{x},\bar{y}) &\iff \bar{y} - T(\bar{x}) \in E_{\sigma}(\mathrm{Im}(F-T), Y_+). \\ &\iff \exists y^* \in Y_+^{*\sigma},\ \bar{y} - T(\bar{x}) \in \operatorname*{arg\,min}_{z \in \mathrm{Im}(F-T)} \langle y^*, z\rangle, \end{aligned}$$

and the other equivalences can be obtained similarly to those in the proof of (3.1). □

For stating our main results, we also will need a new extension of the concept of regular subdifferentiability, due to Raffin [7], in the setting of set-valued mappings.

Definition 3.1. Let $F : X \rightrightarrows Y$ be a set-valued mapping and $(\bar{x},\bar{y}) \in \mathrm{gr}\, F$. F is said to be

(a) regular subdifferentiable at $(\bar{x},\bar{y})$ if

$$\partial(y^* \circ F)(\bar{x}, \langle y^*, \bar{y}\rangle) = y^* \circ \partial^s F(\bar{x},\bar{y}), \qquad \forall y^* \in Y_+^*,$$

where $y^* \circ \partial^s F(\bar{x},\bar{y}) := \{y^* \circ T, T \in \partial^s F(\bar{x},\bar{y})\}$.

(b) σ-regular subdifferentiable at $(\bar{x},\bar{y})$ with $\sigma \in \{p,w\}$ if

$$\partial(y^* \circ F)(\bar{x}, \langle y^*, \bar{y}\rangle) = y^* \circ \partial^s F(\bar{x},\bar{y}), \qquad \forall y^* \in Y_+^{*\sigma},$$

where $y^* \circ \partial^s_{\varepsilon} F(\bar{x},\bar{y}) := \{y^* \circ T, T \in \partial^s_{\varepsilon} F(\bar{x},\bar{y})\}$.

Remark 3.1. (i) The σ-regular η-subdifferentiability concept reposes upon the inclusion "$\subseteq$", the reverse one "$\supseteq$" being trivial. In scalar case, the concept is always fulfilled, i.e., $\partial_{\eta}(\alpha F)(\bar{x}, \alpha\bar{y}) = \alpha \partial_{\frac{\eta}{\alpha}} F(\bar{x},\bar{y}), \forall \alpha \in \mathbb{R}_+^{\sigma} =]0,+\infty[$.

(ii) The concept of regular subdifferentiability (resp. σ-regular subdifferentiability, $\sigma \in \{p,w\}$) of a single vector mapping $f : X \to Y \cup \{+\infty_Y\}$ at $\bar{x}$ can be written simply as $\partial(y^* \circ f)(\bar{x}) = y^* \circ \partial^s f(\bar{x})$ for all $y^* \in Y_+^*$ (resp. $\partial(y^* \circ f)(\bar{x}) = y^* \circ \partial^s f(\bar{x})$ for all $y^* \in Y_+^{\sigma}$, $\sigma \in \{p,w\}$).

We now illustrate the above definition with an example.

Example 3.1. By taking $X=\mathbb{R}$, $Y=\mathbb{R}^2$, $Y_+=\mathbb{R}^2_+$ and the set-valued mapping $F:\mathbb{R}\rightrightarrows\mathbb{R}^2$ defined by $F(x):=\{(a,b)\in\mathbb{R}^2: a\geq x,\ b\geq|x|\}$, we have, for any $y^*=(\alpha,\beta)\in Y_+^w=\mathbb{R}^2_+\backslash\{(0,0)\}$, $(y^*\circ F)(x)=\{\alpha a+\beta b: (a,b)\in F(x)\}$. It is easy to check that

$$\partial(y^*\circ F)(0,0)=\{t\in\mathbb{R}: \alpha-\beta\leq t\leq\alpha+\beta\}, \tag{3.3}$$

and

$$\partial^s F(0,(0,0))=\{(\mu,\nu)\in\mathbb{R}^2: \mu=1,-1\leq\nu\leq 1\},$$

which yields that $y^*\circ\partial^s F(0,(0,0))=\{\alpha+\beta\nu: (1,\nu)\in\partial^s F(0,(0,0))\}$. On the other hand, we claim that $\partial(y^*\circ F)(0,0)\subseteq y^*\circ\partial^s F(0,(0,0))$. Indeed, let $t\in\partial(y^*\circ F)(0,0)$, we will show that there exist $\nu\in\mathbb{R}$ such that $t=\alpha+\beta\nu$, $(1,\nu)\in\partial^s F(0,(0,0))$. In the case $\beta=0$, we obtain by using (3.3) that $t=\alpha$, which means that $t\in y^*\circ\partial^s F(0,(0,0))$. If $\beta\neq 0$, then we can write $t=\alpha+\beta(\frac{t-\alpha}{\beta})$ and from (3.3), we obtain $(1,\frac{t-\alpha}{\beta})\in\partial^s F(0,(0,0))$. Therefore

$$\partial(y^*\circ F)(0,0)\subseteq y^*\circ\partial^s F(0,(0,0)). \tag{3.4}$$

By using Remark 3.1 (i) and (3.4), we deduce that F is w-regular subdifferentiable at $(0,(0,0))$.

Proposition 3.1. *Let $f:X\to Y\cup\{+\infty_Y\}$ be a single vector-valued mapping with $\operatorname{dom}f\neq\emptyset$ and $F:X\rightrightarrows Y$ be defined by*

$$F(x):=\begin{cases}\{y\in Y: f(x)\leq_{Y_+} y\}, & \text{if } x\in\operatorname{dom}f,\\ \emptyset, & \text{otherwise}.\end{cases}$$

If f is regular subdifferentiable (resp. σ-regular subdifferentiable) at $\bar{x}$ ($\sigma\in\{p,w\}$), then F is regular subdifferentiable (resp. σ-regular subdifferentiable) $(\bar{x},f(\bar{x}))$ ($\sigma\in\{p,w\}$).

Proof. Firstly, we prove that $\partial^s F(\bar{x},f(\bar{x}))=\partial^s f(\bar{x})$. Let $T\in\partial^s F(\bar{x},f(\bar{x}))$, then $T(x-\bar{x})\leq_{Y_+} y-f(\bar{x})$, $\forall x\in\operatorname{dom}F$, $\forall y\in F(x)$. As $f(x)\in F(x)$ for all $x\in\operatorname{dom}F=\operatorname{dom}f$, it follows that $T(x-\bar{x})\leq_{Y_+} f(x)-f(\bar{x})$, $\forall x\in\operatorname{dom}f$. Thus $T(x-\bar{x})\leq_{Y_+} f(x)-f(\bar{x})$, $\forall x\in X$, i.e., $T\in\partial^s f(\bar{x})$. For the reverse inclusion, let $T\in\partial^s f(\bar{x})$, i.e., $T(x-\bar{x})\leq_{Y_+} f(x)-f(\bar{x})$, $\forall x\in X$. Hence, for all $x\in\operatorname{dom}f=\operatorname{dom}F$ and $y\in f(x)+Y_+$, $T(x-\bar{x})\leq_{Y_+} y-f(\bar{x})$, which yields that $T\in\partial^s F(\bar{x},f(\bar{x}))$. Thus

$$\partial^s F(\bar{x},f(\bar{x}))=\partial^s f(\bar{x}). \tag{3.5}$$

On the other hand, we can prove similarly that, for all $y^*\in Y_+^*$,

$$\partial(y^*\circ F)(\bar{x},\langle y^*,f(\bar{x})\rangle)=\partial(y^*\circ f)(\bar{x}). \tag{3.6}$$

As f is regular subdifferentiable (resp. σ-regular subdifferentiable) at $\bar{x}$,

$$\partial(y^*\circ f)(\bar{x})=y^*\circ\partial^s f(\bar{x}),\quad \forall y^*\in Y_+^*\ \ (\text{resp.} \forall y^*\in Y_+^\sigma). \tag{3.7}$$

From (3.5), (3.6), and (3.7), we have $\partial(y^*\circ F)(\bar{x},\langle y^*,f(\bar{x})\rangle)=y^*\circ\partial^s F(\bar{x},f(\bar{x}))$, $\forall y^*\in Y_+^*$ (resp. $\forall y^*\in Y_+^\sigma$). The proof is complete. □

It was shown by Valadier [11] that a single continuous convex vector mapping defined on a topological vector space X with values in a topological order complete vector lattice space is regular subdifferentiable at each point. In [12], Zowe extended some results of the regular subdifferentiability of a single convex vector mapping f (see Valadier [11]) from order complete vector lattices to separated locally convex topological vector spaces ordered by closed convex cone.

The next result demonstrates that the concept of σ-regular subdifferentiability play a crucial role for finding the gap between the strong subdifferential and σ-subdifferential ($\sigma \in \{p,w\}$).

Theorem 3.2. *Let $F : X \rightrightarrows Y$ be Y_+-convex and regular subdifferentiable at $(\bar{x},\bar{y}) \in \operatorname{gr} F$, ($\sigma \in \{p,w\}$), then $\partial^\sigma F(\bar{x},\bar{y}) = \partial^s F(\bar{x},\bar{y}) + \vartheta_\sigma(X,Y)$, where $\vartheta_\sigma(X,Y) := \{T \in L(X,Y) : \exists y^* \in Y_+^\sigma,\, y^* \circ T = 0\}$ is the set of σ-zero-like continuous operators, which represents here the gap between $\partial^\sigma F$ and $\partial^s F$.*

Proof. From (3.2) and σ-regular subdifferentiability, we have

$$\begin{aligned} A \in \partial^\sigma F(\bar{x},\bar{y}) &\iff \exists y^* \in Y_+^\sigma : y^* \circ A \in \partial(y^* \circ F)(\bar{x}, \langle y^*, \bar{y}\rangle) = y^* \circ \partial^s F(\bar{x},\bar{y}),\\ &\iff \exists y^* \in Y_+^\sigma, \exists B \in \partial^s F(\bar{x},\bar{y}) : y^* \circ (A-B) = 0,\\ &\iff \exists B \in \partial^s F(\bar{x},\bar{y}) : A - B \in \vartheta_\sigma(X,Y),\\ &\iff A \in \partial^s F(\bar{x},\bar{y}) + \vartheta_\sigma(X,Y), \end{aligned}$$

that is, $\partial^\sigma F(\bar{x},\bar{y}) = \partial^s F(\bar{x},\bar{y}) + \vartheta_\sigma(X,Y)$. ∎

Next, as an application of Theorem 3.2, we present an example in which the weak subdifferential of a mapping at a point is determinated.

Example 3.2. Let $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$ be a set-valued mapping defined as $F(x) := \{(a,b) \in \mathbb{R}^2 : a \geq x,\ \ b \geq |x|\}$. It was proved in Example 3.1 that F is w-regular subdifferentiable at $(0,(0,0))$ and $\partial^s F(0,(0,0)) = \{(\mu,\nu) \in \mathbb{R}^2 :\ \mu = 1, -1 \leq \nu \leq 1\}$, and it is to see that $\vartheta_w(X,Y) = \{(a,b) \in \mathbb{R}^2 : ab \leq 0\}$. Therefore, by Theorem 3.2, it follows that

$$\begin{aligned} \partial^w F(0,(0,0)) &= \partial^s F(0,(0,0)) + \vartheta_w(X,Y),\\ &= \{(a,b+\nu) \in \mathbb{R}^2 : (a-1)b \leq 0, -1 \leq \nu \leq 1\}. \end{aligned}$$

Let Z be a real locally convex topological vector space equipped with a nonempty convex cone Z_+. We need the following definition $L_+(Y,Z) := \{B \in L(Y,Z) : B(Y_+) \subseteq Z_+\}$.

Theorem 3.3. *Let $F : X \rightrightarrows Y$ be Y_+-convex and regular subdifferentiable at $(\bar{x},\bar{y}) \in \operatorname{gr} F$, $A \in L_+(Y,Z)$ and $\sigma \in \{p,w\}$. Then $\partial^\sigma(A \circ F)(\bar{x},A(\bar{y})) = A \circ \partial^s F(\bar{x},\bar{y}) + \vartheta_\sigma(X,Z)$.*

Proof. As F is Y_+-convex and $A \in L_+(Y,Z)$, $A \circ F$ is Z_+-convex. On the other hand, let $z^* \in Z_+^\sigma$. Thus it is easy to check that $z^* \circ A \in Y_+^*$. Hence, by Theorem (3.1) and the regular subdifferentiability of F at $(\bar{x},\bar{y})$, we have

$$\begin{aligned} &B \in \partial^\sigma(A \circ F)(\bar{x},A(\bar{y}))\\ &\iff \exists z^* \in Z_+^\sigma : z^* \circ B \in \partial(z^* \circ A \circ F)(\bar{x}, \langle z^*, A(\bar{y})\rangle) = z^* \circ A \circ \partial^s F(\bar{x},\bar{y}),\\ &\iff \exists C \in \partial^s F(\bar{x},\bar{y}), \exists z^* \in Z_+^\sigma : z^* \circ (B - A \circ C) = 0,\\ &\iff \exists C \in \partial^s F(\bar{x},\bar{y}) : B - A \circ C \in \vartheta_\sigma(X,Z),\\ &\iff B \in A \circ \partial^s F(\bar{x},\bar{y}) + \vartheta_\sigma(X,Z). \end{aligned}$$

The proof is complete. ∎

4. σ-Subdifferential Calculus Rules

In this section, we are concerned with the subdifferential calculus of the sum and/or the composition of convex set-valued mappings.

4.1. **Addition.** The following lemma plays an important role in proving our main results. It is the normal cone intersection formula for two convex subsets under an interior-point-like condition or a simple closure condition. This formula is derived from the subdifferential sum calculus by taking, the indicator functions of two convex subsets C and D, i.e., $\partial(\delta_C + \delta_D) = \partial\delta_C + \partial\delta_D$.

Lemma 4.1. [13, 14] *Let C and D be two convex subsets of X with $\bar{x} \in C \cap D$. Suppose that one of the following conditions is satisfied*

(i) *X is a real topological locally convex vector space and $(\operatorname{int} C) \cap D \neq \emptyset$.*
(ii) *X is a Banach space and both it subsets C and D are closed and the Attouch-Brézis qualification condition holds, i.e., $\mathbb{R}_+[C - D]$ is a closed vector subspace of X. Then we have the following normal cone intersection rule $N_{C\cap D}(\bar{x}) = N_C(\bar{x}) + N_D(\bar{x})$.*

Theorem 4.1. *Let $F_1, F_2 : X \rightrightarrows \mathbb{R}$ be two real set-valued mappings, $(\bar{x}, \bar{u}_1) \in \operatorname{gr} F$, and $(\bar{x}, \bar{u}_2) \in \operatorname{gr} G$. Suppose that one of the following condition is satisfied*

$$(\mathrm{MR}) \quad \begin{cases} X \text{ is a locally convex space}, \\ F_1,\ F_2 \text{ are } \mathbb{R}_+\text{-convex and } \operatorname{int}(\operatorname{epi} F_1) \cap \operatorname{epi} F_2 \neq \emptyset. \end{cases}$$

$$(\mathrm{AB}) \quad \begin{cases} X \text{ is a Banach space}, \\ F_1 \text{ and } F_2 \text{ are } \mathbb{R}_+\text{-convex and } \mathbb{R}_+\text{-epi-closed}, \\ \mathbb{R}_+[\operatorname{dom} F_1 - \operatorname{dom} F_2] \text{ is a closed vector subspace of } X. \end{cases}$$

Then

$$\partial(F_1 + F_2)(\bar{x}, \bar{u}_1 + \bar{u}_2) = \partial F_1(\bar{x}, \bar{u}_1) + \partial F_2(\bar{x}, \bar{u}_2).$$

Proof. Let $x_1^* \in \partial F_1(\bar{x}, \bar{u}_1)$ and $x_2^* \in \partial F_2(\bar{x}, \bar{u}_2)$, that is,

$$\begin{cases} \langle x_1^*, x - \bar{x}\rangle \leq u_1 - \bar{u}_1, \quad \forall (x, u_1) \in \operatorname{gr} F_1, & (4.1) \\ \langle x_2^*, x - \bar{x}\rangle \leq u_2 - \bar{u}_2, \quad \forall (x, u_2) \in \operatorname{gr} F_2. & (4.2) \end{cases}$$

By adding (4.1) and (4.2), we obtain, for any $u_1 \in F_1(x)$ and $u_2 \in F_2(x)$, $\langle x_1^* + x_2^*, x - \bar{x}\rangle \leq u_1 + u_2 - (\bar{u}_1 + \bar{u}_2)$, which means that $x_1^* + x_2^* \in \partial(F_1 + F_2)(\bar{x}, \bar{u}_1 + \bar{u}_2)$. For the reverse inclusion, let $x^* \in \partial(F_1 + F_2)(\bar{x}, \bar{u}_1 + \bar{u}_2)$, i.e., $u - (\bar{u}_1 + \bar{u}_2) - \langle x^*, x - \bar{x}\rangle \geq 0$, $\forall (x, u) \in \operatorname{gr}(F_1 + F_2)$, which yields that, for any $x \in \operatorname{dom} F_1 \cap \operatorname{dom} F_2$, $u_1 \in F_1(x)$, $u_2 \in F_2(x)$, and $\alpha, \beta \in \mathbb{R}_+$, $u_1 + \alpha + u_2 + \beta - (\bar{u}_1 + \bar{u}_2) - \langle x^*, x - \bar{x}\rangle \geq 0$, and thus it follows that, for any $(x, u_1) \in \operatorname{epi} F_1$ and $(x, u_2) \in \operatorname{epi} F_2$, $\langle x^*, x - \bar{x}\rangle - (u_1 - \bar{u}_1) - (u_2 - \bar{u}_2) \leq 0$. Define the following convex subsets of $X \times \mathbb{R} \times \mathbb{R}$

$$\begin{aligned} C &:= \{(x, u_1, u_2) \in X \times \mathbb{R} \times \mathbb{R} : (x, u_1) \in \operatorname{epi} F_1\}, \\ D &:= \{(x, u_1, u_2) \in X \times \mathbb{R} \times \mathbb{R} : (x, u_2) \in \operatorname{epi} F_2\}. \end{aligned}$$

Since $(x, u_1, u_2) \in C \cap D$ is equivalent to $(x, u_i) \in \operatorname{epi} F_i$ for $i \in \{1, 2\}$, it follows that $(x^*, -1, -1) \in N_{C\cap D}(\bar{x}, \bar{u}_1, \bar{u}_2)$. The condition (MR) implies that $(\operatorname{int} C) \cap D \neq \emptyset$. Indeed, since $\operatorname{int}(\operatorname{epi} F_1) \cap \operatorname{epi} F_2 \neq \emptyset$ and $\operatorname{int} C = \operatorname{int}(\operatorname{epi} F_1) \times \mathbb{R}$, it follows that, for any $(x, u) \in \operatorname{int}(\operatorname{epi} F_1) \cap \operatorname{epi} F_2$, $(x, u, u) \in (\operatorname{int} C) \cap D$, which ensures that $(\operatorname{int} C) \cap D \neq \emptyset$. From Lemma 4.1, we deduce that

$$(x^*, -1, -1) \in N_C(\bar{x}, \bar{u}_1, \bar{u}_2) + N_D(\bar{x}, \bar{u}_1, \bar{u}_2).$$

Hence, we deduce that there exist $(x_1^*, \alpha_1, \beta_1) \in N_C(\bar{x}, \bar{u}_1, \bar{u}_2)$ and $(x_2^*, \alpha_2, \beta_2) \in N_D(\bar{x}, \bar{u}_1, \bar{u}_2)$ such that

$$\begin{cases} (x^*, -1, -1) = (x_1^*, \alpha_1, \beta_1) + (x_2^*, \alpha_2, \beta_2), \\ \langle x_1^*, x - \bar{x}\rangle + \alpha_1(u_1 - \bar{u}_1) + \beta_1(u_2 - \bar{u}_2) \le 0, \quad \forall (x, u_1, u_2) \in C, & (4.3) \\ \langle x_2^*, x - \bar{x}\rangle + \alpha_2(u_1 - \bar{u}_1) + \beta_2(u_2 - \bar{u}_2) \le 0, \quad \forall (x, u_1, u_2) \in D. & (4.4) \end{cases}$$

By taking $x = \bar{x}$ and $u_1 = \bar{u}_1$ in (4.3), we obtain $\beta_1(u_2 - \bar{u}_2) \le 0$ for any $u_2 \in \mathbb{R}$ and hence $\beta_1 = 0$. Similarly, by taking $x = \bar{x}$ and $u_2 = \bar{u}_2$ in (4.4), we obtain $\alpha_2(u_1 - \bar{u}_1) \le 0$ for all $u_1 \in \mathbb{R}$ and thus $\alpha_2 = 0$. Consequently, $\alpha_1 = -1$, $\beta_2 = -1$, and

$$\begin{cases} \langle x_1^*, x - \bar{x}\rangle - (u_1 - \bar{u}_1) \le 0, \quad \forall (x, u_1) \in \operatorname{epi} F_1, \\ \langle x_2^*, x - \bar{x}\rangle - (u_2 - \bar{u}_2) \le 0, \quad \forall (x, u_2) \in \operatorname{epi} F_2, \end{cases}$$

which yields that $(x_1^*, -1) \in N_{\operatorname{epi} F_1}(\bar{x}, \bar{u}_1)$ and $(x_2^*, -1) \in N_{\operatorname{epi} F_2}(\bar{x}, \bar{u}_2)$, i.e., $x_1^* \in \partial F_1(\bar{x}, \bar{u}_1)$, $x_2^* \in \partial F_2(\bar{x}, \bar{u}_2)$ and $x^* = x_1^* + x_2^*$. Hence $\partial(F_1 + F_2)(\bar{x}, \bar{u}_1 + \bar{u}_2) \subseteq \partial F_1(\bar{x}, \bar{u}_1) + \partial F_2(\bar{x}, \bar{u}_2)$. Assume now that the Attouch-Brézis condition (AB) is satisfied. At first, let us observe that $C = \operatorname{epi} F_1 \times \mathbb{R}$ and $D = \varphi^{-1}(\operatorname{epi} F_2)$, where φ is a continuous function defined from $X \times \mathbb{R} \times \mathbb{R}$ into $X \times \mathbb{R}$ by $\varphi(x, u_1, u_2) := (x, u_2)$ for all $(x, u_1, u_2) \in X \times \mathbb{R} \times \mathbb{R}$. The subsets C and D are closed since F_1 and F_2 are $\mathbb{R}_+$-epi-closed. Now, let us prove that

$$\mathbb{R}_+[C - D] = \mathbb{R}_+[\operatorname{dom} F_1 - \operatorname{dom} F_2] \times \mathbb{R} \times \mathbb{R}. \tag{4.5}$$

Let $y = t(x_1 - x_2, \alpha_1 - \beta_1, \alpha_2 - \beta_2) \in \mathbb{R}_+[C - D]$, where $t \ge 0$, $(x_1, \alpha_1, \alpha_2) \in C$, and $(x_2, \beta_1, \beta_2) \in D$, i.e., obviously $x_1 \in \operatorname{dom} F_1$, $x_2 \in \operatorname{dom} F_2$. Hence, $y \in \mathbb{R}_+[\operatorname{dom} F_1 - \operatorname{dom} F_2] \times \mathbb{R} \times \mathbb{R}$. For the reverse inclusion, let us note that $0 \in (\operatorname{dom} F_1 - \operatorname{dom} F_2)$ since $\bar{x} \in \operatorname{dom} F_1 \cap \operatorname{dom} F_2$. Hence $\mathbb{R}_+[\operatorname{dom} F_1 - \operatorname{dom} F_2] = \mathbb{R}_+^*[\operatorname{dom} F_1 - \operatorname{dom} F_2]$, where $\mathbb{R}_+^* =]0, +\infty[$. Let $y = (t(x_1 - x_2), \alpha_1, \alpha_2) \in \mathbb{R}_+^*[\operatorname{dom} F_1 - \operatorname{dom} F_2] \times \mathbb{R} \times \mathbb{R}$ with $t > 0, x_1 \in \operatorname{dom} F_1$ and $x_2 \in \operatorname{dom} F_2$. As $y = (t(x_1 - x_2), \alpha_1, \alpha_2) = t((x_1, u_1, t^{-1}\alpha_2 + u_2) - (x_2, -t^{-1}\alpha_1 + u_1, u_2))$ with $u_1 \in F_1(x_1)$ and $u_2 \in F_2(x_2)$, one has that $y \in \mathbb{R}_+[C - D]$, and the relation (4.5) holds. Since $\mathbb{R}_+[\operatorname{dom} F_1 - \operatorname{dom} F_2]$ is a closed vector subspace of X, it follows from (4.5) that $\mathbb{R}_+[C - D]$ is a closed vector subspace of $X \times \mathbb{R} \times \mathbb{R}$. According to Lemma 4.1, we have $(x^*, -1, -1) \in N_C(\bar{x}, \bar{u}_1, \bar{u}_2) + N_D(\bar{x}, \bar{u}_1, \bar{u}_2)$, which yields that $\partial(F_1 + F_2)(\bar{x}, \bar{u}_1 + \bar{u}_2) \subseteq \partial F_1(\bar{x}, \bar{u}_1) + \partial F_2(\bar{x}, \bar{u}_2)$. So, we obtain the desired result. The proof is complete. □

Under the connectedness assumption, we obtain the following result.

Theorem 4.2. *Let $F_1, F_2 : X \rightrightarrows \mathbb{R}$ be two $\mathbb{R}_+$-convex set-valued mappings such that F_1 is connected at some point $x_0 \in \operatorname{dom} F_1 \cap \operatorname{dom} F_2$. Then, for any $(\bar{x}, \bar{u}_1) \in \operatorname{gr} F_1$, $(\bar{x}, \bar{u}_2) \in \operatorname{gr} F_2$, $\partial(F_1 + F_2)(\bar{x}, \bar{u}_1 + \bar{u}_2) = \partial F_1(\bar{x}, \bar{u}_1) + \partial F_2(\bar{x}, \bar{u}_2)$.*

Proof. First, we show that the connectedness assumption implies that $\operatorname{int}(\operatorname{epi} F_1) \cap \operatorname{epi} F_2 \neq \emptyset$. From Theorem 4.1 we obtain the desired result. From the connectedness of F_1 at x_0, there exist some neighborhood U of 0_X and a function $h : X \to \mathbb{R}$, which is continuous at x_0, such that $h(v) \in F_1(v)$ for all $v \in x_0 + U$. It follows that $x_0 + U \subset \operatorname{dom} F_1$ and hence $U = x_0 + U - x_0 \subset \operatorname{dom} F_1 - \operatorname{dom} F_2$, i.e., $0_X \in \operatorname{int}(\operatorname{dom} F_1 - \operatorname{dom} F_2)$. Let us note that the continuity of h at x_0 implies that $\operatorname{int}(\operatorname{epi} h) \neq \emptyset$ which ensures that $\operatorname{int}(\operatorname{epi} F_1) \neq \emptyset$. Now, let us prove that $\operatorname{int}(\operatorname{epi} F_1) \cap \operatorname{epi} F_2 \neq \emptyset$. We proceed by contradiction: Suppose that $\operatorname{int}(\operatorname{epi} F_1) \cap \operatorname{epi} F_2 = \emptyset$. By the separation theorem, there exist a nonzero $(x^*, \alpha) \in X^* \times \mathbb{R}$ and $\beta \in \mathbb{R}$ such that

$$\langle x^*, x_1\rangle + \alpha u_1 \le \beta \le \langle x^*, x_2\rangle + \alpha u_2, \quad \forall (x_1, u_1) \in \operatorname{epi} F_1, \ (x_2, u_2) \in \operatorname{epi} F_2. \tag{4.6}$$

As $x_0 \in \mathrm{dom}F_1 \cap \mathrm{dom}F_2$, we claim that there exists $u_0 \in \mathbb{R}$ such that $(x_0, u_0) \in \mathrm{epi}F_1 \cap \mathrm{epi}F_2$. Indeed, let $u_1 \in F_1(x)$ and $u_2 \in F_2(x)$. By taking $u_0 := \max(u_1, u_2)$, we obtain $u_0 \in F_1(x) + \mathbb{R}_+$ and $u_0 \in F_2(x) + \mathbb{R}_+$, i.e., $(x_0, u_0) \in \mathrm{epi}F_1 \cap \mathrm{epi}F_2$. Since $(x_0, u_0 + 1) \in \mathrm{epi}F_1 \cap \mathrm{epi}F_2$. By taking in relation (4.6) $x_1 = x_2 = x_0, u_1 = u_0$ and $u_2 = u_0 + 1$ (resp. $x_1 = x_2 = x_0, u_1 = u_0 + 1$ and $u_2 = u_0$), we obtain $\alpha \geq 0$ (resp. $\alpha \leq 0$), which yields that $\alpha = 0$. It follows from (4.6) that $\langle x^*, x\rangle \leq 0$ for all $x \in (\mathrm{dom}F_1 - \mathrm{dom}F_2)$, and as $\mathrm{dom}F_1 - \mathrm{dom}F_2$ is a neighborhood of 0, we deduce that $x^* = 0_{X^*}$. This leads to a contradiction. □

Let $F_i : X \rightrightarrows \mathbb{R}$ be a given set-valued mapping, $i = 1, \ldots, n$. The space $\mathbb{R}^n$ is endowed with componentwise order induced the positive orthant $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n, x_i \geq 0, \forall i = 1, \ldots, n\}$. We define the set-valued mappings $F : X \rightrightarrows \mathbb{R}^n$ by

$$F(x) := \begin{cases} \{(y_1, \ldots, y_n) : y_i \in F_i(x),\, i = 1, \ldots, n\}, & \text{if } x \in \bigcap_{i=1}^n \mathrm{dom}F_i, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{4.7}$$

In other words, $F(x) := \prod_{i=1}^n F_i(x)$ if $x \in \bigcap_{i=1}^n \mathrm{dom}F_i$, and $F(x) := \emptyset$ otherwise. It is easy to see that if F_i is connected at $\bar{x}$ (resp. $\mathbb{R}_+$-convex) for any $i \in \{1, \ldots, n\}$, then F is connected at $\bar{x}$ (resp. $\mathbb{R}^n_+$-convex).

Theorem 4.3. *Let $F_i : X \rightrightarrows \mathbb{R}$ be a $\mathbb{R}_+$-convex set-valued mapping and $(\bar{x}, \bar{y}_i) \in \mathrm{gr}\, F_i$, $i = 1, \ldots, n$. If F_i is connected at $\bar{x}$ and $\partial F_i(\bar{x}, \bar{y}_i) \neq \emptyset$ for all $i \in \{1, \ldots, n\}$, then F defined in (4.7) is regular subdifferentiable at $(\bar{x}, (\bar{y}_1, .., \bar{y}_n))$.*

Proof. We can see easily that

$$\prod_{i=1}^n \partial F_i(\bar{x}, \bar{y}_i) = \partial^s F(\bar{x}, (\bar{y}_1, .., \bar{y}_n)). \tag{4.8}$$

As $\partial F_i(\bar{x}, \bar{y}_i) \neq \emptyset$ for all $i \in \{1, \ldots, n\}$, we deduce from (4.8) that $\partial^s F(\bar{x}, (\bar{y}_1, .., \bar{y}_n)) \neq \emptyset$. For the regular subdifferentiability of F at $(\bar{x}, (\bar{y}_1, .., \bar{y}_n))$, we let $y^* = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_+$ to see

$$(y^* \circ F)(x) = (\sum_{i=1}^n \alpha_i F_i)(x), \quad \forall x \in X. \tag{4.9}$$

As F_1,....,F_n satisfy all the assumptions of the Theorem 4.2, we have

$$\partial(\sum_{i=1}^n \alpha_i F_i)(\bar{x}, \alpha_1 \bar{y}_1 + \ldots + \alpha_n \bar{y}_n) = \sum_{i=1}^n \partial(\alpha_i F_i)(\bar{x}, \alpha_i \bar{y}_i).$$

Let us note that $\partial(\alpha_i F_i)(\bar{x}, \alpha_i \bar{y}_i) = \alpha_i \partial F_i(\bar{x}, \bar{y}_i)$ and $\sum_{i=1}^n \alpha_i \partial F_i(\bar{x}, \bar{y}_i) = y^* \circ \prod_{i=1}^n \partial^s F_i(\bar{x}, \bar{y}_i)$. Hence by using (4.8) and (4.9), we have

$$\partial(y^* \circ F)(\bar{x}, \alpha_1 \bar{y}_1 + \ldots + \alpha_n \bar{y}_n) = y^* \circ \partial^s F(\bar{x}, (\bar{x}, (\bar{y}_1, .., \bar{y}_n)).$$

Thus the proof is complete. □

In the following theorem, we demonstrate how the above results can be used to obtain the formula for the Pareto subdifferential of the sum of two convex set-valued mappings taking values in a partially ordered vector space (Y, Y_+). The approach that we will use is to reduce the calculus of the Pareto subdifferential to that of the sum of two $\mathbb{R}_+$-convex set-valued mappings via a scalarization process.

Theorem 4.4. *Let $F_1, F_2 : X \rightrightarrows Y$ be two set-valued mappings and $\sigma \in \{p, w\}$. Then, for all $(\bar{x}, \bar{y}_1) \in \operatorname{gr} F_1$, and $(\bar{x}, \bar{y}_2) \in \operatorname{gr} F_2$, $\partial^\sigma (F_1 + F_2)(\bar{x}, \bar{y}_1 + \bar{y}_2) \supseteq \partial^\sigma F_1(\bar{x}, \bar{y}_1) + \partial^s F_2(\bar{x}, \bar{y}_2)$. Assume that F_2 is σ-regular subdifferentiable at $(\bar{x}, \bar{y}_2)$ and one of the two following qualification conditions is satisfied*

$$(\mathrm{MR})_1 \quad \begin{cases} X \text{ is a locally convex space}, \\ F_1,\ F_2 \text{ are } Y_+\text{-convex}, \\ F_1 \text{ or } F_2 \text{ is connected at some point of } \operatorname{dom} F_1 \cap \operatorname{dom} F_2. \end{cases}$$

$$(\mathrm{AB})_1 \quad \begin{cases} X \text{ is a Banach space}, \\ F_1,\ F_2 \text{ are } Y_+\text{-convex and star } Y_+\text{-epi-closed}, \\ \mathbb{R}_+[\operatorname{dom} F_1 - \operatorname{dom} F_2] \text{ is a closed vector subspace of } X. \end{cases}$$

Then $\partial^\sigma (F_1 + F_2)(\bar{x}, \bar{y}_1 + \bar{y}_2) = \partial^\sigma F_1(\bar{x}, \bar{y}_1) + \partial^s F_2(\bar{x}, \bar{y}_2)$.

Proof. Let us prove the first inclusion for $\sigma = p$. Let $A \in \partial^p F_1(\bar{x}, \bar{y}_1)$ and $B \in \partial^s F_2(\bar{x}, \bar{y}_2)$. Suppose, in the contrary, that $A + B \notin \partial^p (F_1 + F_2)(\bar{x}, \bar{y}_1 + \bar{y}_2)$. Then, for any proper convex cone $\hat{Y}_+ \subset Y$ such that $Y_+ \backslash \{0_Y\} \subseteq \operatorname{int} \hat{Y}_+$, there exist $x_0 \in \operatorname{dom}(F_1 + F_2) = \operatorname{dom} F_1 \cap \operatorname{dom} F_2$, $y_1 \in F_1(x_0)$ and $y_2 \in F_2(x_0)$ satisfying

$$y_1 + y_2 - \bar{y}_1 - \bar{y}_2 - A(x_0 - \bar{x}) - B(x_0 - \bar{x}) \in -\hat{Y}_+ \backslash l(\hat{Y}_+). \tag{4.10}$$

Since $B \in \partial^s F_2(\bar{x}, \bar{y}_2)$, we have $\bar{y}_2 - y_2 + B(x_0 - \bar{x}) \in -Y_+$. If $\bar{y}_2 - y_2 + B(x_0 - \bar{x}) = 0_Y$, we obtain immediately from (4.10) that $y_1 - \bar{y}_1 - A(x_0 - \bar{x}) \in -\hat{Y}_+ \backslash l(\hat{Y}_+)$, which contradicts the fact that $A \in \partial^p F_1(\bar{x}, \bar{y}_1)$. On other hand, Observe that $\bar{y}_2 - y_2 + B(x_0 - \bar{x}) \in -Y_+ \backslash \{0_Y\}$, which together with (4.10) and the fact thats $-Y_+ \backslash \{0_Y\} \subseteq -\operatorname{int} \hat{Y}_+ \subseteq -\hat{Y}_+ \backslash l(\hat{Y}_+)$ and $-\hat{Y}_+ \backslash l(\hat{Y}_+) - \hat{Y}_+ \backslash l(\hat{Y}_+) \subseteq -\hat{Y}_+ \backslash l(\hat{Y}_+)$ yields $y_1 - \bar{y}_1 - A(x_0 - \bar{x}) \in -\hat{Y}_+ \backslash l(\hat{Y}_+)$, which contradicts again the fact that $A \in \partial^p F_1(\bar{x}, \bar{y}_1)$. The case $\sigma = w$ is obtained similarly by taking into account the fact that $-Y_+ - \operatorname{int} Y_+ \subseteq -\operatorname{int} Y_+$. For the reverse inclusion, let $A \in \partial^\sigma (F_1 + F_2)(\bar{x}, \bar{y}_1 + \bar{y}_2)$. By Theorem 3.1, there exists $y^* \in Y_+^\sigma$ such that

$$y^* \circ A \in \partial[y^* \circ (F_1 + F_2)](\bar{x}, \langle y^*, \bar{y}_1 + \bar{y}_2 \rangle) = \partial(y^* \circ F_1 + y^* \circ F_2)(\bar{x}, \langle y^*, \bar{y}_1 \rangle + \langle y^*, \bar{y}_2 \rangle).$$

Following conditions $(\mathrm{MR})_1$ and $(\mathrm{AB})_1$, the real set-valued mappings $y^* \circ F_1$ and $y^* \circ F_2$ satisfy together the qualifications conditions (MR) and (AB) of Theorem 4.1 and hence

$$\partial(y^* \circ F_1 + y^* \circ F_2)(\bar{x}, \langle y^*, \bar{y}_1 \rangle + \langle y^*, \bar{y}_2 \rangle) = \partial(y^* \circ F_1)(\bar{x}, \langle y^*, \bar{y}_1 \rangle) + \partial(y^* \circ F_2)(\bar{x}, \langle y^*, \bar{y}_2 \rangle).$$

As F_2 is σ-regular subdifferentiable at $(\bar{x}, \bar{y}_2)$, we have

$$\partial(y^* \circ F_1 + y^* \circ F_2)(\bar{x}, \langle y^*, \bar{y}_1 \rangle + \langle y^*, \bar{y}_2 \rangle) = \partial(y^* \circ F_1)(\bar{x}, \langle y^*, \bar{y}_1 \rangle) + y^* \circ \partial^s F_2(\bar{x}, \bar{y}_2).$$

Thus we deduce the existence of $B \in \partial^s F_2(\bar{x}, \bar{y}_2)$ such that $y^* \circ (A - B) \in \partial(y^* \circ F_1)(\bar{x}, \langle y^*, \bar{y}_1 \rangle)$ which yields, thanks to Theorem 3.1, that $A - B \in \partial^\sigma F_1(\bar{x}, \bar{y}_1)$ i.e., $A \in \partial^\sigma F_1(\bar{x}, \bar{y}_1) + \partial^s F_2(\bar{x}, \bar{y}_2)$. Thus $\partial^\sigma (F_1 + F_2)(\bar{x}, \bar{y} + \bar{y}_2) = \partial^\sigma F_1(\bar{x}, \bar{y}_1) + \partial^s F_2(\bar{x}, \bar{y}_2)$. □

Now, we illustrate our man results with the help of the following example.

Example 4.1. By taking $X = \mathbb{R}$, $Y = \mathbb{R}^2$ and $Y_+ = \mathbb{R}^2_+$, consider the following set-valued mappings $F_1, F_2 : X \rightrightarrows Y$ defined, respectively, by $F_1(x) := \{(a, b) \in \mathbb{R}^2 : a \geq -2x,\ b \geq 3x\}$ and $F_2(x) := \{(a, b) \in \mathbb{R}^2 : a \geq 3x,\ b \geq -2x\}$. It is easy to check that F_1 and F_2 are $\mathbb{R}^2_+$-convex, connected at 0, and w-regular subdifferentiable at $(0, (0, 0))$. Thus

$$\partial^s F_1(0, (0, 0)) = \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha x \leq a,\ \beta x \leq b,\ \forall (x, (a, b)) \in \operatorname{gr} F_1\} = \{(-2, 3)\},$$

$$\partial^s F_2(0,(0,0)) = \{(\alpha,\beta)\in\mathbb{R}^2 : \alpha x \le a,\ \beta x \le b,\ \forall (x,(a,b))\in \mathrm{gr} F_2\} = \{(3,-2)\},$$

and

$$\partial^s (F_1+F_2)(0,(0,0)) = \{(\alpha,\beta)\in\mathbb{R}^2 : \alpha x \le a,\ \beta x \le b,\ \forall (x,(a,b))\in \mathrm{gr}(F_1+F_2)\} = \{(1,1)\}.$$

It is easy to see that $\vartheta_w(\mathbb{R},\mathbb{R}^2)=\{(a,b)\in\mathbb{R}^2 : ab\le 0\}$. According to Corollary 3.2, we obtain

$$\partial^w F_1((0,0),(0,0)) = \partial^s F_1((0,0),(0,0)) + \vartheta_w(\mathbb{R},\mathbb{R}^2) = \{(\alpha,\beta)\in\mathbb{R}^2 |\ (\alpha+2)(\beta-3)\le 0\},$$

$$\partial^w F_2((0,0),(0,0)) = \partial^s F_1((0,0),(0,0)) + \vartheta_w(\mathbb{R},\mathbb{R}^2) = \{(\alpha,\beta)\in\mathbb{R}^2 |\ (\alpha-3)(\beta+2)\le 0\},$$

and

$$\begin{aligned}\partial^w (F_1+F_2)((0,0),(0,0)) &= \partial^s (F_1+F_2)((0,0),(0,0)) + \vartheta_w(\mathbb{R},\mathbb{R}^2),\\ &= \{(\alpha,\beta)\in\mathbb{R}^2 |\ (\alpha-1)(\beta-1)\le 0\}.\end{aligned}$$

Observe that $(-3,3)\in\partial^w_{\varepsilon_1}F_1(0,(0,0))$ and $(3,-3)\in\partial^w_{\varepsilon_2}G(0,(0,0))$, which yield that $(0,0)\in \partial^w F_1(0,(0,0))+\partial^w F_2(0,(0,0))$. But $(0,0)\notin\partial^w(F_1+F_2)(0,(0,0))$ which proves that the reverse inclusion of (1.2) fails. In fact, the presence of the strong subdifferential enables us to obtain the desired equality, i.e.,

$$\begin{aligned}\partial^w F_1(0,(0,0))+\partial^s F_2(0,(0,0)) &= \{(\alpha,\beta)\in\mathbb{R}^2 |\ (\alpha+2)(\beta-3)\le 0\}+\{(3,-2)\}\\ &= \partial^w(F_1+F_2)(0,(0,0)).\end{aligned}$$

Remark 4.1. Consider a single vector valued mapping $f : X \longrightarrow Y\cup\{+\infty_Y\}$, and the Pareto subdifferential of f at $\bar{x}\in \mathrm{dom} f = \{x\in X : f(x)\in Y\}$ is defined for $\sigma\in\{s,e,p,w\}$ following [1] as $\partial^\sigma f(\bar{x}) = \{T\in L(X,Y) : f(\bar{x})-T(\bar{x})\in E_\sigma((f-T)(X),Y_+)\}$. Define the following set-valued mapping $F_f : X \rightrightarrows Y$ associated to f by

$$F_f(x) := \begin{cases} \{f(x)\} & \text{if } x\in \mathrm{dom}\, f,\\ \emptyset & \text{otherwise.}\end{cases}$$

It is obvious that $\mathrm{dom}\, F_f = \mathrm{dom}\, f$ and $\partial^\sigma f(\bar{x}) = \partial^\sigma F_f(\bar{x}, f(\bar{x}))$. Furthermore, if we suppose that f is continuous at $\bar{x}$, then F_f is connected at $\bar{x}$.

According to Remark 4.1, we obtain the following corollary which is a result obtained in [1].

Corollary 4.1. *Let $f_1,\ f_2 : X\longrightarrow Y\cup\{+\infty_Y\}$ be two single vector valued mappings and $\sigma\in\{p,w\}$. Assume that f_2 is σ-regular subdifferentiable at $\bar{x}\in \mathrm{dom}\, f_1\cap \mathrm{dom}\, f_2$ and one of the two following qualification conditions is satisfied*

$$(\mathrm{MR})_2 \quad \begin{cases} X \text{ is a locally convex space},\\ f_1,\ f_2 \text{ are } Y_+\text{-convex},\\ \text{one of the two functions is continuous at some point of } \mathrm{dom}\, f_1\cap\mathrm{dom}\, f_2.\end{cases}$$

$$(\mathrm{AB})_2 \quad \begin{cases} X \text{ is a Banach space},\\ f_1,\ f_2 \text{ are } Y_+\text{-convex and star } Y_+\text{-epi-closed},\\ \mathbb{R}_+[\mathrm{dom}\, f_1 - \mathrm{dom}\, f_2] \text{ is a closed vector subspace of } X.\end{cases}$$

Then $\partial^\sigma(f_1+f_2)(\bar{x}) = \partial^\sigma f_1(\bar{x})+\partial^s f_2(\bar{x})$.

Proof. Consider the following set-valued mappings

$$F_1(x)=\begin{cases}\{f_1(x)\}, & \text{if } x\in\operatorname{dom} f_1,\\ \emptyset, & \text{otherwise},\end{cases} \quad \textit{and} \quad F_2(x)=\begin{cases}\{f_2(x)\}, & \text{if } x\in\operatorname{dom} f_2,\\ \emptyset, & \text{otherwise}.\end{cases}$$

It is easy to check that if f_1 is continuous at some point of $\operatorname{dom} f_1\cap\operatorname{dom} f_2$, then the set-valued mapping F_1 is connected at some point of $\operatorname{dom} F_1\cap\operatorname{dom} F_2$ and also we easily check that the set-valued mappings F_1 and F_2 satisfy all the assumptions of Theorem 4.4. Thus

$$\partial^\sigma(F_1+F_2)(\bar{x},f_1(\bar{x})+f_2(\bar{x}))=\partial^\sigma F_1(\bar{x},f_1(\bar{x}))+\partial^s F_2(\bar{x},f_1(\bar{x})).$$

According to Remark 4.1, we obtain the desired result. □

4.2. **Composition.** In this subsection, we develop the σ-subdifferential calculus of the composite of two set-valued mappings. The approach that we will use for computing the Pareto subdifferential of the composition set-valued mappings is to transform it as the Pareto subdifferential of the sum of two set-valued mappings. In what follows, we assume that Z is separated and the convex cone Z_+ is pointed with nonempty topological interior. We shall work also with the following definitions. For $(x,z)\in X\times Z$, $(A,B)\in L(X,Y)\times L(Z,Y)$ and $y^*\in Y_+^*$, we set $(A,B)(x,z):=A(x)+B(z)$ and $y^*\circ(A,B):=(y^*\circ A,y^*\circ B)$. Let $F:X\rightrightarrows Y$, $G:X\rightrightarrows Z$ and $H:Z\rightrightarrows Y$ be three set-valued mappings and consider the following set-valued mappings

$$\begin{aligned}\tilde{F}_1: X\times Z &\rightrightarrows Y\\ (x,z)&\mapsto F(x)+R^v_{\operatorname{epi} G}(x,z),\\ \tilde{F}_2: X\times Z &\rightrightarrows Y\\ (x,z)&\mapsto H(z).\end{aligned}$$

Note that $\operatorname{dom}\tilde{F}_1=(\operatorname{dom} F\times Z)\cap\operatorname{epi} G$, $\operatorname{dom}\tilde{F}_2=X\times\operatorname{dom} H$, and $\operatorname{gr}\tilde{F}_2=X\times\operatorname{gr} H$. In addition

$$\begin{aligned}\operatorname{epi}\tilde{F}_2 &= \{(x,z,y)\in X\times Z\times Y:\ y\in\tilde{F}_2(x,z)+Y_+\},\\ &= \{(x,z,y)\in X\times Z\times Y:\ y\in H(z)+Y_+\},\\ &= X\times\operatorname{epi} H,\end{aligned}$$

and

$$\begin{aligned}\operatorname{epi}\tilde{F}_1 &= \{((x,z,y)\in X\times Z\times Y:\ y\in\tilde{F}_1(x,z)+Y_+\},\\ &= \{((x,z,y)\in X\times Z\times Y:\ y\in F(x)+R^v_{\operatorname{epi} G}(x,z)+Y_+\},\\ &= \{((x,z,y)\in X\times Z\times Y:\ (x,z)\in\operatorname{epi} G\ \textit{and}\ (x,y)\in\operatorname{epi} F\},\\ &= (\operatorname{epi} G\times Y)\cap\varphi^{-1}(\operatorname{epi} F),\end{aligned}$$

where φ is a continuous function defined from $X\times Z\times Y$ into $X\times Y$ by $\varphi(x,z,y):=(x,y)$ for all $(x,z,y)\in X\times Z\times Y$. It is easy to see that if F and H are Y_+-convex and G is Z_+-convex, then $\tilde{F}_1$ and $\tilde{F}_2$ are Y_+-convex. Now, we are going to present that the study of the formula $\partial^\sigma(F+H\circ G)$ can be reduced to that for $\partial^\sigma(\tilde{F}_1+\tilde{F}_2)$. For this, we need two lemmas: the first studies the relationship between the subdifferentials of $\tilde{F}_1$, $\tilde{F}_2$ and the subdifferentials of F, G and H, and the second studies the qualification conditions for the computation of $\partial^\sigma(\tilde{F}_1+\tilde{F}_2)$.

Lemma 4.2. *Let $\bar{x}\in\operatorname{dom} F\cap\operatorname{dom}(H\circ G)$, $\bar{y}_1\in F(\bar{x})$, $\bar{z}\in G(\bar{x})$, and $\bar{y}_2\in H(\bar{z})$. Then*

(i) *If H is (Z_+,Y_+)-nondecreasing, then*

$$A \in \partial^{\sigma}(F+H\circ G)(\bar{x},\bar{y}_1+\bar{y}_2) \quad \Longleftrightarrow \quad (A,0)\in \partial^{\sigma}(\tilde{F}_1+\tilde{F}_2)((\bar{x},\bar{z}),\bar{y}_1+\bar{y}_2);$$

(ii) $\partial^s \tilde{F}_2((\bar{x},\bar{z}),\bar{y}_2)=\{0\}\times\partial^s H(\bar{z},\bar{y}_2)$;

(iii) *If H is σ-regular subdifferentiable at $(\bar{z},\bar{y}_2)$, ($\sigma\in\{p,w\}$), then $\tilde{F}_2$ is σ-regular subdifferentiable at $((\bar{x},\bar{z}),\bar{y}_2)$.*

Proof. (i) First case: $\sigma=w$. Letting $A\in\partial^w(F+H\circ G)(\bar{x},\bar{y}_1+\bar{y}_2)$, we have

$$F(x)+(H\circ G)(x)-\bar{y}_1-\bar{y}_2-A(x-\bar{x})\subseteq Y\backslash-\operatorname{int}Y_+,\ \ \forall x\in X,$$

and then

$$F(x)+R^v_{\operatorname{epi}G}(x,z)+(H\circ G)(x)-\bar{y}_1-\bar{y}_2-A(x-\bar{x})\subseteq Y\backslash-\operatorname{int}Y_+,\ \ \forall(x,z)\in X\times Z$$

which implies that

$$\tilde{F}_1(x,z)+(H\circ G)(x)-\bar{y}_1-\bar{y}_2-A(x-\bar{x})\subseteq Y\backslash-\operatorname{int}Y_+,\ \ \forall(x,z)\in\operatorname{epi}G. \tag{4.11}$$

As H is (Z_+,Y_+)-nondecreasing, for any $(x,z)\in\operatorname{epi}G$, we have $H(z)\subset(H\circ G)(x)+Y_+$. From relation (4.11), it follows that $\tilde{F}_1(x,z)+H(z)-\bar{y}_1-\bar{y}_2-A(x-\bar{x})\subseteq(Y\backslash-\operatorname{int}Y_+)+Y_+$. Observe that $(Y\backslash-\operatorname{int}Y_+)+Y_+\subseteq Y\backslash-\operatorname{int}Y_+$. Indeed, let $y=y_1+y_2$ with $y_1\in Y\backslash-\operatorname{int}Y_+$ and $y_2\in Y_+$. Suppose that $y\notin Y\backslash-\operatorname{int}Y_+$. It follows that $y_1=y-y_2\in-\operatorname{int}Y_+-Y_+\subseteq-\operatorname{int}Y_+$. This contradicts the fact that $y_1\in Y\backslash-\operatorname{int}Y_+$ and hence we have

$$\tilde{F}_1(x,z)+\tilde{F}_2(x,z)-\bar{y}_1-\bar{y}_2-A(x-\bar{x})\subseteq Y\backslash-\operatorname{int}Y_+,\ \ \forall(x,z)\in X\times Z,$$

which yields $(A,0)\in\partial^w(\tilde{F}_1+\tilde{F}_2)((\bar{x},\bar{z}),\bar{y}_1+\bar{y}_2)$. Second case: $\sigma=p$. Let $A\in\partial^p(F+H\circ G)(\bar{x},\bar{y}_1+\bar{y}_2)$. There exists a proper convex cone $\hat{Y}_+\subset Y$ such that $Y_+\backslash\{0_Y\}\subseteq\operatorname{int}\hat{Y}_+$ and

$$F(x)+(H\circ G)(x)-\bar{y}_1-\bar{y}_2-A(x-\bar{x})\subseteq(Y\backslash(-\hat{Y}_+\backslash l(\hat{Y}_+)),\ \ \forall x\in X.$$

With a similar reasoning as that in the proof of the first case, we obtain that, for all $(x,z)\in X\times Z$,

$$\tilde{F}_1(x,z)+\tilde{F}_2(x,z)-\bar{u}-\bar{v}-A(x-\bar{x})\subseteq(Y\backslash(-\hat{Y}_+\backslash l(\hat{Y}_+))+Y_+. \tag{4.12}$$

We claim that $\tilde{F}_1(x,z)+\tilde{F}_2(x,z)-\bar{y}_1-\bar{y}_2-A(x-\bar{x})\subseteq Y\backslash(-\hat{Y}_+\backslash l(\hat{Y}_+)),\ \forall(x,z)\in X\times Z$, i.e., $(A,0)\in\partial^p(\tilde{F}_1+\tilde{F}_2)((\bar{x},\bar{z}),\bar{y}_1+\bar{y}_2)$. In fact, if there exists $(x_0,z_0)\in X\times Z$ such that

$$\tilde{F}_1(x_0,z_0)+\tilde{F}_2(x_0,z_0)-\bar{y}_1-\bar{y}_2-A(x_0-\bar{x})\nsubseteq Y\backslash(-\hat{Y}_+\backslash l(\hat{Y}_+)),$$

which asserts the existence of $w\in\tilde{F}_1(x_0,z_0)+\tilde{F}_2(x_0,z_0)-\bar{y}_1-\bar{y}_2-A(x_0-\bar{x})$ such that $w\in-\hat{Y}_+\backslash l(\hat{Y}_+)$, by using (4.12), we see that there exist $u\in(Y\backslash(-\hat{Y}_+\backslash l(\hat{Y}_+))$ and $v\in Y_+$ such that $w=u+v$. If we suppose $v=0_Y$, then $w\in(Y\backslash(-\hat{Y}_+\backslash l(\hat{Y}_+))$ and this contradicts the fact that $w\in-\hat{Y}_+\backslash l(\hat{Y}_+)$. As $v\in Y_+\backslash\{0_Y\}\subseteq\operatorname{int}\hat{Y}_+\subseteq\hat{Y}_+\backslash l(\hat{Y}_+)$, we can write $w\in(Y\backslash(-\hat{Y}_+\backslash l(\hat{Y}_+)))+\hat{Y}_+\backslash l(\hat{Y}_+)$ and $w-v\in(Y\backslash(-\hat{Y}_+\backslash l(\hat{Y}_+))$. This contradicts the fact that $w-v\in-(\hat{Y}_+\backslash l(\hat{Y}_+))-(\hat{Y}_+\backslash l(\hat{Y}_+))\subseteq-\hat{Y}_+\backslash l(\hat{Y}_+)$. Conversely, for $\sigma=w$, let us take any $(A,0)\in\partial^w(\tilde{F}_1+\tilde{F}_2)((\bar{x},\bar{z}),\bar{y}_1+\bar{y}_2)$. Then $\tilde{F}_1(x,z)+\tilde{F}_2(x,z)-\bar{y}_1-\bar{y}_2-A(x-\bar{x})\subseteq Y\backslash-\operatorname{int}Y_+$, $\forall(x,z)\in X\times Z$, i.e.,

$$F(x)+R^v_{\operatorname{epi}G}(x,z)+H(z)-\bar{y}_1-\bar{y}_2-A(x-\bar{x})\subseteq Y\backslash-\operatorname{int}Y_+,\ \ \forall(x,z)\in X\times Z.$$

Therefore, for all $(x,z)\in\operatorname{epi}G$, $F(x)+H(z)-\bar{y}_1-\bar{y}_2-A(x-\bar{x})\subseteq Y\backslash-\operatorname{int}Y_+$, which implies that, for all $x\in X$, $F(x)+\bigcup_{z\in G(x)}H(z)-\bar{y}_1-\bar{y}_2-A(x-\bar{x})\subseteq Y\backslash-\operatorname{int}Y_+$, i.e., $F(x)+(H\circ G)(x)-\bar{y}_1-\bar{y}_2-A(x-\bar{x})\subseteq Y\backslash-\operatorname{int}Y_+$, $\forall x\in X$.

Finally, $A\in\partial^w(F+H\circ G)(\bar{x},\bar{y}_1+\bar{y}_2)$. The converse for the case $\sigma=p$ is obtained similarly as the case $\sigma=w$.

(ii) Let $(A,B) \in \partial^s \tilde{F}_2((\bar{x},\bar{z}),\bar{y}_2)$. Then, for all $((x,z),y_2) \in \mathrm{gr}\tilde{F}_2 = X \times \mathrm{gr}H$, $A(x-\bar{x}) + B(z-\bar{z}) \leq_{Y_+} y_2 - \bar{y}_2$, By taking $z = \bar{z}$ and $y_2 = \bar{y}_2$ in the inequality above, we find, for all $x \in X$, $A(x-\bar{x}) \leq_{Y_+} 0$. Hence $A = 0$, and $\partial^s \tilde{F}_2((\bar{x},\bar{z}),\bar{y}_2) \subseteq \{0\} \times \partial^s H(\bar{z},\bar{y}_2)$. For the reverse inclusion, let $B \in \partial^s H(\bar{z},\bar{y}_2)$, i.e., $B(z-\bar{z}) \leq_{Y_+} y_2 - \bar{y}_2$, $\forall (z,y_2) \in \mathrm{gr}H$. As $\mathrm{gr}\tilde{F}_2 = X \times \mathrm{gr}H$, we deduce that $\{0\} \times \partial^s H(\bar{z},\bar{y}_2) \subseteq \partial^s \tilde{F}_2((\bar{x},\bar{z}),\bar{y}_2)$.

(iii) Suppose that H is σ-regular subdifferentiable at $(\bar{z},\bar{y}_2)$ and let $y^* \in Y_+^{\sigma}$. By using (ii), we have

$$(A,B) \in \partial(y^* \circ \tilde{F}_2)((\bar{x},\bar{z}),\langle y^*,\bar{y}_2\rangle) \iff A = 0 \textit{ and } B \in \partial(y^* \circ H)(\bar{z},\langle y^*,\bar{y}_2\rangle).$$

Since H is σ-regular subdifferentiable at $(\bar{z},\bar{y}_2)$, there exists $T \in \partial^s H(\bar{z},\bar{y}_2))$ such that $B = y^* \circ T$. As $\partial^s \tilde{F}_2((\bar{x},\bar{z}),\bar{y}_2) = \{0\} \times \partial^s H(\bar{z},\bar{y}_2)$, we can write $(0,T) \in \partial^s \tilde{F}_2((\bar{x},\bar{z}),\bar{y}_2)$ and $(A,B) = y^* \circ (0,T) \in y^* \circ \partial^s \tilde{F}_2((\bar{x},\bar{z}),\bar{y}_2)$. Therefore, we obtain $\partial(y^* \circ \tilde{F}_2)((\bar{x},\bar{z}),\langle y^*,\bar{y}_2\rangle) = y^* \circ \partial^s \tilde{F}_2((\bar{x},\bar{z}),\bar{y}_2)$, $\forall y^* \in Y_+^{\sigma}$, which means that $\tilde{F}_2$ is σ-regular subdifferentiable at $((\bar{x},\bar{z}),\bar{y}_2)$. □

Let us consider the following conditions.

$$(\mathrm{MR})_3 \quad \begin{cases} X, Z \text{ are locally convex spaces,} \\ F, H \text{ are } Y_+\text{-convex and } G \text{ is } Z_+\text{-convex,} \\ \exists a \in \mathrm{dom}F \cap \mathrm{dom}G \text{ such that } H \text{ is connected at some point } b \in G(a). \end{cases}$$

$$(\mathrm{AB})_3 \quad \begin{cases} X, Z \text{ are Banach spaces,} \\ F, H \text{ are } Y_+\text{-convex, star } Y_+\text{-epi-closed.} \\ G \text{ is } Z_+\text{-convex and } Z_+\text{-epi-closed,} \\ W = \mathbb{R}_+[G(\mathrm{dom}\,F \cap \mathrm{dom}\,G) - \mathrm{dom}\,H] \text{ is a closed vector subspace of } X. \end{cases}$$

Lemma 4.3. (i) *If condition* $(\mathrm{MR})_3$ *holds, then* $\tilde{F}_2$ *is connected at* $(a,b) \in \mathrm{dom}\,\tilde{F}_1$*;*
(ii) *If condition* $(\mathrm{AB})_3$ *holds, then* $X \times W = \mathbb{R}_+[\mathrm{dom}\,\tilde{F}_1 - \mathrm{dom}\,\tilde{F}_2]$ *is a closed vector subspace of* $X \times Z$*, and* $\tilde{F}_1$ *and* $\tilde{F}_2$ *are star* Y_+*-epi-closed.*

Proof. i) As H is connected at $b \in G(a)$, there exist a neighborhood V of b and a mapping $f : Z \to Y$ such that $f(v) \in H(v)$ for all $v \in V$ and f is continuous at b. Define the following function

$$\begin{aligned} \tilde{f} : X \times Z &\to Y \\ (x,z) &\mapsto f(z). \end{aligned}$$

It is clear that $\tilde{f}$ is continuous at (a,b) and $\tilde{f}(x,v) \in \tilde{F}_2(x,v)$ for all $(x,v) \in X \times V$. Hence, $\tilde{F}_2$ is connected at $(a,b) \in \mathrm{dom}\,\tilde{F}_1$.

ii) The equality $X \times W = \mathbb{R}_+[\mathrm{dom}\,\tilde{F}_1 - \mathrm{dom}\,\tilde{F}_2]$ was proved in [3]. The star Y_+-epi-closedness of $\tilde{F}_1$ and $\tilde{F}_2$ follows directly from the fact that, for all $y^* \in Y_+^*$, $\mathrm{epi}(y^* \circ \tilde{F}_2) = X \times \mathrm{epi}(y^* \circ H)$ and $\mathrm{epi}(y^* \circ \tilde{F}_1) = (\mathrm{epi}G \times Y) \cap \varphi^{-1}(\mathrm{epi}(y^* \circ F))$, where φ is a continuous function defined from $X \times Z \times Y$ into $X \times Y$ by $\varphi(x,z,y) := (x,y)$ for all $(x,z,y) \in X \times Z \times Y$. □

Now, we are ready to state our main results in this subsection.

Theorem 4.5. *Let* $F : X \rightrightarrows Y$*,* $G : X \rightrightarrows Z$*, and* $H : Z \rightrightarrows Y$ *be three set-valued mappings, and let* $(\bar{x},\bar{y}_1) \in \mathrm{gr}F$*,* $(\bar{x},\bar{z}) \in \mathrm{gr}G$*, and* $(\bar{z},\bar{y}_2) \in \mathrm{gr}H$ $(\sigma \in \{p,w\})$*. Then*

$$\partial^{\sigma}(F + H \circ G)(\bar{x},\bar{y}_1 + \bar{y}_2) \supseteq \bigcup_{A \in \partial^s H(\bar{z},\bar{y}_2)} \partial^{\sigma}(F + A \circ G)(\bar{x},\bar{y}_1 + A(\bar{z})).$$

Let H be (Z_+,Y_+)-nondecreasing, and σ-regular subdifferentiable at $(\bar{z},\bar{y}_2)$. Let the condition $(\mathrm{MR})_3$ *or* $(\mathrm{AB})_3$ *hold. Then*

$$\partial^{\sigma}(F+H\circ G)(\bar{x},\bar{y}_1+\bar{y}_2)=\bigcup_{A\in\partial^{s}H(\bar{z},\bar{y}_2)}\partial^{\sigma}(F+A\circ G)(\bar{x},\bar{y}_1+A(\bar{z})).$$

Proof. Let us prove the first inclusion for $\sigma=w$. Let $A\in\partial^{s}H(\bar{z},\bar{y}_2)$ and $B\in\partial^{w}(F+A\circ G)(\bar{x},\bar{y}_1+A(\bar{z}))$. Suppose by contradiction that $B\notin\partial^{w}(F+H\circ G)(\bar{x},\bar{y}_1+\bar{y}_2)$. Then there exist $x_0\in\operatorname{dom}F\cap\operatorname{dom}(H\circ G)$, $y_1\in F(x_0)$, $z_0\in G(x_0)$, and $y_2\in H(z_0)$ such that

$$y_1+y_2-\bar{y}_1-\bar{y}_2-B(x_0-\bar{x})\in-\operatorname{int}Y_+. \tag{4.13}$$

As $A\in\partial^{s}H(\bar{z},\bar{y}_2)$, we have

$$\bar{y}_2-y_2+A(z_0)-A(\bar{z})\in-Y_+. \tag{4.14}$$

Adding (4.13) and (4.14) term by term and using $-Y_+-\operatorname{int}Y_+\subseteq-\operatorname{int}Y_+$, we arrive at $y_1+A(z_0)-\bar{y}_1-A(\bar{z})-B(x_0-\bar{x})\in-\operatorname{int}Y_+$, which contradicts the fact that $B\in\partial^{w}(F+A\circ G)(\bar{x},\bar{y}_1+A(\bar{z}))$. The case $\sigma=p$ is similarly obtained by the same arguments as in Theorem 4.4. For the reverse inclusion, take any $B\in\partial^{\sigma}(F+H\circ G)(\bar{x},\bar{y}_1+\bar{y}_2)$. According to Lemma 4.2 (i), we have

$$(B,0)\in\partial^{\sigma}(\tilde{F}_1+\tilde{F}_2)((\bar{x},\bar{z}),\bar{y}_1+\bar{y}_2). \tag{4.15}$$

Under (MR_3) or (AB_3), successively using Lemma 4.2 (iii), Lemma 4.3, and Theorem 4.4, we obtain that

$$\partial^{\sigma}(\tilde{F}_1+\tilde{F}_2)((\bar{x},\bar{z}),\bar{y}_1+\bar{y}_2)=\partial^{\sigma}\tilde{F}_1((\bar{x},\bar{z}),\bar{y}_1)+\partial^{s}\tilde{F}_2((\bar{x},\bar{z}),\bar{y}_2).$$

Then there exists $(T,A)\in\partial^{s}\tilde{F}_2((\bar{x},\bar{z}),\bar{y}_2)$ such that $(B-T,-A)\in\partial^{\sigma}\tilde{F}_1((\bar{x},\bar{z}),\bar{y}_1)$. By virtue of Lemma 4.2 (ii), we obtain $T=0$ and $A\in\partial^{s}H(\bar{z},\bar{y}_2)$. Now, let us prove that $B\in\partial^{\sigma}(F+A\circ G)(\bar{x},\bar{y}_1+A(\bar{z}))$. Case $\sigma=p$. As $(B,-A)\in\partial^{p}\tilde{F}_1((\bar{x},\bar{z}),\bar{y}_1)$, there exists a proper convex cone $\hat{Y}_+\subset Y$ with $Y_+\backslash\{0_Y\}\subseteq\operatorname{int}\hat{Y}_+$ such that, for all $(x,z)\in X\times Z$,

$$F(x)+R^{v}_{\operatorname{epi}G}(x,z)-\bar{y}_1-B(x-\bar{x})+A(z-\bar{z})\subseteq Y\backslash(-\hat{Y}_+\backslash l(\hat{Y}_+)),$$

which implies that, for all $(x,z)\in\operatorname{epi}G$,

$$F(x)-\bar{y}_1-B(x-\bar{x})+A(z-\bar{z})\subset Y\backslash(-\hat{Y}_+\backslash l(\hat{Y}_+)).$$

Hence,

$$(F+A\circ G)(x)-(\bar{y}_1+A(\bar{z}))-B(x-\bar{x})\subset Y\backslash(-\hat{Y}_+\backslash l(\hat{Y}_+)),\ \ \forall x\in X.$$

Thus $B\in\partial^{p}(F+A\circ G)(\bar{x},\bar{y}_1+A(\bar{z}))$. The case $\sigma=w$ is obtained similarly. Therefore, the proof of theorem is complete. □

By taking $F(x)=\{0_Y\}$ for any $x\in X$, we obtain the following corollary.

Corollary 4.2. *Let $G:X\rightrightarrows Z$ and $H:Z\rightrightarrows Y$ be two set-valued mappings, and let $\bar{z}\in G(\bar{x})$ and $\bar{y}\in H(\bar{z})$. Let H be (Z_+,Y_+)-nondecreasing and σ-regular subdifferentiable at $(\bar{z},\bar{y})$. Let one of the following conditions hold*

$$(\mathrm{MR})_4\quad\begin{cases}X,Z\text{ are locally convex spaces},\\ G\text{ is }Z_+\text{-convex and }H\text{ is }Y_+\text{-convex},\\ H\text{ is connected at some point of }\operatorname{Im}G.\end{cases}$$

$$(\mathrm{AB})_4 \quad \begin{cases} X,\ Z \text{ are Banach spaces}, \\ H \text{ is } Y_+\text{-convex, star } Y_+\text{-epi-closed}, \\ G \text{ is } Z_+\text{-convex and } Z_+\text{-epi-closed}, \\ \mathbb{R}_+[G(\operatorname{dom} G) - \operatorname{dom} H] \text{ is a closed vector subspace of } X. \end{cases}$$

Then $\partial^\sigma(H \circ G)(\bar{x},\bar{y}) = \bigcup_{A \in \partial^s H(\bar{z},\bar{y})} \partial^\sigma(A \circ G)(\bar{x}, A(\bar{z}))$.

Consider now the case of composition with a linear operator. Let $A : X \to Z$ be a linear operator and $H : Z \rightrightarrows Y$ be a Y_+-convex set-valued mapping. By putting $Z_+ = \{0_Z\}$, the function H is obviously (Z_+,Y_+)-nondecreasing and A is Z_+-convex . Applying Corollary 4.2, we have the following result.

Corollary 4.3. *Let* $\bar{x} \in X$ *and* $\bar{y} \in H(A(\bar{x}))$. *Let* H *be* σ*-regular subdifferentiable at* $(A(\bar{x}),\bar{y})$ *and one of the following conditions hold*

$$(\mathrm{MR})_5 \quad \begin{cases} X,\ Z \text{ are locally convex spaces}, \\ H \text{ is } Y_+\text{-convex}, \\ H \text{ is connected at some point of } \operatorname{Im} A. \end{cases}$$

$$(\mathrm{AB})_5 \quad \begin{cases} X,\ Z \text{ are Banach spaces}, \\ H \text{ is } Y_+\text{-convex and star } Y_+\text{-epi-closed}, \\ \mathbb{R}_+[\operatorname{Im} A - \operatorname{dom} H] \text{ is a closed vector subspace of } X. \end{cases}$$

Then $\partial^\sigma(H \circ A)(\bar{x},\bar{y}) = \partial^s H(A(\bar{x}),\bar{y}) \circ A + \vartheta_\sigma(X,Y)$.

Proof. It follows from Corollary 4.2 and Theorem 3.2 that

$$\begin{aligned} \partial^\sigma(H \circ A)(\bar{x},\bar{y}) &= \bigcup_{B \in \partial^s H(A(\bar{x}),\bar{y})} \partial^\sigma(B \circ A)(\bar{x}, A(\bar{z})) \\ &= \bigcup_{B \in \partial^s H(A(\bar{x}),\bar{y})} \partial^s(B \circ A)(A(\bar{x}),\bar{y}) + \vartheta_\sigma(X,Y) \\ &= \bigcup_{B \in \partial^s H(A(\bar{x}),\bar{y})} B \circ A + \vartheta_\sigma(X,Y) \\ &= \partial^s H(A(\bar{x}),\bar{y}) \circ A + \vartheta_\sigma(X,Y). \end{aligned}$$

□

Corollary 4.4. *Under the assumptions of Theorem 4.5, if* F *or* G *is connected at some point of* $\operatorname{dom} F \cap \operatorname{dom} G$ *and* H *is* (Z_+,Y_+)*-nondecreasing, then*

(1) *if* G *is regular subdifferentiable at* $(\bar{x},\bar{z})$*, then*

$$\partial^\sigma(F + H \circ G)(\bar{x},\bar{y}_1 + \bar{y}_2) = \partial^\sigma F(\bar{x},\bar{y}_1) + \bigcup_{A \in \partial^s H(\bar{z},\bar{y}_2)} \partial^s(A \circ G)(\bar{x}, A(\bar{z}));$$

(2) *if* F *is* σ*-regular subdifferentiable at* $(\bar{x},\bar{u})$*, then*

$$\partial^\sigma(F + H \circ G)(\bar{x},\bar{y}_1 + \bar{y}_2) = \partial^s F(\bar{x},\bar{y}_1) + \bigcup_{A \in \partial^s H(\bar{z},\bar{y}_2)} \partial^\sigma(A \circ G)(\bar{x}, A(\bar{z})).$$

Proof. According to Theorem 4.5, we have

$$\partial^\sigma(F + H \circ G)(\bar{x},\bar{y}_1 + \bar{y}_2) = \bigcup_{A \in \partial^s H(\bar{z},\bar{y}_2)} \partial^\sigma(F + A \circ G)(\bar{x},\bar{y}_1 + A(\bar{z})).$$

Let us note that $A\circ G$ is Y_+-convex since $A\in L_+(Z,Y)$. As F or G is connected at some point of $\operatorname{dom}F\cap\operatorname{dom}G$, we can see easily that F or $A\circ G$ is connected at some point of $\operatorname{dom}F\cap\operatorname{dom}(A\circ G)=\operatorname{dom}F\cap\operatorname{dom}G$. In order to apply Theorem 4.4 for $\partial^\sigma(F+A\circ G)(\bar{x},A(\bar{z}))$, it remains only to prove that $A\circ G$ is σ-regular subdifferentiable at $(\bar{x},A(\bar{z}))$. In fact, since $y^*\circ A\in Z_+^*$ for any $y^*\in Y_+^*$, it follows by the regular subdifferentiability of G at $(\bar{x},\bar{z})\in\operatorname{gr}G$ that

$$\partial(y^*\circ A\circ G)(\bar{x},\langle y^*\circ A,\bar{z}\rangle)=y^*\circ A\circ\partial^s G(\bar{x},\bar{z}).$$

Since $A\in L_+(Z,Y)$, we have $A\circ\partial^s G(\bar{x},\bar{z})\subseteq\partial^s(A\circ G)(\bar{x},A(\bar{z}))$ and hence

$$\partial(y^*\circ A\circ G)(\bar{x},\langle y^*\circ A,\bar{z}\rangle)\subseteq y^*\circ\partial^s(A\circ G)(\bar{x},A(\bar{z})).$$

For the reverse inclusion, let $T\in\partial^s(A\circ G)(\bar{x},A(\bar{z}))$ and $(x,\alpha)\in\operatorname{gr}(y^*\circ A\circ G)$. It is easy to check that $\alpha=\langle y^*,y\rangle$ for some $y\in(A\circ G)(x)$. Then

$$\begin{aligned}
T\in\partial^s(A\circ G)(\bar{x},A(\bar{z})) &\implies \langle T,x-\bar{x}\rangle\leq_{Y_+} y-A(\bar{z}),\quad \forall(x,y)\in\operatorname{gr}(A\circ G)\\
&\implies \langle y^*\circ T,x-\bar{x}\rangle\leq\langle y^*,y\rangle-\langle y^*,A(\bar{z})\rangle,\quad \forall(x,y)\in\operatorname{gr}(A\circ G)\\
&\implies \langle y^*\circ T,x-\bar{x}\rangle\leq\alpha-\langle y^*\circ A,\bar{z}\rangle,\quad \forall(x,\alpha)\in\operatorname{gr}(y^*\circ A\circ G),
\end{aligned}$$

which implies that $y^*\circ T\in\partial(y^*\circ A\circ G)(\bar{x},\langle y^*\circ A,\bar{z}\rangle)$ and hence we have the following inclusion

$$y^*\circ\partial^s(A\circ G)(\bar{x},A(\bar{z}))\subseteq\partial(y^*\circ A\circ G)(\bar{x},\langle y^*\circ A,\bar{z}\rangle),$$

which yields the desired equality. □

By using the same arguments used in Corollary 4.1 and Remark 4.1, we recapture a result established in [1].

Corollary 4.5. *Let $f:X\to Y\cup\{+\infty_Y\}$, $g:X\to Z\cup\{+\infty_Y\}$ and $h:Z\to Y\cup\{+\infty_Y\}$ be three single vector-valued mappings, and $\sigma\in\{p,w\}$. Let h be (Z_+,Y_+)-nondecreasing, σ-regular subdifferentiable at $g(\bar{x})$ and one of the following conditions hold*

$$(\mathrm{MR})_6\quad\begin{cases} X,\ Z \text{ are locally convex spaces},\\ f,\ h \text{ are } Y_+\text{-convex, } g \text{ is } Z_+\text{-convex},\\ h \text{ is continuous at some point of } g(\operatorname{dom}f\cap\operatorname{dom}g).\end{cases}$$

$$(\mathrm{AB})_6\quad\begin{cases} X,\ Z \text{ are Banach spaces},\\ f,\ h \text{ are } Y_+\text{-convex, star } Y_+\text{-epi-closed},\\ g \text{ is } Z_+\text{-convex and } Z_+\text{-epi-closed},\\ W=\mathbb{R}_+[g(\operatorname{dom}f\cap\operatorname{dom}g)-\operatorname{dom}h] \text{ is a closed vector subspace of } X.\end{cases}$$

Then, for any $\bar{x}\in X$, $\partial^\sigma(f+h\circ g)(\bar{x})=\bigcup_{A\in\partial^s h(g(\bar{x}))}\partial^\sigma(f+A\circ g)(\bar{x})$.

5. σ-EFFICIENCY OPTIMALITY CONDITIONS

In this section, we consider the following constrained vector set-valued optimization problem

$$(\mathrm{P})\quad\begin{cases}\text{Minimize F(x)},\\ x\in S,\end{cases}$$

where $F : X \rightrightarrows Y$ is a set-valued mapping, and S is a nonempty convex closed subset of X. By using the set-valued indicator mapping R^v_S of the nonempty subset $S \subseteq X$, problem (P_1) becomes equivalent to the unconstrained set-valued minimization problem

$$(\mathrm{Q}) \quad \begin{cases} \text{Minimize } (F + R^v_S)(x), \\ x \in X, \end{cases}$$

in the following sense

$$K_\sigma(F(S), Y_+) = K_\sigma((F + R^v_S)(X), Y_+). \tag{5.1}$$

Lemma 5.1. (i) *If S is convex and closed, then R^v_S is proper, Y_+-convex, and star Y_+-epi-closed. Furthermore, for all $\bar{x} \in S$, $\partial^s R^v_S(\bar{x}, 0_Y) = N^v_S(\bar{x})$, where*

$$N^v_S(\bar{x}) := \{A \in L(X,Y) : A(x - \bar{x}) \leq_{Y_+} 0_Y,\ \forall x \in S\}$$

is the vector normal cone at $\bar{x} \in S$.

(ii) *If $\mathrm{int}(S) \neq \emptyset$, then, R^v_S is connected on $\mathrm{int}(S)$.*

(iii) *R^v_S is σ-regular (resp. regular) subdifferentiable on $S \times \{0_Y\}$ ($\sigma \in \{p, w\}$).*

Proof. (i) The properness of R^v_S is immediate since $\mathrm{dom}\,(R^v_S) = S \neq \emptyset$. The epigraph of R^v_S is given by

$$\mathrm{epi}\, R^v_S = \{(x,y) \in X \times Y : y \in R^v_S(x) + Y_+\} = S \times Y_+,$$

and its Y_+-convexity follows easily from the convexity of S and Y_+. Let us note that for any $y^* \in Y^*_+$

$$\mathrm{epi}(y^* \circ R^v_S) = \{(x,\alpha) \in X \times \mathbb{R} : \alpha \in (y^* \circ R^v_S)(x) + \mathbb{R}_+\} = S \times \mathbb{R}_+,$$

and therefore the star Y_+-epi-closedness of R^v_S comes from the closedness of S.
(ii) Let us consider the following single mapping $h : Z \longrightarrow Y$ defined by $h(z) := 0_Y$ for all $z \in Z$. Since $0_Y \in R^v_S(z)$, for any $z \in S$, it follows that $h(z) \in R^v_S(z)$, for any $z \in \mathrm{int}(S)$, which ensures that R^v_S is connected on $\mathrm{int}(S)$.
(iii) Let us first show that R^v_S is σ-regular subdifferentiable on $S \times \{0_Y\}$, i.e., $\partial(y^* \circ R^v_S)(\bar{x}, 0) = y^* \circ \partial^s R^v_S(\bar{x}, 0_Y)$ for all $y^* \in Y^{*\sigma}_+$ and $\bar{x} \in S$. Let $A \in \partial^s R^v_S(\bar{x}, 0_Y)$. By virtue of Theorem 3.1, we have $y^* \circ A \in \partial(y^* \circ R^v_S)(\bar{x}, 0)$ for any $y^* \in Y^{*\sigma}_+$, i.e., $\langle y^* \circ A, x - \bar{x}\rangle = \langle y^*, A(x - \bar{x})\rangle \leq 0$ for all $x \in S$, which yields that $y^* \circ \partial^s R^v_S(\bar{x}, 0_Y) \subseteq \partial(y^* \circ R^v_S)(\bar{x}, 0)$. Conversely, let $y^* \in Y^{*\sigma}_+$ and $x^* \in \partial(y^* \circ R^v_S)(\bar{x}, 0)$. There exists $e \in \mathrm{int}\, Y_+$ such that $\langle y^*, e\rangle = 1$ (see [1]). By taking the following operator $A_{y^*} : X \to Y$ defined by $A_{y^*}(x) = \langle x^*, x\rangle e$, we easily check that $A_{y^*} \in L(X,Y)$ and $y^* \circ A_{y^*} = x^*$. On the other hand, we have $A_{y^*}(x - \bar{x}) = \langle x^*, x - \bar{x}\rangle e \leq_{Y_+} 0_Y$, $\forall x \in S$, that is, $A_{y^*} \in \partial^s R^v_S(\bar{x}, 0_Y)$. Consequently, $y^* \circ \partial^s R^v_S(\bar{x}, 0_Y) = \partial(y^* \circ R^v_S)(\bar{x}, 0)$. This proves the σ-regular subdifferentiability. The above equality holds for $y^* = 0$ and hence R^v_S is regular subdifferentiable on $S \times \{0_Y\}$. □

We are now in a position to establish σ-efficient optimality conditions for problem (P_1) with $\sigma \in \{p, w\}$.

Theorem 5.1. *Let $F : X \rightrightarrows Y$ be a set-valued mapping, S a nonempty convex closed subset of X and $(\bar{x}, \bar{y}) \in \mathrm{gr} F$ with $\bar{x} \in S$. Let one of the following qualification conditions hold*

$$(\mathrm{MR})_7 \quad \begin{cases} X \text{ is a locally convex space}, \\ F \text{ is } Y_+\text{-convex}, \\ \mathrm{dom}\, F \cap \mathrm{int}(S) \neq \emptyset \text{ or } F \text{ is connected at some point of } \mathrm{dom}\, F \cap S. \end{cases}$$

$$(\mathrm{AB})_7 \quad \begin{cases} X \textit{ is a Banach space}, \\ F \textit{ is } Y_+\textit{-convex, star } Y_+\textit{-epi-closed}, \\ \mathbb{R}_+[\operatorname{dom} F - S] \textit{ is a closed vector subspace of } X. \end{cases}$$

Then $(\bar{x},\bar{y})$ *is a* σ*-efficient solution for* (P_1) *with respect to* Y_+ *if and only if there exists* $A \in \partial^\sigma F(\bar{x},\bar{y})$ *such that* $-A \in N_S^v(\bar{x})$.

Proof. According to (5.1), we have

$$(\bar{x},\bar{y}) \in K_\sigma(F(S),Y_+) \quad \Longleftrightarrow \quad (\bar{x},\bar{y}) \in K_\sigma((F+R_S^v)(X),Y_+).$$

From relation (2.1), we can write

$$(\bar{x},\bar{y}) \in K_\sigma(F(S),Y_+) \quad \Longleftrightarrow \quad 0 \in \partial^\sigma(F+R_S^v)(\bar{x},\bar{y}+0_Y). \tag{5.2}$$

The conditions $(\mathrm{MR})_7$, $(\mathrm{AB})_7$, and Lemma 5.1 indicate that mappings F and R_S^v satisfy all the hypotheses of Theorem 4.4. Hence, $0 \in \partial^\sigma F(\bar{x},\bar{y}) + \partial^s R_S^v(\bar{x},0_Y)$, i.e., there exists $A \in \partial^\sigma F(\bar{x},\bar{y})$ such that $-A \in \partial^s R_S^v(\bar{x},0_Y) = N_S^v(\bar{x})$. □

Previously, in Theorem 3.2, we have studied the gap between Pareto σ-subdifferential ($\sigma \in \{p,w\}$) and strong subdifferential. This result enables us to derive the relationship between the set of σ-efficient solutions and the set of strong solutions of the problem (P).

Corollary 5.1. *In addition to the assumptions of Theorem 5.1, assume that* F *is* σ*-regular subdifferentiable at* $(\bar{x},\bar{y}) \in \operatorname{gr} F$, $(\sigma \in \{p,w\})$. *Then*

$$(\bar{x},\bar{y}) \in E_\sigma^{set}(F(S),Y_+) \quad \Longleftrightarrow \quad \exists B \in \vartheta_\sigma(X,Y): \quad (\bar{x},\bar{y}) \in E_s^{set}((F-B)(S),Y_+).$$

Proof. By virtue of Theorem 5.1, we have $(\bar{x},\bar{y}) \in E_\sigma^{set}(F(S),Y_+)$ if and only if there exists $A \in \partial^\sigma F(\bar{x},\bar{y})$ such that $-A \in \partial^s R_S^v(\bar{x},0_Y)$. Since F is σ-regular subdifferentiable at $(\bar{x},\bar{y})$, we obtain according to Theorem 3.2 that $\partial^\sigma F(\bar{x},\bar{y}) = \partial^s F(\bar{x},\bar{y}) + \vartheta_\sigma(X,Y)$. It is obvious that $\vartheta_\sigma(X,Y) = -\vartheta_\sigma(X,Y)$. Hence, there exists $B \in \vartheta_\sigma(X,Y)$ such that

$$B \in -A + \partial^s F(\bar{x},\bar{y}) \subseteq \partial^s R_S^v(\bar{x},0_Y) + \partial^s F(\bar{x},\bar{y}) \subseteq \partial^s(F+R_S^v)(\bar{x},\bar{y}).$$

By using the relation (2.1) and (5.1), we obtain

$$(\bar{x},\bar{y}) \in K_s((F-B+R_S^v)(X),Y_+) = K_s((F-B)(S),Y_+).$$

Conversely, let $(\bar{x},\bar{y}) \in K_s((F-B)(S),Y_+) = K_s((F-B+R_S^v)(X),Y_+)$ with $B \in \vartheta_\sigma(X,Y)$. It follows from relation (2.1) that $0 \in \partial^s(F+R_S^v-B)(\bar{x},\bar{y})$, i.e., $B \in \partial^s(F+R_S^v)(\bar{x},\bar{y})$. By Theorem 3.1, we have

$$y^* \circ B \in \partial(y^* \circ F + y^* \circ R_S^v)(\bar{x},\langle y^*,\bar{y}\rangle), \quad \forall y^* \in Y_+^* \backslash \{0\}. \tag{5.3}$$

Since $B \in \vartheta_\sigma(X,Y)$, there exists $v^* \in Y_+^\sigma$ such $v^* \circ B = 0$. Hence, we obtain from (5.3) that

$$0 \in \partial(v^* \circ F + v^* \circ R_S^v)(\bar{x},\langle v^*,\bar{y}\rangle),$$

which means by virtue of Theorem 3.1 that $0 \in \partial^\sigma(F+R_S^v)(\bar{x},\bar{y})$. From (5.2), we have $(\bar{x},\bar{y}) \in K_\sigma(F(S),Y_+)$. □

Example 5.1. Let us consider the following constrained vector set-valued optimization problem

$$(\mathrm{Q}) \quad \begin{cases} \text{Min F(x)}, \\ x \in [1,2], \end{cases}$$

where $F:\mathbb{R}\rightrightarrows\mathbb{R}^2$ is defined by $F(x):=\{(a,b)\in\mathbb{R}^2 : a\geq x,\;\; b\geq |x|\}$. It is easy to see that F satisfies the condition $(\mathrm{MR})_7$ of Theorem 5.1. It was proved in Example 3.2 that

$$\partial^w F(0,(0,0)) \;\; = \;\; \{(a,b+\nu)\in\mathbb{R}^2 : (a-1)b\leq 0, \quad -1\leq\nu\leq 1\}.$$

In addition, we have

$$N^v_{[1,2]}(0)=\{(\alpha,\beta)\in\mathbb{R}^2 : \alpha\leq 0, \quad \beta\leq 0\}=-\mathbb{R}^2.$$

Obviously, $(1,1)\in\partial^w F(0,(0,0))$ and $(-1,-1)\in N^v_{[1,2]}(0)$. Hence, $(1,1)$ is a w-efficient solution for (Q).

In the sequel, we establish the σ-efficient optimality conditions in terms of the Lagrange-Kuhn-Tucker multipliers and the vector normal cone of the following general convex set-valued mathematical programming problem

$$(\mathrm{P}_3) \quad \begin{cases} \text{Minimize F(x)}, \\ G(x)\cap -Z_+\neq\emptyset, \\ x\in C, \end{cases}$$

where $F:X\rightrightarrows Y$ and $G:X\rightrightarrows Z$ are two set-valued mappings, Z is a real locally convex topological vector space, Z_+ is a closed convex pointed cone with nonempty topological interior, and C is a nonempty closed convex set of X. For establishing the σ-efficient optimality conditions of this problem, we need the following lemma.

Lemma 5.2. (i) *If Z is a real locally convex topological vector space and $Z_+\subseteq Z$ is a closed convex cone, then the strong subdifferential of the indicator set-valued mapping $R^v_{-Z_+}:Z\rightrightarrows Y$ is given by $\partial^s R^v_{-Z_+}(\bar{z},0_Y)=\{A\in L_+(Z,Y) : A(\bar{z})=0_Y\}$.*

(ii) *The indicator set-valued mapping $R^v_{-Z_+}$ is (Z_+,Y_+)-nondecreasing on Z.*

Proof. (i) Let $\bar{z}\in -Z_+$. Obviously, $(\bar{z},0_Y)\in\mathrm{gr}R^v_{-Z_+}=-Z_+\times\{0_Y\}$. From Lemma 5.1 (i), we have $\partial^s R^v_{-Z_+}(\bar{z},0_Y)=N^v_{-Z_+}(\bar{z})$. Now, let us prove the first inclusion

$$\partial^s R^v_{-Z_+}(\bar{z},0_Y)\subseteq\{A\in L_+(Z,Y) : \; A(\bar{z})=0_Y\}.$$

We have

$$A\in\partial^s R^v_{-Z_+}(\bar{z},0_Y)\Longleftrightarrow A(z-\bar{z})\leq_{Y_+} 0_Y,\; \forall z\in -Z_+. \tag{5.4}$$

By taking successively $z=0$ and $z=2\bar{z}$ in (5.4), we have $A(\bar{z})\in Y_+\cap -Y_+=\{0_Y\}$, i.e., $A(\bar{z})=0_Y$. Consequently, we deduce from (5.4) that $A(z)\in Y_+$, for all $z\in Z_+$, which means that $A\in L_+(Z,Y)$. For the reverse inclusion, since $A(\bar{z})=0_Y$ and $A\in L_+(Z,Y)$, then it follows that $A(z-\bar{z})=A(z)\leq_{Y_+} 0_Y$ for all $z\in -Z_+$. Hence, the equality holds.

(ii) has been proved in [6]. □

Now, we are ready to state σ-efficient optimality conditions of the problem (P_3) in terms of the Lagrange−Kuhn−Tucker multipliers and the vector normal cone.

Theorem 5.2. *Let $F:X\rightrightarrows Y$ and $G:X\rightrightarrows Z$ be two set-valued mappings. Let $(\bar{x},\bar{y})\in\mathrm{gr}F$ with $\bar{x}\in C$ and $G(\bar{x})\cap(-Z_+)\neq\emptyset$. Assume that one of the following conditions holds*

$$(\mathrm{MR})_8 \quad \begin{cases} X,\, Z \text{ are locally convex vector spaces}, \\ F \text{ is } Y_+\text{-convex and } G \text{ is } Z_+\text{-convex}, \\ \mathrm{int}(-Z_+)\cap G(C\cap\mathrm{dom}F\cap\mathrm{dom}G)\neq\emptyset. \end{cases}$$

$$(\mathrm{AB})_8 \quad \begin{cases} X,\, Z \text{ are Banach spaces}, \\ F \text{ is } Y_+\text{-convex and star } Y_+\text{-epi-closed}, \\ G \text{ is } Z_+\text{-convex and epi-closed }, \\ \mathbb{R}_+[G(\mathrm{dom}\, F \cap C \cap \mathrm{dom}\, G) + Z_+] \text{ is a closed vector subspace of } Z. \end{cases}$$

Then $(\bar{x},\bar{y})$ is a σ-efficient solution of problem (P_3) *if and only if, for any $\bar{z} \in G(\bar{x}) \cap -Z_+$, there exists $A \in L_+(Z,Y)$ such that $A(\bar{z}) = 0_Y$ and $0 \in \partial^\sigma(F + A \circ G + R_C^v)(\bar{x},\bar{y})$.*

Proof. The feasible set associated to problem (P_3) is given by $S = \{x \in X : G(x) \cap -Z_+ \neq \emptyset\} \cap C$, and it is easy to check that $R_S^v = R_C^v + R_{-Z_+}^v \circ G$. Thus problem (P_3) becomes equivalent to the unconstrained set-valued minimization problem

$$\begin{cases} \text{Minimize } (\mathrm{F} + R_C^v + R_{-Z_+}^v \circ G)(x), \\ x \in X. \end{cases}$$

By virtue of (5.1), we have $K_\sigma(F(S),Y_+) = K_\sigma((F + R_C^v + R_{-Z_+}^v \circ G)(X),Y_+)$. From relation (2.1), we can write

$$(\bar{x},\bar{y}) \in K_\sigma(F(S),Y_+) \Longleftrightarrow 0 \in \partial^\sigma(F + R_C^v + R_{-Z_+}^v \circ G)(\bar{x},\bar{y}).$$

Observe that $\mathrm{epi}(F + R_C^v) = \mathrm{epi}\, F \cap (C \times Y)$, which assert that the convexity of the set-valued mapping $F + R_C^v$, follows from the convexity of the epigraph of F and the convexity of C. Also, let us note that, for any $y^* \in Y_+^*$,

$$\mathrm{epi}(y^* \circ (F + R_C^v)) = \mathrm{epi}(y^* \circ F + y^* \circ R_C^v) = \mathrm{epi}(y^* \circ F) \cap (C \times \mathbb{R}).$$

Thus the star Y_+-epi-closedness of the set-valued mapping $F + R_C^v$ comes from the star Y_+-epi-closedness of F and the closedness of the subset C. Note that the conditions $(\bar{x},\bar{y}) \in \mathrm{gr}\, F$ with $\bar{x} \in C$ and $G(\bar{x}) \cap (-Z_+) \neq \emptyset$ may be written equivalently as $(\bar{x},\bar{y}) \in \mathrm{gr}(F + R_C^v)$, $(\bar{x},\bar{z}) \in \mathrm{gr}\, G$, and $(\bar{z},0_Y) \in \mathrm{gr} R_{-Z_+}^v$ for any $\bar{z} \in G(\bar{x}) \cap (-Z_+)$. According to Lemma 5.1 and Lemma 5.2, the set-valued mappings $F + R_C^v$, G, and $R_{-Z_+}^v$ satisfy all the assumptions of Theorem 4.5. Thus we obtain $(\bar{x},\bar{y}) \in E_\sigma^{set}(F(S),Y_+)$ if and only if there exists $A \in \partial^s R_{-Z_+}^v(\bar{z},0_Y) = \{A \in L_+(Z,Y) : \; A(\bar{z}) = 0_Y\}$ such that $0 \in \partial^\sigma(F + R_C^v + A \circ G)(\bar{x},\bar{y} + A(\bar{z}))$. The proof of theorem is complete. □

Corollary 5.2. *Under the assumptions of Theorem 5.2, we assume in addition that, for $\sigma \in \{p,w\}$, F is σ-regular subdifferentiable at $(\bar{x},\bar{y})$ and connected at some point of C, and G is regular subdifferentiable at $(\bar{x},\bar{z})$, star Z_+-epi-closed and connected at some point of C . Then $(\bar{x},\bar{y})$ is a σ-efficient solution of problem* (P_3) *if and only if, for any $\bar{z} \in G(\bar{x}) \cap -Z_+$, there exist $A \in L_+(Z,Y)$, $B \in \partial^\sigma F(\bar{x},\bar{y})$, and $T \in \partial^s G(\bar{x},\bar{z})$ such that $A(\bar{z}) = 0_Y$ and $-A \circ T - B \in N_C^v(\bar{x})$.*

Proof. According to Theorem 5.2, we have that $(\bar{x},\bar{y})$ is a σ-efficient solution of problem (P_3) if and only if, for any $\bar{z} \in G(\bar{x}) \cap -Z_+$, there exists $A \in L_+(Z,Y)$ such that $A(\bar{z}) = 0_Y$ and $0 \in \partial^\sigma(F + A \circ G + R_C^v)(\bar{x},\bar{y})$. Since $A \in L_+(Z,Y)$, we deduce $y^* \circ A \in Z_+^*$ for any $y^* \in Y_+^*$. Since G is Z_+-star epi-closed, it follows that $A \circ G$ is star Y_+-epi-closed. Also, we deduce that $A \circ G$ is Y_+-convex. As G is connected at some point of C, it is easy to check that $A \circ G$ is connected at some point of C. Since $\mathrm{epi}(A \circ G + R_C^v) = \mathrm{epi}(A \circ G) \cap (C \times Y)$, then the convexity of the set-valued mapping $A \circ G + R_C^v$ follows from the convexity of the epigraph of $A \circ G$ and the convexity of the subset C. Note that, for any $y^* \in Y_+^*$,

$$\mathrm{epi}(y^* \circ (A \circ G + R_C^v)) = \mathrm{epi}(y^* \circ A \circ G + y^* \circ R_C^v) = \mathrm{epi}(y^* \circ A \circ G) \cap (C \times \mathbb{R}).$$

Thus the star Y_+-epi-closedness of the set-valued mapping $A \circ G + R_C^v$ follows from the star Y_+-epi-closedness of $A \circ G$ and the closedness of the subset C. The set-valued mappings F and $A \circ G + R_C^v$ satisfy together all the assumptions of Theorem 4.4 and hence we obtain

$$0 \in \partial^\sigma(F + A \circ G + R_C^v)(\bar{x},\bar{y}) = \partial^\sigma(A \circ G + R_C^v)(\bar{x},0_Y) + \partial^s F(\bar{x},\bar{y}).$$

According to Lemma 5.1, the set-valued indicator mapping R_C^v is σ-regular subdifferentiable on $C \times \{0_Y\}$. By using the fact that $A \circ G$ is connected at some point of C, we claim that the set-valued mappings $A \circ G$ and R_C^v satisfy together all the hypothesis of Theorem 4.4. Hence

$$0 \in \partial^\sigma(F + A \circ G + R_C^v)(\bar{x},\bar{y}) = \partial^\sigma(A \circ G)(\bar{x},0_Y) + \partial^s R_C^v(\bar{x},0_Y) + \partial^s F(\bar{x},\bar{y}).$$

By virtue of Theorem 3.3, we have

$$0 \in \partial^\sigma(F + A \circ G + R_C^v)(\bar{x},\bar{y}) = A \circ \partial^s G(\bar{x},\bar{z}) + \vartheta_\sigma(X,Y) + \partial^s F(\bar{x},\bar{y}) + N_C^v(\bar{x}).$$

Following Theorem 3.2, we have $\partial^\sigma F(\bar{x},\bar{y}) = \vartheta_\sigma(X,Y) + \partial^s F(\bar{x},\bar{y})$. This implies

$$0 \in \partial^\sigma(F + A \circ G + R_C^v)(\bar{x},\bar{y}) = A \circ \partial^s G(\bar{x},\bar{z}) + \partial^\sigma F(\bar{x},\bar{y}) + N_C^v(\bar{x}),$$

i.e., there exist $B \in \partial^\sigma F(\bar{x},\bar{y})$ and $T \in \partial^s G(\bar{x},\bar{z})$ such that $-A \circ T - B \in N_C^v(\bar{x})$. This completes the proof. □

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