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CONVERGENCE OF INEXACT ITERATES OF STRICT CONTRACTIONS IN METRIC SPACES WITH GRAPHS

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Abstract. In our joint 2006 paper with Dan Butnariu, it was shown that if all exact orbits of a nonexpansive mapping converge to a fixed point, then this convergence property also holds for all the inexact orbits with summable errors. In the present paper, we establish an analog of this result for the inexact iterates of strict contractions in complete metric spaces with graphs.

Keywords. Complete metric space; Fixed point; Graph; Inexact iterate; Nonexpansive mapping, Strict contraction.

1. Introduction

For almost sixty years now, there has been a lot of research activities regarding the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings; see, e.g., [1]-[21] and the references cited therein.

These activities stem from Banach's classical theorem [22] concerning the existence of a unique fixed point for a strict contraction. It also concerns the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this field including, in particular, the studies of common fixed point problems, feasibility, projecyion methods, and variational inequalities, which find important applications in engineering, medical, and the natural sciences [20, 21, 23, 24, 25, 26, 27].

In [2], it was shown that if all exact orbits of a nonexpansive mapping converge to a fixed point, then this convergence property also holds for all its inexact orbits with summable errors. This result was obtained for a nonexpansive self-mapping of a complete metric space. In the present paper, we establish an analog of this result for inexact iterates of strict contractions defined on complete metric spaces with graphs. Note that the study of contractive and nonexpansive mappings in metric spaces with graphs has recently become a rapidly growing area of research [8, 12, 13, 28, 29, 30].

2. Main Results

Let (X,ρ) be a complete metric space, and let G be a graph such that its set of vertices $V(G)$ is a subset of X and its set of edges $E(G)$ is closed in $X\times X$. We identify the graph G with the

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pair $(V(G), E(G))$. Assume further that $\alpha \in (0,1)$ and that a mapping $T : X \to X$ satisfies the following assumption

(A) for all $x, y \in X$, if $(x,y) \in E(G)$, then $(T(x), T(y)) \in E(G)$ and

$$\rho(T(x), T(y)) \le \alpha \rho(x,y). \tag{2.1}$$

Note that the assumption (A) appears in [28] where the mappings satisfy it are called G-contractions (see also [29, 30]).

In the following section, we establish the following result.

Theorem 2.1. *Assume that a sequence $\{x_i\}_{i=0}^{\infty} \subset X$ satisfies*

$$(x_i, x_{i+1}) \in E(G),\ i = 0, 1, \dots, \tag{2.2}$$

and

$$\sum_{i=0}^{\infty} \rho(T(x_i), x_{i+1}) < \infty. \tag{2.3}$$

Then there exists the limit $x_* = \lim_{i\to\infty} x_i$ so that $(x_*, x_*) \in E(G)$. If T is continuous at x_*, then $T(x_*) = x_*$.

Note that a similar result in the case of weakly connected graphs was already obtained in [28].

Corollary 2.1. *Assume that a point $x \in X$ satisfies $(x, T(x)) \in E(G)$. Then there exists $x_* = \lim_{i\to\infty} T^i(x)$ so that $(x_*, x_*) \in E(G)$. If T is continuous at x_*, then $T(x_*) = x_*$.*

As was pointed out by the referee, the continuity at x^* can be replaced by G-continuity, that is, the sequential continuity on paths in G (see [30, Definition 2]).

3. Proof of Theorem 2.1

We use the convention that any sum over the empty set equals zero.

Let $q \ge 0$ be an integer. In view of (2.1) and (2.2), we have

$$\begin{aligned} \rho(x_{q+2}, x_{q+1}) &\le \rho(x_{q+2}, T(x_{q+1})) + \rho(T(x_{q+1}), T(x_q)) + \rho(T(x_q), x_{q+1}) \\ &\le \rho(x_{q+2}, T(x_{q+1})) + \rho(T(x_q), x_{q+1}) + \alpha\rho(x_{q+1}, x_q). \end{aligned} \tag{3.1}$$

Using (2.1), (2.2), and (3.1), we obtain

$$\begin{aligned} \rho(x_{q+3}, x_{q+2}) &\le \rho(x_{q+3}, T(x_{q+2})) + \rho(x_{q+2}, T(x_{q+1})) + \alpha\rho(x_{q+2}, x_{q+1}) \\ &\le \alpha^2\rho(x_{q+1}, x_q) + \alpha\rho(T(x_q), x_{q+1}) + \alpha\rho(T(x_{q+1}), x_{q+2}) \\ &\quad + \rho(x_{q+2}, T(x_{q+1})) + \rho(x_{q+3}, T(x_{q+2})) \\ &= \alpha^2\rho(x_{q+1}, x_q) + \alpha\rho(T(x_q), x_{q+1}) \\ &\quad + (1+\alpha)\rho(T(x_{q+1}), x_{q+2}) + \rho(x_{q+3}, T(x_{q+2})). \end{aligned}$$

By (2.1), (2.2), and the above relation, we also have

$$\begin{aligned}
\rho(x_{q+4},x_{q+3}) &\le \rho(x_{q+4},T(x_{q+3}))+\rho(T(x_{q+3}),T(x_{q+2}))+\rho(T(x_{q+2}),x_{q+3})\\
&\le \rho(x_{q+4},T(x_{q+3}))+\rho(x_{q+3},T(x_{q+2}))+\alpha\rho(x_{q+3},x_{q+2})\\
&\le \alpha^3\rho(x_{q+1},x_q)+\rho(x_{q+4},T(x_{q+3}))\\
&\quad+\alpha^2\rho(T(x_q),x_{q+1})+\alpha(1+\alpha)\rho(T(x_{q+1}),x_{q+2})\\
&\quad+(1+\alpha)\rho(x_{q+3},T(x_{q+2})).
\end{aligned}$$

Next, we use mathematical induction to show that, for all integers $p\ge 2$,

$$\begin{aligned}
&\rho(x_{q+p},x_{q+p+1})\\
&\le \alpha^p\rho(x_{q+1},x_q)+\alpha^{p-1}\rho(T(x_q),x_{q+1})\\
&\quad+(1+\alpha)\sum_{i=1}^{p-1}\rho(x_{q+i+1},T(x_{q+i}))\alpha^{p-i-1}+\rho(T(x_{q+p}),x_{q+p+1}).
\end{aligned}\tag{3.2}$$

In view of the above relations, (3.2) is true for $p=2,3$. Now, we assume that $p\ge 3$ is an integer and that (3.2) holds true. By (2.1), (2.2), and (3.2), we have

$$\begin{aligned}
&\rho(x_{q+p+1},x_{q+p+2})\\
&\le \rho(x_{q+p+1},T(x_{q+p}))+\rho(T(x_{q+p}),T(x_{q+p+1}))+\rho(T(x_{q+p+1}),x_{q+p+2})\\
&\le \rho(x_{q+p+1},T(x_{q+p}))+\rho(T(x_{q+p+1}),x_{q+p+2})+\alpha\rho(x_{q+p},x_{q+p+1})\\
&\le \rho(T(x_{q+p+1}),x_{q+p+2})+\rho(x_{q+p+1},T(x_{q+p}))\\
&\quad+\alpha^{p+1}\rho(x_{q+1},x_q)+\alpha^p\rho(T(x_q),x_{q+1})\\
&\quad+(1+\alpha)\sum_{i=1}^{p-1}\rho(x_{q+i+1},T(x_{q+i}))\alpha^{p-i}+\alpha\rho(T(x_{q+p}),x_{q+p+1})\\
&\le \alpha^{p+1}\rho(x_{q+1},x_q)+\alpha^p\rho(T(x_q),x_{q+1})\\
&\quad+\rho(T(x_{q+p+1}),x_{q+p+2})+(1+\alpha)\sum_{i=1}^{p}\rho(x_{q+i+1},T(x_{q+i}))\alpha^{p-i}.
\end{aligned}$$

Thus (3.2) holds for $p+1$ too. Therefore (3.2) indeed holds for all integers $p\ge 2$. Let $q\ge 0$ and $n\ge 1$ be integers. By (3.1) and (3.2), we have

$$\begin{aligned}
&\rho(x_q,x_{q+n})\\
&\le \sum_{i=0}^{n-1}\rho(x_{q+i},x_{q+i+1})\\
&\le \rho(x_{q+1},x_q)\sum_{i=0}^{n-1}\alpha^i+\rho(T(x_q),x_{q+1})\sum\{\alpha^{i-1}:\ i \text{ is an integer and } 1\le i\le n-1\}\\
&\quad+2(1+\alpha)\sum_{j=0}^{\infty}\alpha^j\sum_{i=0}^{\infty}\rho(T(x_{q+i}),x_{q+i+1})\\
&\le \rho(x_{q+1},x_q)(1-\alpha)^{-1}+\rho(T(x_q),x_{q+1})(1-\alpha)^{-1}\\
&\quad+2(1+\alpha)(1-\alpha)^{-1}\sum_{i=0}^{\infty}\rho(T(x_{q+i}),x_{q+i+1}).
\end{aligned}\tag{3.3}$$

Set

$$\Delta := \sup\{\rho(T(x_i), x_{i+1}) : \ i = 0, 1, \dots\}. \tag{3.4}$$

By (2.3), Δ is finite. We claim that $\lim_{i\to\infty}\rho(x_i, x_{i+1}) = 0$. Applying (3.2) with $q = 0$, we see that, for each integer $p \geq 2$,

$$\begin{aligned}\rho(x_p, x_{p+1}) &\leq \alpha^p \rho(x_0, x_1) + \alpha^{p-1}\rho(T(x_0), x_1) \\ &\quad + \rho(T(x_p), x_{p+1}) + (1+\alpha)\sum_{i=1}^{p-1}\rho(x_{i+1}, T(x_i))\alpha^{p-i-1}.\end{aligned} \tag{3.5}$$

Let $\varepsilon > 0$ be given. By (2.3), there exists an integer $k_1 \geq 3$ such that

$$\alpha^{k_1}\rho(x_0, x_1) \leq \varepsilon/8, \ \alpha^{k_1-1}\rho(T(x_0), x_1) \leq \varepsilon/8 \tag{3.6}$$

and such that, for all integers $p \geq k_1$,

$$\rho(T(x_p), x_{p+1}) \leq 8^{-1}\varepsilon(1+\alpha)^{-1}(1-\alpha). \tag{3.7}$$

Choose a natural number $k_2 > k_1 + 1$ such that

$$\alpha^{k_2-k_1-1} \leq 8^{-1}\varepsilon(1+\alpha)^{-1}(1+\Delta)^{-1}k_1^{-1}. \tag{3.8}$$

Equations (3.4)-(3.8) imply that, for all integers $p \geq k_2$,

$$\begin{aligned}&\rho(x_p, x_{p+1}) \\ &\leq \varepsilon/8 + \varepsilon/8 + \varepsilon/8 + (1+\alpha)(\sum_{i=1}^{k_1}\rho(x_{i+1}, T(x_i))\alpha^{p-i-1} + \sum_{i=k_1}^{p-1}\rho(x_{i+1}, T(x_i))\alpha^{p-i-1}) \\ &\leq 3\varepsilon/8 + (1+\alpha)\Delta k_1\alpha^{p-1-k_1} \\ &\quad + (1+\alpha)\sum_{j=0}^{\infty}\alpha^j \sup\{\rho(x_{i+1}, T(x_i)) : \ i \geq k_1 \text{ is an integer}\} \\ &\leq 3\varepsilon/8 + (1+\alpha)\Delta k_1\alpha^{k_2-k_1} + \varepsilon/8 \leq \varepsilon.\end{aligned}$$

Since ε is any positive number, we conclude that $\lim_{i\to\infty}\rho(x_i, x_{i+1}) = 0$ as claimed. From (2.3), we find that there exists an integer $q_0 \geq 1$ such that, for all integers $q \geq q_0$,

$$\rho(x_q, x_{q+1}) \leq \varepsilon(1-\alpha)/8, \ \rho(T(x_q), x_{q+1}) \leq \varepsilon(1-\alpha)/8, \tag{3.9}$$

and

$$\sum_{i=q_0}^{\infty}\rho(T(x_q), x_{q+1}) \leq 8^{-1}\varepsilon(1-\alpha). \tag{3.10}$$

It now follows from (3.3), (3.9), and (3.10) that, for every integer $q \geq q_0$ and any integer $n \geq 1$,

$$\rho(x_q, x_{q+n}) \leq \varepsilon/8 + \varepsilon/8 + \varepsilon/8.$$

Thus we see that $\{x_i\}_{i=0}^{\infty}$ is a Cauchy sequence and there exists the limit $x_* = \lim_{i\to\infty} x_i$. Since the set of edges $E(G)$ is closed in $X \times X$, it follows that $(x_*, x_*) \in E(G)$. In view of (2.3) and the continuity of T at x_*, we also have $T(x_*) = x_*$, as asserted. This completes the proof of the theorem.

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