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THE TSENG'S EXTRAGRADIENT METHOD FOR SEMISTRICTLY QUASIMONOTONE VARIATIONAL INEQUALITIES

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Abstract. In this paper, we investigate the weak convergence of an iterative method for solving classical variational inequalities problems with semistrictly quasimonotone and Lipschitz-continuous mappings in real Hilbert space. The proposed method is based on Tseng's extragradient method and uses a set stepsize rule that is dependent on the Lipschitz constant as well as a simple self-adaptive stepsize rule that is independent of the Lipschitz constant. We proved a weak convergence theorem for our method without requiring any additional projections or the knowledge of the Lipschitz constant of the involved mapping. Finally, we offer some numerical experiments that demonstrate the efficiency and benefits of the proposed method.

Keywords. Variational inequality; Semistrictly quasimonotone operator; Tseng's extragradient method; Weak convergence.

1. INTRODUCTION

Let $\mathscr{H}$ be a real Hilbert space with inner product $\langle \cdot,\cdot \rangle$ and induced norm $\|\cdot\|$. The weakly convergent sequence $\{x_n\}$ to a point x is denoted by $x_n \rightharpoonup x$. For a given closed and convex subset $\mathscr{C} \subset \mathscr{H}$, the variational inequality problem, denoted by $VI(\mathscr{C},\mathscr{T})$, is to find $x^* \in \mathscr{C}$ such that

$$\left\langle \mathscr{T}(x^*), y - x^* \right\rangle \geq 0, \ \forall y \in \mathscr{C}, \quad \text{(VIP)}$$

where $\mathscr{T} : \mathscr{H} \to \mathscr{H}$ is an operator. It is known (see, e.g., [1]) that (VIP) is closely related to the problem of finding a point $x^* \in \mathscr{C}$ such that

$$\left\langle \mathscr{T}(y), y - x^* \right\rangle \geq 0, \ \forall y \in \mathscr{C}. \quad \text{(DVIP)}$$

In view of [1], Problem (DVIP) is called the dual variational inequality problem, denoted by $DVI(\mathscr{C},\mathscr{T})$, of (VIP). For a closed and convex $\mathscr{C} \subset \mathscr{H}$, the metric projection $P_{\mathscr{C}} : \mathscr{H} \to \mathscr{C}$

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is defined by $P_{\mathscr{C}}(x) = \arg\min\{\|x-y\| : y \in \mathscr{C}\}, \quad \forall x \in \mathscr{H}$. In this paper, $\mathbb{R}$ and $\mathbb{N}$ are used to denote the sets of real numbers and natural numbers, respectively. It is obvious to see that problem (VIP) is equivalent to solving the following problem:

$$\text{Finding } x^* \in \mathscr{C} \text{ such that } x^* = P_{\mathscr{C}}[x^* - \tau \mathscr{T}(x^*)],$$

where τ is a positive real number. The theory of variational inequalities has been employed as an important tool to study a wide range of problems in physics, engineering, economics, and computer science. It was first presented by Stampacchia [2] in 1964. Problem (VIP) is an important mathematical problem that includes several important problems in applied mathematics, such as network equilibrium problems, complementarity problems, saddle problems, inclusion problems, and the systems of nonlinear equations (for more details, see, e.g., [3, 4, 5, 6, 7]). Recently, many iterative methods for solving problem (VIP) were proposed and analyzed; see, e..g, [8, 9, 10, 11, 12, 13, 14, 15] and the references therein. The metric projection is essential to solve various variational inequality problems. The extragradient method, introduced in [16, 17], has the following form:

$$\begin{cases} x_0 \in \mathscr{C}, \\ y_n = P_{\mathscr{C}}[x_n - \tau \mathscr{T}(x_n)], \\ x_{n+1} = P_{\mathscr{C}}[x_n - \tau \mathscr{T}(y_n)], \end{cases} \tag{1.1}$$

where $\mathscr{T}$ is a Lipschitz continuous operator with modulus L, and τ is a constant with $0 < \tau < \frac{1}{L}$. Here, two projections on set $\mathscr{C}$ are needed at each iteration. However, if $\mathscr{C}$ has a complicated structure, this might have an impact on the computing efficacy of the method. We restrict our interest to some methods which can overcome this drawback. The first is the following subgradient extragradient method, which is due to Censor et al. [8]. This method takes the form

$$\begin{cases} x_0 \in \mathscr{C}, \\ y_n = P_{\mathscr{C}}[x_n - \tau \mathscr{T}(x_n)], \\ x_{n+1} = P_{\mathscr{H}_n}[x_n - \tau \mathscr{T}(y_n)], \end{cases} \tag{1.2}$$

where $0 < \tau < \frac{1}{L}$ and $\mathscr{H}_n = \{z \in \mathscr{H} : \langle x_n - \tau \mathscr{T}(x_n) - y_n, z - y_n \rangle \leq 0\}$.

In this paper, we concentrate on the Tseng's extragradient method [18] that uses only one projection for each iteration:

$$\begin{cases} x_0 \in \mathscr{C}, \\ y_n = P_{\mathscr{C}}[x_n - \tau \mathscr{T}(x_n)], \\ x_{n+1} = y_n + \tau\big[\mathscr{T}(x_n) - \mathscr{T}(y_n)\big], \end{cases} \tag{1.3}$$

where $0 < \tau < \frac{1}{L}$.

The main objective of this work is to study semistrictly quasimonotone variational inequalities in infinite dimensional Hilbert spaces via an iterative method, which is based on Tseng's extragradient method [18] and the subgradient extragradient method [8]. At each iteration, our method only requires one projection onto the feasible set. Under some suitable conditions imposed on control parameters, the iterative sequences generated by our method converge weakly to some solution to the problem. We also give two variants of our method and present examples to explain the computational performance of the new methods.

The paper is organized in the following manner. In Section 2, some preliminary results were presented. Section 3 provides the new iterative methods and their convergence theorems.

Finally, Section 4 presents some numerical results to support the practical efficiency of the proposed method.

2. Preliminaries

Let $\mathscr{H}$ be a real Hilbert space. Given $x,y \in \mathscr{H}$, the closed line segment consisting of x and y is defined by $[x,y] = \{tx+(1-t)y : 0 \le t \le 1\}$. The segments $(x,y]$, $[x,y)$, and (x,y) are defined similarly. In the framework of Hilbert space, the following equality and inequality are known

(i) $\|\ell x+(1-\ell)y\|^2 = \ell\|x\|^2+(1-\ell)\|y\|^2-\ell(1-\ell)\|x-y\|^2$ for any $\ell\ in (0,1)$;
(ii) $\|x+y\|^2 \le \|x\|^2+2\langle y,x+y\rangle$.

Let $\mathscr{C}$ be a nonempty, closed and convex subset of $\mathscr{H}$, and let $P_{\mathscr{C}} : \mathscr{H} \to \mathscr{C}$ be the metric projection from $\mathscr{H}$ onto $\mathscr{C}$. Then $\|x-P_{\mathscr{C}}(y)\|^2+\|P_{\mathscr{C}}(y)-y\|^2 \le \|x-y\|^2$ for any $x \in \mathscr{C}$ and $y \in \mathscr{H}$; $\langle x-z, y-z\rangle \le 0, \forall y \in \mathscr{C}$ if and only if $z = P_{\mathscr{C}}(x)$; $\|x-P_{\mathscr{C}}(x)\| \le \|x-y\|$ for any $y \in \mathscr{C}$ and $x \in \mathscr{H}$.

Definition 2.1. [19, 20] Let $\mathscr{T} : \mathscr{C} \to \mathscr{H}$ be a mapping. Recall that $\mathscr{T}$ is said to be

(a) *strongly monotone* with constant $\gamma > 0$ if, for each pair of points $x,y \in \mathscr{C}$,
$$\big\langle \mathscr{T}(x)-\mathscr{T}(y), x-y\big\rangle \ge \gamma\|x-y\|^2;$$
(b) *strictly monotone* if, for all distinct $x,y \in \mathscr{C}$,
$$\big\langle \mathscr{T}(x)-\mathscr{T}(y), x-y\big\rangle > 0;$$
(c) *monotone* if, for all distinct $x,y \in \mathscr{C}$,
$$\big\langle \mathscr{T}(x)-\mathscr{T}(y), x-y\big\rangle \ge 0;$$
(d) *pseudomonotone* if, for all distinct $x,y \in \mathscr{C}$,
$$\big\langle \mathscr{T}(y), x-y\big\rangle \ge 0 \quad \Longrightarrow \quad \big\langle \mathscr{T}(x), x-y\big\rangle \ge 0;$$
(e) *quasimonotone* if, for all distinct $x,y \in \mathscr{C}$,
$$\big\langle \mathscr{T}(y), x-y\big\rangle > 0 \quad \Longrightarrow \quad \big\langle \mathscr{T}(x), x-y\big\rangle \ge 0;$$
(f) *semistrictly quasimonotone* if $\mathscr{T}$ is quasimonotone and for all distinct of points $x,y \in \mathscr{C}$,
$$\big\langle \mathscr{T}(y), x-y\big\rangle > 0 \quad \Longrightarrow \quad \big\langle \mathscr{T}(z), x-y\big\rangle \ge 0, \text{ for some } z \in \Big(\frac{x+y}{2}, x\Big).$$

Remark 2.1. The implications are as follows:
$$(a) \Longrightarrow (b) \Longrightarrow (c) \Longrightarrow (d) \Longrightarrow (e) \quad \text{and} \quad (f) \Longrightarrow (e).$$
In general, however, the inverse claims are incorrect.

Definition 2.2. A mapping $\mathscr{T} : \mathscr{H} \to \mathscr{H}$ is said to be

(i) *weakly hemicontinuous* if $\mathscr{T}$ is upper semicontinuous from line segments in $\mathscr{H}$ to the weak topology of $\mathscr{H}$;
(ii) *sequentially weakly continuous* if $\{\mathscr{T}(x_n)\}$ converges weakly to $\mathscr{T}(x)$ for every sequence $\{x_n\}$ converges weakly to x.

Remark 2.2. It is easy to prove that if $\mathscr{T} : \mathscr{H} \to \mathscr{H}$ is sequentially weakly continuous, then $\mathscr{T}$ must be weakly hemicontinuous.

Additionally, we also need the following lemmas.

Lemma 2.1. [1] *A solution to problem* (DVIP) *is always a solution to problem* (VIP) *provided that the operator* $\mathscr{T}$ *is, say, weakly hemicontinuous.*

Lemma 2.2. [21] *Let* $\mathscr{C}$ *be a nonempty set of* $\mathscr{H}$*, and let* $\{x_n\}$ *be a sequence in* $\mathscr{H}$*. If the following two conditions hold:*

(i) *for every* $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ *exists;*
(ii) *each sequentially weak cluster point of* $\{x_n\}$ *is in* $\mathscr{C}$*,*

then $\{x_n\}$ *converges weakly to a point in* $\mathscr{C}$*.*

3. Main Results

In this section, we propose our iterative algorithm for semistrictly quasimonotone variational inequalities, which is based on Tseng's extragradient method that does not require either the knowledge of the Lipschitz constant of the operator or additional projection.

Algorithm 1

Step 0: Choose $x_0 \in \mathscr{C}$ and $0 < \tau < \frac{1}{L}$.
Step 1: Compute

$$y_n = P_{\mathscr{C}}(x_n - \tau \mathscr{T}(x_n)).$$

If $x_n = y_n$, then STOP and y_n is a solution. Otherwise, go to **Step 2**.
Step 2: Compute

$$x_{n+1} = y_n + \tau\big[\mathscr{T}(x_n) - \mathscr{T}(y_n)\big].$$

Set $n = n+1$ and go back to **Step 1**.

In order to verify the weak convergence, we assume that the following conditions hold:

($\mathscr{T}$1) the solution set of problem (VIP), denoted by Ω, is nonempty;
($\mathscr{T}$2) $\mathscr{T} : \mathscr{H} \to \mathscr{H}$ is semistrictly quasimonotone if $\mathscr{T}$ is quasimonotone on $\mathscr{C}$ and

$$\big\langle \mathscr{T}(y), x-y \big\rangle > 0 \quad \Longrightarrow \quad \big\langle \mathscr{T}(z), x-y\big\rangle \geq 0, \text{ for some } z \in \left(\frac{x+y}{2}, x\right); \tag{SQM}$$

($\mathscr{T}$3) $\mathscr{T} : \mathscr{H} \to \mathscr{H}$ is *Lipschitz continuous* with constant $L > 0$ such that

$$\|\mathscr{T}(x) - \mathscr{T}(y)\| \leq L\|x-y\|, \ \forall x, y \in \mathscr{C}; \tag{LC}$$

($\mathscr{T}$4) $\mathscr{T} : \mathscr{H} \to \mathscr{H}$ is *sequentially weakly continuous* if $\{\mathscr{T}(x_n)\}$ converges weakly to $\mathscr{T}(x)$ for every sequence $\{x_n\}$ converges weakly to x.

In what following, we show certain lemma in the case of semistrictly quasimonotone and Lipschitz-continuous operators. It is identical to the one in [22].

Lemma 3.1. *Let* $\mathscr{T} : \mathscr{H} \to \mathscr{H}$ *be an operator that satisfies conditions* ($\mathscr{T}$1)-($\mathscr{T}$4)*, and let* $\{x_n\}$ *be a sequence generated by Algorithm* 1*. Then,*

$$\|x_{n+1} - x^*\|^2 \leq \big\|x_n - x^*\big\|^2 - \big(1 - \tau^2 L^2\big)\big\|x_n - y_n\big\|^2.$$

Proof. Since $x^* \in \Omega$, we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&= \|y_n + x_n - x_n - x^*\|^2 + \tau^2 \|\mathscr{T}(x_n) - \mathscr{T}(y_n)\|^2 + 2\tau\langle y_n - x^*, \mathscr{T}(x_n) - \mathscr{T}(y_n)\rangle \\
&= \|y_n - x_n\|^2 + \|x_n - x^*\|^2 + 2\langle y_n - x_n, x_n - x^*\rangle \\
&\quad + \tau^2 \|\mathscr{T}(x_n) - \mathscr{T}(y_n)\|^2 + 2\tau\langle y_n - x^*, \mathscr{T}(x_n) - \mathscr{T}(y_n)\rangle \\
&= \|x_n - x^*\|^2 + \|y_n - x_n\|^2 + 2\langle y_n - x_n, y_n - x^*\rangle + 2\langle y_n - x_n, x_n - y_n\rangle \\
&\quad + \tau^2 \|\mathscr{T}(x_n) - \mathscr{T}(y_n)\|^2 + 2\tau\langle y_n - x^*, \mathscr{T}(x_n) - \mathscr{T}(y_n)\rangle. \qquad (3.1)
\end{aligned}$$

It indicates that $y_n = P_{\mathscr{C}}[x_n - \tau\mathscr{T}(x_n)]$, which further implies that

$$\langle x_n - \tau\mathscr{T}(x_n) - y_n, y - y_n\rangle \leq 0, \ \forall y \in \mathscr{C}.$$

Thus,

$$\langle x_n - y_n, x^* - y_n\rangle \leq \tau\langle \mathscr{T}(x_n), x^* - y_n\rangle. \qquad (3.2)$$

Combining (3.1) and (3.2), we have

$$\begin{aligned}
&\|x_{n+1} - x^*\|^2 \\
&\leq \|x_n - x^*\|^2 + \|y_n - x_n\|^2 + 2\tau\langle \mathscr{T}(x_n), x^* - y_n\rangle - 2\langle x_n - y_n, x_n - y_n\rangle \\
&\quad + \tau^2 \|\mathscr{T}(x_n) - \mathscr{T}(y_n)\|^2 - 2\tau\langle \mathscr{T}(x_n) - \mathscr{T}(y_n), x^* - y_n\rangle \\
&= \|x_n - x^*\|^2 - \|x_n - y_n\|^2 + \tau^2 \|\mathscr{T}(x_n) - \mathscr{T}(y_n)\|^2 - 2\tau\langle \mathscr{T}(y_n), y_n - x^*\rangle. \qquad (3.3)
\end{aligned}$$

Since x^* is the solution of problem (VIP), we have $\langle \mathscr{T}(x^*), y - x^*\rangle \geq 0, \forall y \in \mathscr{C}$. It follows that $\langle \mathscr{T}(y), y - x^*\rangle \geq 0, \forall y \in \mathscr{C}$. Substituting $y = y_n \in \mathscr{C}$, we have

$$\langle \mathscr{T}(y_n), y_n - x^*\rangle \geq 0. \qquad (3.4)$$

From (3.3) and (3.4), we obtain

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - \tau^2 L^2)\|x_n - y_n\|^2.$$

□

Theorem 3.1. *Let $\mathscr{T} : \mathscr{H} \to \mathscr{H}$ be an operator that satisfies conditions ($\mathscr{T}$1)–($\mathscr{T}$4). Then, the sequence $\{x_n\}$ generated by the Algorithm 1 converges weakly to $x^* \in \Omega$.*

Proof. By using Lemma 3.1, we have

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 - (1 - \tau^2 L^2)\|x_n - y_n\|^2. \qquad (3.5)$$

Since $0 < \tau < \frac{1}{L}$, we obtain $\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2$, which implies that $\lim_{n\to\infty} \|x_n - x^*\| = l$ for some finite $l \geq 0$. From (3.5), we have

$$(1 - \tau^2 L^2)\|x_n - y_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.$$

Due to the existence of $\lim_{n\to\infty} \|x_n - x^*\| = l$, we infer that $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Thus, we have $\lim_{n\to\infty} \|y_n - x^*\| = l$. It follows that

$$\|x_{n+1} - y_n\| = \|y_n + \tau[\mathscr{T}(x_n) - \mathscr{T}(y_n)] - y_n\| \leq \tau L\|x_n - y_n\|.$$

The above expression implies that $\lim_{n\to\infty}\|x_{n+1}-y_n\|=0$. It follows that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. This implies that the sequences $\{x_n\}$ and $\{y_n\}$ are bounded.

Now, we show that the sequence $\{x_n\}$ converges weakly to $x^*\in\Omega$. Indeed, since $\{x_n\}$ is bounded, we assume that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\rightharpoonup\hat{x}$. Next, we prove that $\hat{x}\in\Omega$. Indeed, we have $y_{n_k}=P_{\mathscr{C}}[x_{n_k}-\tau_{n_k}\mathscr{T}(x_{n_k})]$, which is equivalent to

$$\langle x_{n_k}-\tau_{n_k}\mathscr{T}(x_{n_k})-y_{n_k},y-y_{n_k}\rangle\le 0,\ \forall y\in\mathscr{C}.$$

This implies that

$$\langle x_{n_k}-y_{n_k},y-y_{n_k}\rangle\le\tau_{n_k}\langle\mathscr{T}(x_{n_k}),y-y_{n_k}\rangle,\ \forall y\in\mathscr{C}.$$

Thus,

$$\frac{1}{\tau_{n_k}}\langle x_{n_k}-y_{n_k},y-y_{n_k}\rangle+\langle\mathscr{T}(x_{n_k}),y_{n_k}-x_{n_k}\rangle\le\langle\mathscr{T}(x_{n_k}),y-x_{n_k}\rangle,\ \forall y\in\mathscr{C}. \tag{3.6}$$

Observe that $\min\left\{\frac{\mu}{L},\tau_1\right\}\le\tau\le\tau_1$ and $\{x_{n_k}\}$ is a bounded sequence. Using $\lim_{k\to\infty}\|x_{n_k}-y_{n_k}\|=0$ and letting $k\to\infty$ in (3.6), we obtain $\liminf_{k\to\infty}\langle\mathscr{T}(x_{n_k}),y-x_{n_k}\rangle\ge 0$, $\forall y\in\mathscr{C}$. Moreover, we have

$$\begin{aligned}&\langle\mathscr{T}(y_{n_k}),y-y_{n_k}\rangle\\&=\langle\mathscr{T}(y_{n_k})-\mathscr{T}(x_{n_k}),y-x_{n_k}\rangle+\langle\mathscr{T}(x_{n_k}),y-x_{n_k}\rangle+\langle\mathscr{T}(y_{n_k}),x_{n_k}-y_{n_k}\rangle.\end{aligned} \tag{3.7}$$

Since $\lim_{k\to\infty}\|x_{n_k}-y_{n_k}\|=0$ and $\mathscr{T}$ is L-Lipschitz continuity on $\mathscr{H}$, we have

$$\lim_{k\to\infty}\|\mathscr{T}(x_{n_k})-\mathscr{T}(y_{n_k})\|=0, \tag{3.8}$$

which together with (3.7) and (3.8) obtains that

$$\liminf_{k\to\infty}\langle\mathscr{T}(y_{n_k}),y-y_{n_k}\rangle\ge 0,\ \forall y\in\mathscr{C}. \tag{3.9}$$

To prove further, let us take a positive sequence $\{\varepsilon_k\}$ that is convergent to zero and decreasing. For each $\{\varepsilon_k\}$, we denote by m_k the smallest positive integer such that

$$\langle\mathscr{T}(x_{n_i}),y-x_{n_i}\rangle+\varepsilon_k>0,\ \forall i\ge m_k, \tag{3.10}$$

where the existence of m_k follows from (3.9). Since $\{\varepsilon_k\}$ is decreasing, it is easy to see that m_k is increasing.

Case I. If there exists a $x_{n_{m_{k_j}}}$ subsequence of $x_{n_{m_k}}$ such that $\mathscr{T}(x_{n_{m_{k_j}}})=0$ $(\forall j)$. Letting $j\to\infty$, we obtain $\langle\mathscr{T}(\hat{x}),y-\hat{x}\rangle=\lim_{j\to\infty}\langle\mathscr{T}(x_{n_{m_{k_j}}}),y-\hat{x}\rangle=0$. Thus, $\hat{x}\in\mathscr{C}$ and this implies that $\hat{x}\in VI(\mathscr{C},\mathscr{T})$.

Case II. If there exits $N_0\in\mathbb{N}$ such that, for all $n_{m_k}\ge N_0$, $\mathscr{T}(x_{n_{m_k}})\ne 0$. Consider

$$\digamma_{n_{m_k}}=\frac{\mathscr{T}(x_{n_{m_k}})}{\|\mathscr{T}(x_{n_{m_k}})\|^2},\ \forall n_{m_k}\ge N_0.$$

From the above definition, we obtain $\langle\mathscr{T}(x_{n_{m_k}}),\digamma_{n_{m_k}}\rangle=1$, $\forall 4n_{m_k}\ge N_0$, which together with (3.10) yields that, for all $n_{m_k}\ge N_0$, $\langle\mathscr{T}(x_{n_{m_k}}),\ y+\varepsilon_k\digamma_{n_{m_k}}-x_{n_{m_k}}\rangle>0$. Since $\mathscr{T}$ is quasimonotone, then

$$\langle\mathscr{T}(y+\varepsilon_k\digamma_{n_{m_k}}),\ y+\varepsilon_k\digamma_{n_{m_k}}-x_{n_{m_k}}\rangle>0.$$

For all $n_{m_k}\ge N_0$, we have

$$\langle\mathscr{T}(y),y-x_{n_{m_k}}\rangle\ge\langle\mathscr{T}(y)-\mathscr{T}(y+\varepsilon_k\digamma_{n_{m_k}}),\ y+\varepsilon_k\digamma_{n_{m_k}}-x_{n_{m_k}}\rangle-\varepsilon_k\langle\mathscr{T}(y),\digamma_{n_{m_k}}\rangle. \tag{3.11}$$

Since $\{x_{n_k}\}$ weakly converges to $\hat{x} \in \mathscr{C}$ and $\mathscr{T}$ is sequentially weakly continuous on the set $\mathscr{C}$, we obtain that $\{\mathscr{T}(x_{n_k})\}$ weakly converges to $\mathscr{T}(\hat{x})$. Letting $\mathscr{T}(\hat{x}) \neq 0$, we have $\|\mathscr{T}(\hat{x})\| \leq \liminf_{k\to\infty} \|\mathscr{T}(x_{n_k})\|$. Since $\{x_{n_{m_k}}\} \subset \{x_{n_k}\}$ and $\lim_{k\to\infty} \varepsilon_k = 0$, we have

$$0 \leq \lim_{k\to\infty} \|\varepsilon_k F_{n_{m_k}}\| = \lim_{k\to\infty} \frac{\varepsilon_k}{\|\mathscr{T}(x_{n_{m_k}})\|} \leq \frac{0}{\|\mathscr{T}(\hat{x})\|} = 0.$$

Letting $k \to \infty$ in (3.11), we obtain $\langle \mathscr{T}(y), y - \hat{x} \rangle \geq 0, \forall y \in \mathscr{C}$. Thus, we infer that $\hat{x} \in VI(\mathscr{C}, \mathscr{T})$. Therefore, we proved that

(1) for every $x^* \in VI(\mathscr{C}, \mathscr{T})$, then $\lim_{n\to\infty} \|x_n - x^*\|$ exists;
(2) every sequential weak cluster point of the sequence $\{x_n\}$ is in $VI(\mathscr{C}, \mathscr{T})$.

By using Lemma 2.2, we obtain that $\{x_n\}$ converges weakly to $x^* \in VI(\mathscr{C}, \mathscr{T})$. □

Next, we introduce the first variant of Algorithm 1 in which the constant step size τ is chosen adaptively and thus produced a sequence τ_n that does not require the knowledge of the Lipschitz-like constant L.

Algorithm 2

Step 0: Choose $x_0 \in \mathscr{C}$, $\mu \in (0,1)$ and $\tau_0 > 0$.
Step 1: Compute

$$y_n = P_{\mathscr{C}}(x_n - \tau_n \mathscr{T}(x_n)).$$

If $x_n = y_n$, then STOP and y_n is a solution. Otherwise, go to **Step 2**.
Step 2: Compute

$$x_{n+1} = y_n + \tau_n \left[\mathscr{T}(x_n) - \mathscr{T}(y_n)\right].$$

Set $n = n+1$ and go back to **Step 1**.
Step 3: Compute

$$\tau_{n+1} = \begin{cases} \min\left\{\tau_n, \frac{\mu\|x_n - y_n\|}{\|\mathscr{T}(x_n) - \mathscr{T}(y_n)\|}\right\} & \text{if} \quad \mathscr{T}(x_n) - \mathscr{T}(y_n) \neq 0, \\ \tau_n & otherwise. \end{cases} \tag{3.12}$$

The following lemma is useful for showing that the step size sequence formed by expression (3.12) is properly defined, decreasing monotonically, and convergent to a fixed number.

Lemma 3.2. *The sequence* $\{\tau_n\}$ *generated by* (3.12) *is monotonically decreasing and converges to* $\tau > 0$.

Proof. It is given that $\mathscr{T}$ is Lipschitz-continuous with constant $L > 0$. Let $\mathscr{T}(x_n) \neq \mathscr{T}(y_n)$ such that

$$\frac{\mu\|x_n - y_n\|}{\|\mathscr{T}(x_n) - \mathscr{T}(y_n)\|} \geq \frac{\mu\|x_n - y_n\|}{L\|x_n - y_n\|} = \frac{\mu}{L}.$$

The above expression implies that $\lim_{n\to\infty} \tau_n = \tau$. □

The following lemma is important for proving the boundedness of a sequence generated by Algorithm 2 and serves the same purpose as Lemma 3.1 in the proof of Theorem 3.1.

Lemma 3.3. *Let $\mathscr{T} : \mathscr{H} \to \mathscr{H}$ be an operator that satisfies conditions ($\mathscr{T}1$)-($\mathscr{T}4$), and let x_n be a sequence generated by Algorithm 2. Then,*

$$\|x_{n+1} - x^*\|^2 \leq \left\|x_n - x^*\right\|^2 - \left(1 - \mu^2 \frac{\tau_n^2}{\tau_{n+1}^2}\right) \left\|x_n - y_n\right\|^2.$$

Theorem 3.2. *Let $\mathscr{T} : \mathscr{H} \to \mathscr{H}$ be an operator that satisfies conditions ($\mathscr{T}1$)–($\mathscr{T}4$). Then, the sequence $\{x_n\}$ generated by the Algorithm 2 converges weakly to $x^* \in \Omega$.*

Proof. The proof is analogous to the proof of Theorem 3.1. □

Next, we introduce the second variant of Algorithm 1 in which the constant step size τ is chosen adaptively and thus produced a sequence τ_n that does not require the knowledge of the Lipschitz-like constant L.

Algorithm 3

Step 0: Choose $x_0 \in \mathscr{C}$, $\mu \in (0,1)$, $\tau_0 > 0$ and select a nonnegative real sequence $\{\varphi_n\}$ such that $\sum_{n=1}^{+\infty} \varphi_n < +\infty$.

Step 1: Compute

$$y_n = P_{\mathscr{C}}(x_n - \tau_n \mathscr{T}(x_n)).$$

If $x_n = y_n$, then STOP and y_n is a solution. Otherwise, go to **Step 2**.

Step 2: Compute

$$x_{n+1} = y_n + \tau_n \left[\mathscr{T}(x_n) - \mathscr{T}(y_n)\right].$$

Set $n = n+1$ and go back to **Step 1**.

Step 3: Compute

$$\tau_{n+1} = \begin{cases} \min\left\{\tau_n + \varphi_n, \frac{\mu\|x_n - y_n\|}{\|\mathscr{T}(x_n) - \mathscr{T}(y_n)\|}\right\} & \text{if} \quad \mathscr{T}(x_n) - \mathscr{T}(y_n) \neq 0, \\ \tau_n + \varphi_n & otherwise. \end{cases} \tag{3.13}$$

The following lemma can be used to show that the step size sequence created by expression (3.13) is well defined, monotonically decreasing, and convergent to a fixed value.

Lemma 3.4. *The sequence $\{\tau_n\}$ generated by (3.13) is convergent to τ and also satisfies the following inequality*

$$\min\left\{\frac{\mu}{L}, \tau_0\right\} \leq \tau \leq \tau_0 + P, \quad where \quad P = \sum_{n=1}^{+\infty} \varphi_n.$$

Proof. Due to the Lipschitz continuity of a mapping $\mathscr{T}$, there exists a fixed number $L > 0$. Consider $\mathscr{T}(x_n) - \mathscr{T}(y_n) \neq 0$ such that

$$\frac{\mu\|x_n - y_n\|}{\|\mathscr{T}(x_n) - \mathscr{T}(y_n)\|} \geq \frac{\mu\|x_n - y_n\|}{L\|x_n - y_n\|} = \frac{\mu}{L}.$$

By using mathematical induction on the definition of τ_{n+1}, we have

$$\min\left\{\frac{\mu}{L}, \tau_0\right\} \leq \tau_n \leq \tau_0 + P.$$

Let $[\tau_{n+1}-\tau_n]^+ = \max\{0, \tau_{n+1}-\tau_n\}$ and $[\tau_{n+1}-\tau_n]^- = \max\{0, -(\tau_{n+1}-\tau_n)\}$. From the definition of $\{\tau_n\}$, we have

$$\sum_{n=1}^{+\infty}(\tau_{n+1}-\tau_n)^+ = \sum_{n=1}^{+\infty}\max\{0, \tau_{n+1}-\tau_n\} \le P < +\infty.$$

That is, the series $\sum_{n=1}^{+\infty}(\tau_{n+1}-\tau_n)^+$ is convergent.

Next we need to prove the convergence of $\sum_{n=1}^{+\infty}(\tau_{n+1}-\tau_n)^-$. Let $\sum_{n=1}^{+\infty}(\tau_{n+1}-\tau_n)^- = +\infty$. Due to the reason that $\tau_{n+1}-\tau_n = (\tau_{n+1}-\tau_n)^+ - (\tau_{n+1}-\tau_n)^-$, we have

$$\tau_{k+1}-\tau_0 = \sum_{n=0}^{k}(\tau_{n+1}-\tau_n) = \sum_{n=0}^{k}(\tau_{n+1}-\tau_n)^+ - \sum_{n=0}^{k}(\tau_{n+1}-\tau_n)^-. \tag{3.14}$$

Letting $k \to +\infty$ in (3.14), we have $\tau_k \to -\infty$ as $k \to \infty$. This is a contradiction. Due to the convergence of the series $\sum_{n=0}^{k}(\tau_{n+1}-\tau_n)^+$ and $\sum_{n=0}^{k}(\tau_{n+1}-\tau_n)^-$ taking $k \to +\infty$ in (3.14), we obtain $\lim_{n\to\infty}\tau_n = \tau$. This completes the proof. □

Theorem 3.3. *Let $\mathscr{T} : \mathscr{H} \to \mathscr{H}$ be an operator that satisfies conditions ($\mathscr{T}$1)–($\mathscr{T}$4). Then, the sequence $\{x_n\}$ generated by the Algorithm 3 converges weakly to $x^* \in \Omega$.*

Proof. The proof is analogous to the proof of Theorem 3.1. □

4. Numerical Illustrations

The computational results of the proposed algorithms are described in this section in contrast to some related algorithms in the literature and also in the analysis of how they variations in control parameters affect the numerical effectiveness of the proposed algorithms. All computations are done in MATLAB R2018b and run on HP i- 5 Core(TM)i5-6200 8.00 GB (7.78 GB usable) RAM laptop.

Example 4.1. Consider that $\mathscr{H} = l_2$ is a real Hilbert space with sequences of real numbers satisfying the following condition $\|x_1\|^2 + \|x_2\|^2 + \cdots + \|x_n\|^2 + \cdots < +\infty$. Assume that $\mathscr{T} : \mathscr{C} \to \mathscr{C}$ is defined by $G(x) = (5-\|x\|)x, \forall x \in \mathscr{H}$, where $C = \{x \in \mathscr{H} : \|x\| \le 3\}$. It is easy to see that $\mathscr{T}$ is weakly sequentially continuous on $\mathscr{H}$ and $VI(\mathscr{C}, \mathscr{T}) = \{0\}$. For any $x, y \in \mathscr{H}$, we have

$$\begin{aligned}\big\|\mathscr{T}(x)-\mathscr{T}(y)\big\| &= \big\|(5-\|x\|)x-(5-\|y\|)y\big\| \\ &= \big\|5(x-y)-\|x\|(x-y)-(\|x\|-\|y\|)y\big\| \\ &\le 5\|x-y\| + \|x\|\|x-y\| + \big|\|x\|-\|y\|\big|\|y\| \\ &\le 11\|x-y\|,\end{aligned}$$

that is, $\mathscr{T}$ is L-Lipschitz continuous with $L = 11$. For any $x, y \in \mathscr{H}$, let $\langle \mathscr{T}(x), y-x\rangle > 0$ such that $(5-\|x\|)\langle x, y-x\rangle > 0$. Since $\|x\| \le 3$ implies that $\langle x, y-x\rangle > 0$, we have

$$\begin{aligned}\langle \mathscr{T}(y), y-x\rangle &= (5-\|y\|)\langle y, y-x\rangle \\ &\ge (5-\|y\|)\langle y, y-x\rangle - (5-\|y\|)\langle x, y-x\rangle \\ &\ge 2\|x-y\|^2 \ge 0.\end{aligned}$$

Thus, we shown that $\mathscr{T}$ is quasimonotone on $\mathscr{C}$. Let $x = (\frac{5}{2}, 0, 0, \cdots, 0, \cdots)$ and $y = (3, 0, 0, \cdots, 0, \cdots)$ such that $\langle \mathscr{T}(x) - \mathscr{T}(y), x - y \rangle = (2.5 - 3)^2 < 0$. A projection on the set C is computed explicitly as follows:

$$P_C(x) = \begin{cases} x, & \|x\| \leq 3, \\ \frac{3x}{\|x\|}, & \text{otherwise.} \end{cases}$$

The control conditions were taken as $\tau = \frac{1}{2L}$, $\tau_0 = \frac{5}{11}$, $\mu = 0.44$, and $\varphi_n = \frac{100}{(n+3)^2}$.

TABLE 1. Numerical Results of Example 4.1.

	Number of Iterations			Elapsed time in seconds		
x_0	**Algorithm 1**	**Algorithm 2**	**Algorithm 3**	**Algorithm 1**	**Algorithm 2**	**Algorithm 3**
$(2, 2, \cdots, 2_{10000}, 0, 0, \cdots)$	34	28	26	1.30281	1.7876019	1.7080414
$(1, 2, \cdots, 10000, 0, 0, \cdots)$	43	36	30	1.79729	2.5490706	1.9033160
$(7, 7, \cdots, 7_{100000}, 0, 0, \cdots)$	56	43	37	2.39814	2.8637895	2.3877556
$(50, 50, \cdots, 50_{100000}, 0, 0, \cdots)$	70	67	46	5.46383	4.9474732	4.4748392

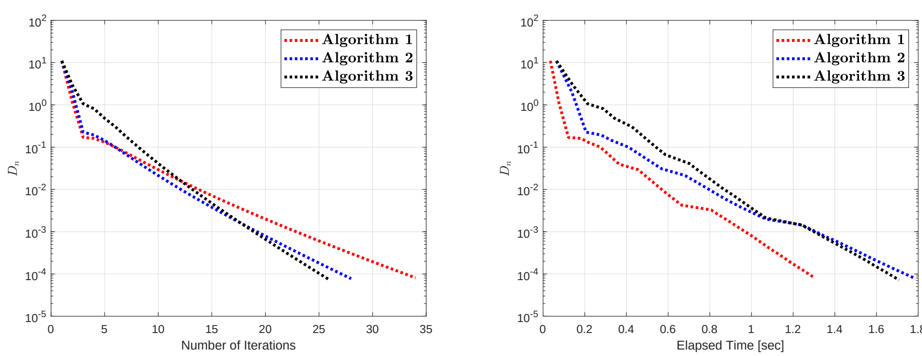

FIGURE 1. Numerical illustration when $u_1 = (2, 2, \cdots, 2_{10000}, 0, 0, \cdots)$.

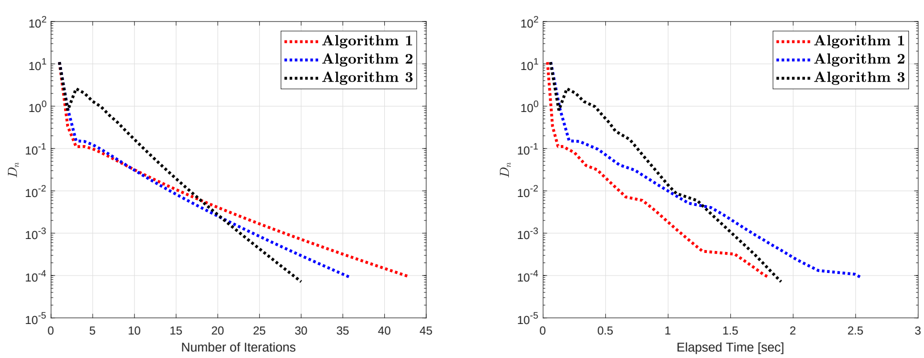

FIGURE 2. Numerical illustration when $u_1 = (1, 2, \cdots, 10000, 0, 0, \cdots)$.

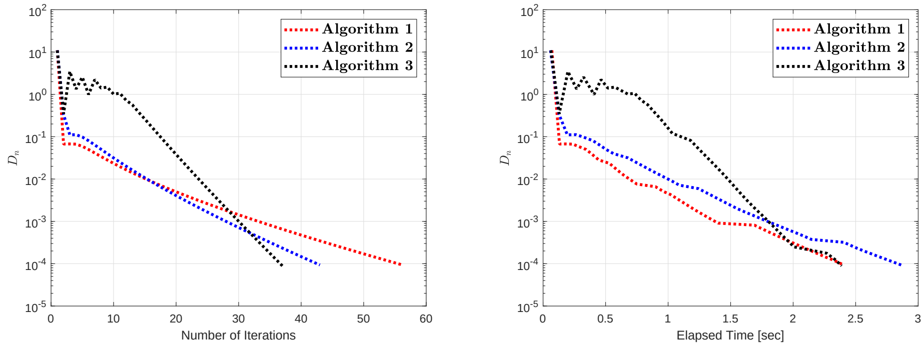

FIGURE 3. Numerical illustration when $u_1 = (7, 7, \cdots, 7_{100000}, 0, 0, \cdots)$.

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