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PARALLEL PROJECTION METHOD FOR SOLVING SPLIT EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we study a class of bilevel multiple-sets split pseudomonotone equilibrium problems in real Hilbert spaces. A new parallel projection method is presented and a strong convergence theorem for solving the problem is established under mild assumptions. Our result is general and extends several existing results in the literature.

Keywords. Bilevel problems; Parallel projections; Pseudomonotone bifunctions; Split equilibrium problems.

1. INTRODUCTION

Censor et al. in [1, Section 2] introduced a prototypical *split inverse problem* (SIP). Let X and Y be two given vector spaces, and let $A : X \to Y$ be a linear operator. In addition, two inverse problems are involved. The first one, denoted by IP_1, is formulated in space X and the second one, denoted by IP_2, is formulated in space Y. Given these data, the split inverse problem is formulated as follows:

$$\text{find a point } x^* \in X \text{ that solves } \mathrm{IP}_1$$
$$\text{such that}$$
$$\text{the point } y^* = Ax^* \in Y \text{ solves } \mathrm{IP}_2.$$

The first instance of the SIP is the *split convex feasibility problem* (SCFP) due to Censor and Elfving [2] in which the IP_1 and IP_2 are feasibility problems, that is, $x^* \in C$ and $y^* \in Q$, where C and Q are nonempty, closed, and convex sets in the corresponding spaces. The theory of the SCFP attracted much attention due to its wide applications in the real world. Another special case of the SIP is the *multiple sets split feasibility problem* (MSSFP) [3] of the form:

$$\text{find } x^* \in C := \cap_{i=1}^{N} C_i \quad \text{such that } \; A(x^*) \in Q := \cap_{j=1}^{M} Q_j,$$

where $\{C_i\}_{i=1}^{N} \subset \mathrm{H}_1$ and $\{Q_j\}_{j=1}^{M} \subset \mathrm{H}_2$ are nonempty, closed, and convex sets in real Hilbert spaces H_1 and H_2, respectively, and $A : \mathrm{H}_1 \to \mathrm{H}_2$ is a bounded linear operator. In [4], the

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MSSFP was successfully applied to inverse problems arising in radiation treatment planning, image reconstruction and so on; see, e.g., [3, 4, 5, 6, 7].

The classical *equilibrium problem* (EP) [8] associated with a nonempty, closed, and convex set C and a bifunction $f : C \times C \to \mathbb{R}$ consists of finding a point $x^* \in C$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The equilibrium problem provides tools to model many nonlinear operator problems with applications; see, e.g., [9, 10] and the references therein. For a given mapping $F : C \to H$, if $f(x,y) := \langle F(x), y-x \rangle$, then problem (1.1) is equivalent to the classical *variational inequality problem* (VIP) of Fichera [11, 12] and Stampacchia [13]. The VIP is formulated as follows; find a point $x^* \in C$ such that $\langle F(x^*), x-x^* \rangle \geq 0$, $\forall x \in C$. In 2018, Quy [14] introduced the *bilevel split equilibrium problem* (BSEP). Let C be a nonempty, closed, and convex set, $f : C \times C \to \mathbb{R}$ a bifunction, and $B : C \to C$ a strongly monotone mapping. The BSEP consists of finding a point $x^* \in \Omega^0$ such that

$$\langle B(x^*), y - x^* \rangle \geq 0, \ \forall y \in \Omega^0 := \Big\{ z \in C : A(z) \in Q \text{ and } f(z,y) \geq 0, \ \forall y \in C \Big\}. \tag{1.2}$$

Imposing some conditions on sequences $\{\gamma_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}$, and $\{\rho_n\}_{n=1}^{\infty}$, Quy [14] established the following iterative procedure for solving (1.2)

$$\begin{cases} y_n = P_C\big(x_n - \gamma_n A^*(I-P_Q)Ax_n\big), \\ v_n = \operatorname*{argmin}_{y \in C} \big\{ f(y_n, y) + \dfrac{1}{2\rho_n}\|y - y_n\|^2 \big\}, \\ z_n = \operatorname*{argmin}_{y \in C} \big\{ f(v_n, y) + \dfrac{1}{2\rho_n}\|y - y_n\|^2 \big\}, \\ x_{n+1} := \beta_n x_n + (1-\beta_n) z_n - \alpha_n \mu B(z_n). \end{cases} \tag{1.3}$$

Bilevel problems attracted considerable research interests due to their wide applications in various fields ranging from transportation engineering (optimal design, optimal chemical equilibria, and network design [15, 16, 17]) to management economics (network facility location, Stackelberg games, principal-agent problem, taxation, optimal pricing, and policy decisions [15, 17, 18, 19, 20]). In [21], Suleiman et al. modified algorithm (1.3) and introduced a self-adaptive extragradient-CQ method (BiS-ECQM) that clearly demonstrates good theoretical and practical features.

The purpose of this paper is to introduce a bilevel problem, *bilevel multiple sets split equilibrium problem* (BMSSEP). Let $\{C_i\}_{i=1}^N \subset \mathrm{H}_1$ and $\{Q_j\}_{j=1}^M \subset \mathrm{H}_2$ be nonempty, closed, and convex subsets, $f_i : C_i \times C_i \to \mathbb{R}$ a bifunction, and $B : C \to C$ a strongly monotone mapping. The BMSSEP is formulated as follows

$$\text{find } x^* \in \Omega := \cap_{i=1}^N \mathrm{SOL}(f_i, A, C_i) \ \text{ such that } \Big\langle B(x^*), y - x^* \Big\rangle \geq 0, \ \forall y \in \Omega. \tag{1.4}$$

To solve (1.4), we propose a parallel self-adaptive extragradient-CQ method (BiP-SECQM). The computational advantages of our scheme lies in the fact that, per each iteration, the vectors move in parallel from $i = 1$ through N and from $j = 1$ through M until a common solution is reached at the lower level before it can move to the upper level. This clearly links our result and the results of Quy [14], and also generalizes other important results, such as the system of equilibrium problem [22] and the multiple-sets split equilibrium problem [6].

The paper is organized as follows. We first recall some basic definitions and results in Section 2. In Section 3, we present our new iterative method for solving bilevel multiple-sets split equilibrium problems and prove its norm convergence. In Section 4, we illustrate the performances of our proposed algorithm via a numerical example. Concluding remarks and conclusions are given in Section 5.

2. Preliminaries

In this section, we give some definitions and basic results, that will be used in our convergence analysis. Let H a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. Let $C \subseteq H$ be a nonempty, closed, and convex set. $\rightharpoonup$ and $\longrightarrow$ are denoted for weak and strong convergence of a sequence, respectively.

Recall that a bifunction $f : C \times C \subset\!\!\to \mathbb{R}$ is said to be

(i) *monotone* if $f(x,y) + f(y,x) \leq 0, \forall\, x,y \in C$;
(ii) *pseudomonotone* with respect to $x \in C$ if $f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0, \forall y \in C$;
(iii) *pseudomonotone* with respect to $\emptyset \neq \Omega \subset C$ if, $\forall x^* \in \Omega$,

$$f(x^*,y) \geq 0 \;\Rightarrow\; f(y,x^*) \leq 0, \;\forall\, y \in C;$$

(iv) *Lipschitz-type continuous* if there are two positive constants L_1 and L_2 such that

$$f(x,y) + f(y,z) \geq f(x,z) - L_1\|x-y\|^2 - L_2\|y-z\|^2, \;\forall\, x,y,z \in C;$$

(v) *jointly weakly continuous* $\lim_{n\to\infty} f(x_n,y_n) = f(x,y)$. where $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are two sequences in C converging weakly to x and y, respectively.

Recall that a single-valued mapping $f : \mathrm{H} \to \mathrm{H}$ with domain $D(f)$ in H is said to be

(vi) β*-strongly monotone* if there exists $\beta > 0$ such that

$$\langle f(x) - f(y), x-y \rangle \geq \beta \|x-y\|^2, \;\forall\, x,y \in D(f);$$

(vii) *monotone* if

$$\langle f(x) - f(y), x-y \rangle \geq 0, \;\forall\, x,y \in D(f);$$

(viii) *pseudomonotone* if

$$\langle f(x), y-x \rangle \geq 0 \;\Rightarrow\; \langle f(y), y-x \rangle \geq 0, \;\forall\, x,y \in D(f);$$

(ix) *L-Lipschitz continuous* if there exists a positive constant L such that $\|f(x)-f(y)\| \leq L\|x-y\|, \;\forall\, x,y \in D(f)$.

A metric projection onto C is the mapping $\mathrm{P_C} : H \to C$, which assigns each $x \in H$ to the (nearest) unique point $\mathrm{P_C}(x)$ in C, i.e., $\|x - \mathrm{P_C}(x)\| = \min\{\|x-y\| : y \in C\}$.

Next, we recall some useful lemmas.

Lemma 2.1. [23] *Let $B : \mathrm{H} \to \mathrm{H}$ be β- strongly monotone and L-Lipschitz continuous on* H *with $0 < \theta < 1,\; 0 \leq \sigma \leq 1-\theta$, and $0 < \mu < \frac{2\beta}{L^2}$. Then, for all $x,y \in H$,*

$$\left\| \big((1-\sigma)x - \theta\mu B(x)\big) - \big((1-\sigma)y - \theta\mu B(y)\big) \right\| \leq (1-\sigma-\theta\tau)\,\|x-y\|,$$

where $\tau = 1 - \sqrt{1-\mu(2\beta-\mu L^2)} \in (0,1]$.

Lemma 2.2. [24] *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_{i+1}}$ for all $i \geq 0$. Then there exists an increasing sequence $\{m_k\}_{k=1}^{\infty} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$: $a_{m_k} \leq a_{m_{k+1}}$ and $a_k \leq a_{m_{k+1}}$. In fact, m_k is the largest number n in the set $\{1,2,\ldots,k\}$ such that the condition $a_n \leq a_{n+1}$ holds.*

Lemma 2.3. [25] *For each $x_1, x_2, \ldots, x_m \in \mathrm{H}$ and $\alpha_1, \alpha_2, \ldots, \alpha_m \in [0,1]$ with $\sum_{i=1}^{m} \alpha_i = 1$. Then $\|\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_m x_m\|^2 = \sum_{i=1}^{m} \alpha_i \left\|x_i\right\|^2 - \sum_{1 \leq i < j \leq m} \alpha_i \alpha_j \left\|x_i - x_j\right\|^2$.*

Lemma 2.4. [26] *Let $\{\gamma_n\}_{n=1}^{\infty}$ be a sequences in $(0,1)$, and let $\{\delta_n\}_{n=1}^{\infty}$ be in $\mathbb{R}$ satisfying $\sum_{n=1}^{\infty} \gamma_n = \infty$ and $\limsup_{n\to\infty} \delta_n \leq 0$ or $\sum_{n=1}^{\infty} |\gamma_n \delta_n| < \infty$. If $\{a_n\}_{n=1}^{\infty}$ is a sequence of nonnegative real numbers such that $a_{n+1} \leq (1-\gamma_n) a_n + \gamma_n \delta_n$, $\forall n \geq 0$, then $\lim_{n\to\infty} a_n = 0$.*

3. Main Results

For the convergence analysis of our method, we denote by

$$\mathrm{SOL}(f_i, A, B, C_i) = \left\{ x \in \Omega : \left\langle B(x^*), y - x^* \right\rangle \geq 0, \ \forall y \in \Omega \right\},$$

the solutions set of (1.4) and assume the following assumptions on the BMSSEP data.

Assumption 3.1.

- (A1) B is β-strongly monotone and L-Lipschitz continuous on every bounded subset of H;
- (A2) f_i is pseudomonotone, jointly weakly and Lipschitz-type continuous with constants L_1 and L_2, for each $i = 1, 2, \ldots, N$;
- (A3) $f_i(x, \cdot)$ is convex and subdifferentiable on C_i for each $i = 1, 2, \ldots, N$;
- (A4) $\mathrm{SOL}(f_i, A, B, C_i) \neq \emptyset$.

Remark 3.1. To initialise our procedure, we choose parameter sequences as follows:

- (B1) $\{\delta_n^i\}_{n=1}^{\infty} \subset [\underline{\delta}, \bar{\delta}] \subset (0,1]$ such that $\sum_{i=1}^{N} \delta_n^i = 1$;
- (B2) $\{\rho_n^i\}_{n=1}^{\infty} \subset [\underline{\rho}, \bar{\rho}]$ such that $[\underline{\rho}, \bar{\rho}] \subset (0, \min\{\frac{1}{2L_1}, \frac{1}{2L_2}\})$;
- (B3) $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty} \subset (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $0 \leq \beta_n \leq 1 - \alpha_n$ satisfying $\lim_{n\to\infty} \beta_n = K < 1$.

Algorithm 3.1 (BiP-SECQM)**.**
Initialization: Choose $x_0 \in C := \cap_{i=1}^{N} C_i$.
Iterative steps: Given the current iterate x_n, the next iterate x_{n+1} is updated according to the following steps.
Step 1: Solve N strongly convex programs in parallel:

$$y_n^i = \underset{y \in C_i}{\operatorname{argmin}} \left\{ f_i(x_n, y) + \frac{1}{2\rho_n^i} \|y - x_n\|^2 \right\}, \qquad z_n^i = \underset{y \in C_i}{\operatorname{argmin}} \left\{ f_i(y_n^i, y) + \frac{1}{2\rho_n^i} \|y - x_n\|^2 \right\},$$

and set $\overline{z_n} = \sum_{i=1}^{N} \delta_n^i z_n^i$.

Step 2: Compute $F(\overline{z_n}) := \frac{1}{2} \|A\overline{z_n} - P_Q A\overline{z_n}\|^2$ and $\nabla F(\overline{z_n}) = A^*(A\overline{z_n} - P_Q A\overline{z_n})$.

Step 3: Select a step-size $\{\gamma_n\}_{n=1}^{\infty}$ as follows:

$$\gamma_n = \begin{cases} \sigma_n \frac{F(\overline{z_n})}{\|\nabla F(\overline{z_n})\|^2}, & \text{if } \nabla F(\overline{z_n}) \neq 0; \\ 0, & \text{otherwise,} \end{cases}$$

with $\{\gamma_n\}_{n=1}^{\infty}$ and $\{\sigma_n\}_{n=1}^{\infty} \subset (0,4)$ satisfying the following conditions:

$(C1)$ $0 \leq t_1 \leq \gamma_n \leq t_2$ for some $t_1, t_2 \in \mathbb{R}$ and $\forall n \in \Gamma = \Big\{ n \geq 1 : \nabla F(\overline{z_n}) \neq 0 \Big\}$;

$(C2)$ $\inf\limits_{n \in \Gamma} \{4 - \sigma_n\} > 0$.

Step 4: Compute $u_n = \mathrm{P_C}\Big(\overline{z_n} - \gamma_n \nabla F(\overline{z_n})\Big)$.

Step 5: Update the next iterate as $x_{n+1} = \beta_n x_n + (1-\beta_n) u_n - \alpha_n \mu\ B(u_n)$. If $x_{n+1} = x_n$, then **STOP**, and x_n is a solution of the BMSSEP (1.4); Otherwise, set $n \leftarrow n+1$ and go to **Step 1**.

The following lemma is essential to our the convergence analysis of the above algorithm.

Lemma 3.1 ([27], Lemma 3.1)**.** *Assume that the conditions* $(A1)-(A4)$ *of Assumption 3.1 hold and* $x^* \in \cap_{i=1}^{N} \mathrm{SOL}(f_i, C_i)$. *Let* $\{x_n\}_{n=1}^{\infty}$ *be a sequence generated by Algorithm 3.1. Then*

(a) $\rho_n^i \Big(f_i(x_n, y) - f_i(x_n, y_n^i) \Big) \geq \Big\langle y_n^i - x_n, y_n^i - y \Big\rangle$, $\forall\, y \in C_i$, $i = 1, 2, \ldots, N$;

(b) $\Big\| z_n^i - x^* \Big\|^2 \leq \Big\| x_n - x^* \Big\|^2 - (1 - 2\rho_n^i L_1) \Big\| x_n - y_n^i \Big\|^2 - (1 - 2\rho_n^i L_2) \Big\| y_n^i - z_n^i \Big\|^2$, $\forall\, n \in \mathbb{N}$.

Lemma 3.2. *Assume that the conditions* $(A1)-(A4)$ *of Assumption 3.1 hold. Let* $\{x_n\}_{n=1}^{\infty}$, $\{y_n^i\}_{n=1}^{\infty}$, $\{z_n^i\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, *and* $\{\overline{z_n}\}_{n=1}^{\infty}$ *be the sequences generated by Algorithm 3.1. Then, the following inequalities hold.*

(1) $\sum_{i=1}^{N} \delta_n^i (1 - 2\rho_n^i L_1) \Big\| x_n - y_n^i \Big\|^2 + \sum_{i=1}^{N} \delta_n^i (1 - 2\rho_n^i L_2) \Big\| y_n^i - z_n^i \Big\|^2 \leq \Big\| x_n - x^* \Big\|^2 - \Big\| \overline{z_n} - x^* \Big\|^2, \forall x^* \in \mathrm{SOL}(f_i, A, B, C_i)$;

(2) $\Big\langle \nabla F(\overline{z_n}),\ \overline{z_n} - x^* \Big\rangle \geq 2F(\overline{z_n})$, $\forall x^* \in \mathrm{SOL}(f_i, A, B, C_i)$;

(3) $\sigma_n(4 - \sigma_n) \frac{(F(\overline{z_n}))^2}{\|\nabla F(\overline{z_n})\|^2} \leq \|\overline{z_n} - x^*\|^2 - \|u_n - x^*\|^2$, $\forall x^* \in \mathrm{SOL}(f_i, A, B, C_i)$;

(4) $\{x_n\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, *and* $\{\overline{z_n}\}_{n=1}^{\infty}$ *are bounded.*

Proof. Let x^* be an arbitrary element in $\mathrm{SOL}(f_i, A, B, C_i)$, that is, $x^* \in \cap_{i=1}^{N} \mathrm{SOL}(f_i, C_i)$ and $A(x^*) \in \cap_{j=1}^{M} Q_j$. It follows from Lemma 3.1 that

$$\Big\| z_n^i - x^* \Big\|^2 \leq \Big\| x_n - x^* \Big\|^2 - (1 - 2\rho_n^i L_1) \Big\| x_n - y_n^i \Big\|^2 - (1 - 2\rho_n^i L_2) \Big\| y_n^i - z_n^i \Big\|^2. \tag{3.1}$$

Since inequality (3.1) is valid for each $i \in \{1, 2, 3, \ldots, N\}$ and $\overline{z_n} = \sum_{i=1}^{N} \delta_n^i z_n^i$, we have

$$\Big\| \overline{z_n} - x^* \Big\|^2 = \Big\| \sum_{i=1}^{N} \delta_n^i z_n^i - x^* \Big\|^2 = \Big\| \sum_{i=1}^{N} \delta_n^i (z_n^i - x^*) \Big\|^2. \tag{3.2}$$

Applying Lemma 2.3 to (3.2), we obtain

$$\Big\| \overline{z_n} - x^* \Big\|^2 = \Big\| \sum_{i=1}^{N} \delta_n^i (z_n^i - x^*) \Big\|^2 = \sum_{i=1}^{N} \delta_n^i \Big\| z_n^i - x^* \Big\|^2 - \frac{1}{2} \sum_{i=1}^{N} \sum_{k=1}^{N} \alpha_n^i \delta_n^k \Big\| z_n^i - z_n^k \Big\|^2. \tag{3.3}$$

Substituting (3.1) into (3.3), we have

$$\begin{aligned}\left\|\overline{z_n}-x^*\right\|^2 &= \sum_{i=1}^{N}\delta_n^i\left\|z_n^i-x^*\right\|^2-\frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{N}\delta_n^i\delta_n^k\left\|z_n^i-z_n^k\right\|^2\\ &\leq \sum_{i=1}^{N}\delta_n^i\Big(\left\|x_n-x^*\right\|^2-(1-2\rho_n^iL_1)\left\|x_n-y_n^i\right\|^2\\ &\quad -(1-2\rho_n^iL_2)\left\|y_n^i-z_n^i\right\|^2\Big)-\frac{1}{2}\sum_{i=1}^{N}\sum_{k=1}^{N}\delta_n^i\delta_n^k\left\|z_n^i-z_n^k\right\|^2.\end{aligned}$$

Hence

$$\left\|\overline{z_n}-x^*\right\|^2 \leq \left\|x_n-x^*\right\|^2-\sum_{i=1}^{N}\delta_n^i(1-2\rho_n^iL_1)\left\|x_n-y_n^i\right\|^2-\sum_{i=1}^{N}\delta_n^i(1-2\rho_n^iL_2)\left\|y_n^i-z_n^i\right\|^2.$$

Since $A(x^*)\in\cap_{j=1}^{M}Q_j:=Q$, we have $A(x^*)-\mathrm{P}_Q(Ax^*)=A(x^*)-A(x^*)=0$. It follows from the firm nonexpansiveness of $(I-\mathrm{P}_Q)$ that

$$\begin{aligned}\left\langle\nabla F(\overline{z_n}),\overline{z_n}-x^*\right\rangle &= \left\langle A^*(I-P_Q)A(\overline{z_n}),\overline{z_n}-x^*\right\rangle\\ &= \left\langle (I-P_Q)A(\overline{z_n}),A(\overline{z_n}-x^*)\right\rangle\\ &= \left\langle (I-P_Q)A(\overline{z_n})-(I-P_Q)A(x^*),A(\overline{z_n})-A(x^*)\right\rangle\\ &\geq \|(I-P_Q)A(\overline{z_n})\|^2.\end{aligned}$$

Hence, we have $\langle\nabla F(\overline{z_n}),\overline{z_n}-x^*\rangle\geq 2F(\overline{z_n})$, $\forall\, n\in\mathbb{N}$. From the definition of $\{u_n\}_{n=1}^{\infty}$, we obtain

$$\begin{aligned}\|u_n-x^*\|^2 &= \|\mathrm{P}_C(\overline{z_n}-\gamma_n\nabla F(\overline{z_n}))-\mathrm{P}_C(x^*)\|^2\\ &\leq \|\overline{z_n}-\gamma_n\nabla F(\overline{z_n})-x^*\|^2=\|\overline{z_n}-x^*-\gamma_n\nabla F(\overline{z_n})\|^2\\ &\leq \|\overline{z_n}-x^*\|^2+\|\gamma_n\nabla F(\overline{z_n})\|^2-2\gamma_n\langle\nabla F(\overline{z_n}),\overline{z_n}-x^*\rangle\\ &\leq \|\overline{z_n}-x^*\|^2+\gamma_n^2\|\nabla F(\overline{z_n})\|^2-4\gamma_nF(\overline{z_n})\\ &= \|\overline{z_n}-x^*\|^2+\sigma_n^2\frac{(F(\overline{z_n}))^2}{\|\nabla F(\overline{z_n})\|^4}\|\nabla F(\overline{z_n})\|^2-4\sigma_n\frac{F(\overline{z_n})}{\|\nabla F(\overline{z_n})\|^2}F(\overline{z_n})\\ &= \|\overline{z_n}-x^*\|^2+\sigma_n^2\frac{(F(\overline{z_n}))^2}{\|\nabla F(\overline{z_n})\|^2}-4\sigma_n\frac{(F(\overline{z_n}))^2}{\|\nabla F(\overline{z_n})\|^2}.\end{aligned}$$

Therefore, we have

$$\|u_n-x^*\|^2\leq\|\overline{z_n}-x^*\|^2-\sigma_n(4-\sigma_n)\frac{(F(\overline{z_n}))^2}{\|\nabla F(\overline{z_n})\|^2},\ \forall\, n\in\mathbb{N}. \tag{3.4}$$

From conditions $(C1)-(C2)$ on $\{\sigma_n\}_{n=1}^{\infty}$ and (3.4), we have

$$\|u_n-x^*\|^2\leq\|\overline{z_n}-x^*\|^2,\ \forall\, n\in\mathbb{N}. \tag{3.5}$$

In view of (1) and (3.5), we conclude

$$\|u_n-x^*\|^2\leq\|\overline{z_n}-x^*\|^2\leq\|x_n-x^*\|^2,\ \forall\, n\in\mathbb{N}. \tag{3.6}$$

Using Algorithm 3.1, we have

$$\begin{aligned}\|x_{n+1}-x^*\| \le \|&\big((1-\beta_n)u_n-\alpha_n\mu B(u_n)\big)-\big((1-\beta_n)x^*-\alpha_n\mu B(x^*)\big)\\&+\beta_n(x_n-x^*)-\alpha_n\mu B(x^*)\|\\ \le&\|\big((1-\beta_n)u_n-\alpha_n\mu B(u_n)\big)-\big((1-\beta_n)x^*-\alpha_n\mu B(x^*)\big)\|\\&+\beta_n\|x_n-x^*\|+\alpha_n\mu\|B(x^*)\|\\ \le&(1-\beta_n-\alpha_n\tau)\|u_n-x^*\|+\beta_n\|x_n-x^*\|+\alpha_n\mu\|B(x^*)\|\\ \le&(1-\alpha_n\tau)\|x_n-x^*\|+\tau\alpha_n\frac{\mu\|B(x^*)\|}{\tau},\end{aligned}$$

where the last two inequalities follow from Lemmas 2.1 and (3.6), and

$$\tau=1-\sqrt{1-\mu(2\beta-\mu L^2)}\in(0,1].$$

Hence,

$$\|x_{n+1}-x^*\|\le\max\left\{\|x_0-x^*\|,\frac{\mu\|B(x^*)\|}{\tau}\right\},\ \forall\, n\in\mathbb{N}.$$

This indicates that $\{x_n\}_{n=1}^{\infty}$ is bounded. From (3.6), we conclude that $\{\overline{z_n}\}_{n=1}^{\infty}$ and $\{u_n\}_{n=1}^{\infty}$ are bounded. By virtue of condition $(A1)$, $\{B(u_n)\}_{n=1}^{\infty}$ is also bounded. □

Now, we are in position to prove the main convergence theorem.

Theorem 3.1. *If the assumptions of Lemma 3.2 hold, then the sequence $\{x_n\}_{n=1}^{\infty}$ generated by Algorithm 3.1 converges to $x^*\in \mathrm{SOL}(f_i,A,B,C_i)$ in norm.*

Proof. Let $x^*\in\mathrm{SOL}(f_i,A,B,C_i)$. Using Lemma 2.1 and the following known subdifferential inequality $\|x-y\|^2\le\|x\|^2-2\langle y,x-y\rangle$, $\forall x,y\in \mathrm{H}_1$, we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2=&\|((1-\beta_n)u_n-\alpha_n\mu B(u_n))-((1-\beta_n)x^*-\alpha_n\mu B(x^*))\\&+\beta_n(x_n-x^*)-\alpha_n\mu B(x^*)\|^2\\ \le&\|((1-\beta_n)u_n-\alpha_n\mu B(u_n))-((1-\beta_n)x^*-\alpha_n\mu B(x^*))\\&+\beta_n(x_n-x^*)\|^2-2\alpha_n\mu\langle B(x^*),x_{n+1}-x^*\rangle\\ \le&\Big(\|((1-\beta_n)u_n-\alpha_n\mu B(u_n))-((1-\beta_n)x^*-\alpha_n\mu B(x^*))\|\\&+\beta_n\|x_n-x^*\|\Big)^2-2\alpha_n\mu\langle B(x^*),x_{n+1}-x^*\rangle\\ \le&\Big((1-\beta_n-\alpha_n\tau)\|u_n-x^*\|+\beta_n\|x_n-x^*\|\Big)^2-2\alpha_n\mu\langle B(x^*),x_{n+1}-x^*\rangle\\ \le&\Big(1-\beta_n-\alpha_n\tau)\|x_n-x^*\|+\beta_n\|x_n-x^*\|\Big)^2-2\alpha_n\mu\langle B(x^*),x_{n+1}-x^*\rangle.\end{aligned}$$

This implies that

$$\|x_{n+1}-x^*\|^2\le(1-\alpha_n\tau)\|x_n-x^*\|^2-2\alpha_n\mu\langle B(x^*),x_{n+1}-x^*\rangle,\ \forall\, n\in\mathbb{N}. \tag{3.7}$$

In order to show that $\{x_n\}_{n=1}^{\infty}$ converges strongly to x^*, we consider two possibilities on $\{\|x_n-x^*\|\}_{n=1}^{\infty}$.

Case 1. Assume that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - x^*\|\}_{n=1}^{\infty}$ is monotone decreasing for $n \geq n_0$. Since $\{x_n\}_{n=1}^{\infty}$ is bounded, the limit of $\{\|x_n - x^*\|\}_{n=1}^{\infty}$ exists. It follows from (3.6) and (3.7) that

$$\begin{aligned} 0 \leq & \|x_n - x^*\|^2 - \|\overline{z_n} - x^*\|^2 \\ \leq & -\frac{\alpha_n \tau}{1-\beta_n}\|\overline{z_n} - x^*\|^2 - \frac{2\alpha_n \mu}{1-\beta_n}\langle B(x^*), x^* - x_{n+1}\rangle + \frac{1}{1-\beta_n}\left(\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2\right). \end{aligned} \tag{3.8}$$

By condition $(B3)$ on $\{\beta_n\}_{n=1}^{\infty}$ and $\{\alpha_n\}_{n=1}^{\infty}$ as well as the fact that limit of $\{\|x_n - x^*\|\}_{n=1}^{\infty}$ exists, we have

$$\lim_{n\to\infty}\left(\|x_n - x^*\|^2 - \|\overline{z_n} - x^*\|^2\right) = 0. \tag{3.9}$$

B combining (3.6), (3.8), and (3.9), we have $\lim_{n\to\infty}\left(\|x_n - x^*\|^2 - \|u_n - x^*\|^2\right) = 0$. Again, by (3.8) and (3.9), we obtain

$$\lim_{n\to\infty}\left(\|u_n - x^*\|^2 - \|\overline{z_n} - x^*\|^2\right) = 0. \tag{3.10}$$

On the other hand, Lemma 3.2 (1) yields that

$$\begin{aligned} 0 \leq & \sum_{i=1}^{N}\delta_n^i(1-2\rho_n^i L_1)\left\|x_n - y_n^i\right\|^2 + \sum_{i=1}^{N}\delta_n^i(1-2\rho_n^i L_2)\left\|y_n^i - z_n^i\right\|^2 \\ \leq & \left\|x_n - x^*\right\|^2 - \left\|\overline{z_n} - x^*\right\|^2. \end{aligned}$$

Using (3.9) and the conditions $(B1)$ and $(B2)$ on $\{\delta_n\}_{n=1}^{\infty}$ and $\{\rho_n\}_{n=1}^{\infty}$, respectively, we deduce that

$$\lim_{n\to\infty}\|x_n - y_n^i\|^2 = 0 \text{ and } \lim_{n\to\infty}\|y_n^i - z_n^i\|^2 = 0, \ i = 1,2,3,\ldots,N. \tag{3.11}$$

In view of Lemma 3.2 (3) and condition $(C1)$, we have

$$0 \leq t_2(4-\sigma_n)F(\overline{z_n}) \leq \|\overline{z_n} - x^*\|^2 - \|u_n - x^*\|^2.$$

It follows from (3.10) and condition $(C2)$ that

$$\lim_{n\to\infty}F(\overline{z_n}) = 0. \text{ Hence } \lim_{n\to\infty}\|(I-P_Q)A\overline{z_n}\| = 0. \tag{3.12}$$

Using (3.12) and definition of $\{u_n\}_{n=1}^{\infty}$, we get

$$\lim_{n\to\infty}\|u_n - \overline{z_n}\| = 0. \tag{3.13}$$

It is easy to see that

$$\left\|y_n^i - \overline{z_n}\right\| = \left\|y_n^i - \sum_{i=1}^{N}\delta_n^i z_n^i\right\| = \left\|\sum_{i=1}^{N}\delta_n^i(y_n^i - z_n^i)\right\| \leq \sum_{i=1}^{N}\delta_n^i\left\|y_n^i - z_n^i\right\|,$$

which together with (3.11) obtains $\lim_{n\to\infty}\left\|y_n^i - \overline{z_n}\right\| = 0$. In view of the triangle inequality, we have $\|x_n - \overline{z_n}\| \leq \|x_n - y_n^i\| + \|y_n^i - \overline{z_n}\|$, which together with (3.11) leads to

$$\lim_{n\to\infty}\left\|x_n - \overline{z_n}\right\| = 0. \tag{3.14}$$

Similarly, from $\|u_n - x_n\| \leq \|u_n - \overline{z_n}\| + \|x_n - \overline{z_n}\|$ and (3.13), we arrive at

$$\lim_{n\to\infty}\left\|u_n - x_n\right\| = 0. \tag{3.15}$$

By using the definition of x_{n+1} and u_n, we have $\|x_{n+1}-u_n\| \leq \alpha_n\mu\|B(u_n)\|$. Since $\{B(u_n)\}_{n=1}^{\infty}$ is bounded and $\alpha_n \to 0$ as $n\to\infty$, we have $\lim_{n\to\infty}\Big\|x_{n+1}-u_n\Big\| = 0$. In view of (3.15) and $\|x_{n+1} - x_n\| \leq \|x_{n+1}-u_n\|+\|u_n-x_n\|$, we deduce that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.16}$$

Since H_1 is reflexive and $\{x_n\}_{n=1}^{\infty}\subset H_1$ is bounded, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $x_{n_k}\rightharpoonup e^*\in H_1$ as $k\to\infty$. So, we may assume that

$$\limsup_{n\to\infty}\langle B(x^*),x^*-x_n\rangle = \lim_{k\to\infty}\langle B(x^*),x^*-x_{n_k}\rangle. \tag{3.17}$$

Thus, it follows from (3.14), (3.15), (3.11), and the boundedness of $\{\overline{z_n}\}_{n=1}^{\infty}$, $\{u_n\}_{n=1}^{\infty}$, $\{y_n^i\}_{n=1}^{\infty}$, and $\{z_n^i\}_{n=1}^{\infty}$ that

$$\bar{z}_{n_k}\rightharpoonup e^*,\ u_{n_k}\rightharpoonup e^*,\ y^i_{n_k}\rightharpoonup e^*,\ \text{and}\ z^i_{n_k}\rightharpoonup e^*\ \text{as}\ \ k\to\infty. \tag{3.18}$$

We next show that $e^*\in\Omega$. Indeed, since C_i, $i=1,2,\ldots,N$ is closed and convex, then $\cap_{i=1}^N C_i$ is closed and convex. Consequently, $\cap_{i=1}^N C_i$ and $\cap_{j=1}^M Q_i$ are weakly closed. Because $\{x_n\}_{n=1}^{\infty}\subset \cap_{i=1}^N C_i$, then it follows that

$$e^*\in\cap_{i=1}^N C_i. \tag{3.19}$$

By Lemma 3.1, we have

$$\rho^i_{n_k}\Big(f_i(x_{n_k},y)+f_i(x_{n_k},y^i_{n_k})\Big)\geq\Big\langle y^i_{n_k}-x_{n_k},y^i_{n_k}-y\Big\rangle,\ \forall\, y\in C_i,\ i=1,2,\ldots N.$$

This implies

$$\rho^i_{n_k}\Big(f_i(x_{n_k},y)+f_i(x_{n_k},y^i_{n_k})\Big)\geq\Big\langle y^i_{n_k}-x_{n_k},y^i_{n_k}-y\Big\rangle\geq-\Big\|y^i_{n_k}-x_{n_k}\Big\|\Big\|y^i_{n_k}-y\Big\|,\ \forall\, y\in C_i. \tag{3.20}$$

By taking limit as $k\to\infty$ in (3.20) and using (3.18) with condition $(A3)$, we obtain

$$f_i\Big(e^*,y\Big)-f_i\Big(e^*,e^*\Big)\geq 0,\ \forall\, y\in C_i,\ i=1,2,\ldots,N,$$

which implies $e^*\in\mathrm{SOL}(f_i,C_i),\ i=1,2,3,\ldots,N$. This means

$$e^*\in\cap_{i=1}^N\mathrm{SOL}(f_i,C_i). \tag{3.21}$$

Since A is linear, bounded, and continuous, then it follows from (3.18) that $A(\bar{z}_{n_k})\rightharpoonup A(e^*)$ as $k\to\infty$. Clearly, it follows from (3.12), and the boundedness of $\{A\overline{z_n}\}_{n=1}^{\infty}$ that $P_Q(A(\bar{z}_{n_k}))\rightharpoonup A(e^*)$ as $k\to\infty$. Similarly, since $\{P_Q(A(\bar{z}_{n_k}))\}_{k=1}^{\infty}\subset\cap_{j=1}^M Q_j$. It follows that $A(e^*)\in\cap_{j=1}^M Q_j$. Thus, from (3.19) and (3.21), we deduce that $e^*\in\Omega$. Hence $\langle B(x^*),e^*-x^*\rangle\geq 0$. From (3.16), (3.17), and (3.18), we have

$$\limsup_{n\to\infty}\langle B(x^*),x^*-x_{n+1}\rangle\leq\limsup_{n\to\infty}\langle B(x^*),x^*-x_n\rangle=\lim_{k\to\infty}\langle B(x^*),x^*-x_{n_k}\rangle\leq 0.$$

This implies $\limsup_{n\to\infty} r_n\leq 0$, where $r_n=\frac{2\mu}{\tau}\langle B(x^*),x^*-x_{n+1}\rangle$ and $\tau=1-\sqrt{1-\mu(2\beta-\mu L^2)}\in(0,1]$. In view of this and (3.7), we have $\|x_{n+1}-x^*\|^2\leq(1-\alpha_n\tau)\|x_n-x^*\|^2+\tau\alpha_n r_n$. By Lemma 2.4, we conclude that $\lim_{n\to\infty}\|x_n-x^*\|=0$. Thus, $x_n\to x^*$ in $\mathrm{SOL}(f_i,A,B,C_i)$ as $n\to\infty$.

CASE 2. Suppose that the sequence $\{\|x_n - x^*\|\}_{n=1}^{\infty}$ is not-decreasing. That is, there exists a subsequence $\{x_{n_m}\}_{m=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\|x_{n_m} - x^*\| \leq \|x_{n_{m+1}} - x^*\|$, $\forall m \in \mathbb{N}$. By Lemma 2.2, there exists a nondecreasing sequence $\{\tau(n)\} \subset \mathbb{N}$ such that $\lim_{n\to\infty} \tau(n) = \infty$,

$$\|x_{\tau(n)} - x^*\| \leq \|x_{\tau(n)+1} - x^*\| \text{ and } \|x_n - x^*\| \leq \|x_{\tau(n)+1} - x^*\|, \text{ for sufficiently large } n \in \mathbb{N}. \tag{3.22}$$

By Lemma 3.2 (4), the sequences $\{x_{\tau(n)}\}_{n=1}^{\infty}$, $\{u_{\tau(n)}\}_{n=1}^{\infty}$, and $\{z_{\bar{\tau}(n)}\}_{n=1}^{\infty}$ are bounded. Thus, following an argument similar to that in **Case 1**, we deduce that $\lim_{n\to\infty} \|x_{\tau(n)+1} - x_{\tau(n)}\| = 0$. Also, we have

$$\limsup_{n\to\infty} 2\mu \langle B(x^*), x^* - x_{\tau(n)+1} \rangle \leq 0. \tag{3.23}$$

Thus, it follows from equation (3.7) and (3.22) that

$$\|x_{\tau(n)+1} - x^*\|^2 \leq (1 - \alpha_{\tau(n)}\tau)\|x_{\tau(n)+1} - x^*\|^2 + 2\alpha_{\tau(n)}\mu \langle B(x^*), x^* - x_{\tau(n)+1} \rangle. \tag{3.24}$$

Combining (3.22) and (3.24), we have

$$\|x_n - x^*\|^2 \leq \|x_{\tau(n)+1} - x^*\|^2 \leq 2\mu \langle B(x^*), x^* - x_{\tau(n)+1} \rangle \text{ since } \alpha_{\tau(n)} > 0. \tag{3.25}$$

By taking limsup in (3.25) and using (3.23) as $n \to \infty$, we obtain $\limsup_{n\to\infty} \|x_n - x^*\|^2 \leq 0$. Therefore, in both cases, $x_n \to x^*$ in $\mathrm{SOL}(f_i, A, B, C_i)$ and this ends the proof. □

4. ILLUSTRATION

In this section, we consider a special case of the bilevel multiple sets split equilibrium problem (BMSSEP) (1.4) in which we aim to find the projection of a given point p onto the solution set of the linear split feasibility problem taken from Dang and Xue [28]. The problem is defined as

$$\begin{aligned} &\min \tfrac{1}{2}\|x - p\|^2 \\ &\text{such that } x \in \cap_{i=1}^{N} C_i \quad \text{and } \; A(x) \in \cap_{j=1}^{M} Q_j. \end{aligned} \tag{4.1}$$

Observe that (4.1) is obtained from (1.4) by taking $B(x) = \nabla\left(\frac{1}{2}\|x - p\|^2\right) = x - p$ and $f_i \equiv 0$. Consider the Euclidean spaces $\mathbb{R}^5$ and $\mathbb{R}^4$ and the constraints sets

$$\begin{aligned} C_1 &= \{x \in \mathbb{R}^5 \mid x_1 + 2x_2 + x_3 + x_4 \leq 5\}, \\ C_2 &= \{x \in \mathbb{R}^5 \mid x_2 + 4x_4 + 4x_5 \leq 1\} \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} Q_1 &= \{y \in \mathbb{R}^4 \mid y_1 + y_4 \leq 1\}, \\ Q_2 &= \{y \in \mathbb{R}^4 \mid 2y_2 + 3y_3 \leq 6\}, \\ Q_3 &= \{y \in \mathbb{R}^4 \mid y_3 + 2y_4 \leq 10\}. \end{aligned} \tag{4.3}$$

In addition,

$$A = \begin{pmatrix} 2 & -1 & 3 & 2 & 3 \\ 1 & 2 & 5 & 2 & 1 \\ 2 & 0 & 2 & 1 & -2 \\ 2 & -1 & 0 & -3 & 5 \end{pmatrix}. \tag{4.4}$$

The exact solution of the problem is $(0.2645, -0.6568, 0.4890, -0.7548, -0.3836)$. In Algorithm 3.1, we solve the strongly convex programs and compute the projection onto C and

Q via the Matlab routine `fmincon` and CVX http://cvxr.com/cvx/). We have $N = 2, M = 3$, $x_0 = (0,0,0,0,0), p = (1,-1,1,-1,1)$, $\sigma_n = 3 - \frac{1}{n+1}$, $\delta_n^1 = \delta_n^2 = 0.5, \rho_n^i = 1, \alpha_n = 1/n$, and $\beta_n = (n-1)/2n$. We define the function TOL_n by $TOL_n := \|x_{n+1} - x_n\|$ and use the stopping rule $TOL_n < \varepsilon$ for the iterative process, where ε is the predetermined error. The performance of Algorithm 3.1 is illustrated next in Table 1.

TABLE 1. Performance of Algorithm 3.1

Iterations	x_1	x_2	x_3	x_4	x_5
1	1	-1	1	-1	1
2	0.9489	-0.9788	0.9658	-0.98	0.9117
3	0.90177426	-0.95898452	0.93444442	-0.9614787	0.82964858
4	0.85832628	-0.940456015	0.905712211	-0.944333172	0.753390518
5	0.818281537	-0.923123976	0.879399031	-0.928467843	0.682503979
6	0.781385862	-0.906904421	0.855315995	-0.913793957	0.616598063
⋮	⋮	⋮	⋮	⋮	⋮
40	0.300783969	-0.675872355	0.517753296	-0.761538378	-0.307677191
⋮	⋮	⋮	⋮	⋮	⋮
100	0.264826996	-0.656992447	0.48929388	-0.754801155	-0.382801781

5. CONCLUSION

In this paper, we studied a class of bilevel multiple sets split equilibrium problems in real Hilbert spaces. We introduced and analyzed a new strong convergence iterative procedure for solving the problem and our results improve some related results recently announced in the literature, such as [1, 21, 29, 30, 31]. For the convenience of the readers, we list the novelty and main contributions of our work as follows.

(i) We extend the results in [14, 21] from bilevel split equilibrium problems to bilevel multiple-sets split equilibrium problems.
(ii) In the case that $H_1 = H_2$, $M = N = 1$, $C = C_1 = Q$, and $B \equiv 0$ are in Theorem 3.1, we obtain the algorithm in [30] for solving pseudomonotone equilibrium problems.
(iii) By setting $f_i(x,y) = \langle F_i(x), y - x\rangle \geq 0$, $\forall x,y \in H_1$, where, F_i is a strongly monotone on C_i in Theorem 3.1, we recover the algorithm of Censor et al. [1] for solving multiple sets split variational inequality problems.
(iv) Setting $B \equiv 0$ and $f \equiv 0$ in Theorem 3.1, we recover the self-adaptive method for solving multiple-sets split feasibility problems [31].
(v) Fixing $H_1 = H_2$, $C = \cap_{i=1}^{N} = Q$, and $B \equiv 0$ in Theorem 3.1, we obtain the system of pseudomonotone equilibrium problems [22].

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