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ON ANGLES BETWEEN CONVEX CONES

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Abstract. There are two basic angles associated with a pair of linear subspaces: the Diximier angle and the Friedrichs angle. The Dixmier angle of the pair of orthogonal complements is the same as the Dixmier angle of the original pair provided that the original pair gives rise to a direct (not necessarily orthogonal) sum of the underlying Hilbert space. The Friedrichs angles of the original pair and the pair of the orthogonal complements always coincide. These two results are due to Krein, Krasnoselskii, and Milman and to Solmon, respectively. In 1995, Deutsch provided a very nice survey with complete proofs and interesting historical comments. One key result in Deutsch's survey was an inequality for Dixmier angles provided by Hundal. In this paper, we present the extensions of these results to the case when the linear subspaces are only required to be convex cones. It turns out that Hundal's result has a nice conical extension while the situation is more technical for the results by Krein et al. and by Solmon. Our analysis is based on Deutsch's survey and our recent work on angles between convex sets. Throughout, we also provide examples illustrating the sharpness of our results.
Keywords. Angle between closed convex cones; Dixmier angle; Dual cone; Friedrichs angle; Polar cone.

1. INTRODUCTION

Throughout this paper, we assume that

$$\mathcal{H} \text{ is a real Hilbert space,}$$

with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$.

Following Deutsch and Hundal [1], we now recall two notions of angles between two nonempty convex sets:

Definition 1.1. (Dixmier and Friedrichs angle) [1, Definitions 2.3 and 3.2] Let C and D be nonempty convex sets in $\mathcal{H}$. The *minimal angle* or *Dixmier angle* between C and D is the angle in $\left[0, \frac{\pi}{2}\right]$ whose cosine is given by

$$c_0(C,D) := \sup \left\{ \langle x,y \rangle \;\middle|\; x \in \overline{\operatorname{cone}}(C) \cap \mathbf{B}_{\mathcal{H}},\ y \in \overline{\operatorname{cone}}(D) \cap \mathbf{B}_{\mathcal{H}} \right\}, \tag{1.1}$$

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where the cone and $\overline{\text{cone}}$ operator returns the smallest cone and closed cone containing its argument, respectively, and where $\mathbf{B}_{\mathscr{H}}$ denotes the closed unit ball in $\mathscr{H}$. In addition, the *angle* or *Friedrichs angle* between C and D is the angle in $\left[0, \frac{\pi}{2}\right]$ whose cosine is given by

$$c(C,D) := c_0\big((\operatorname{cone} C) \cap \overline{C^\ominus + D^\ominus},\ (\operatorname{cone} D) \cap \overline{C^\ominus + D^\ominus}\big), \tag{1.2}$$

and where $C^\ominus := \{y \in \mathscr{H} \mid \sup\langle C, y\rangle \leq 0\}$ is the polar cone of C and $C^\oplus := -C^\ominus$ is the dual cone of C.

The cosine of these angles plays a key role in describing convergence rates for projection methods such as the standard cyclic projection algorithm. For a taste of such results, we refer the reader to, for example, [1], [2], and [3, Example 9.40].

When M and N are closed linear subspaces of $\mathscr{H}$, then the cosines of their Dixmier and Friedrichs angles — which were first studied in [4] and [5] — can be written more succinctly as

$$c_0(M,N) = \sup\big\{\langle x,y\rangle \bigm| x \in M \cap \mathbf{B}_{\mathscr{H}},\ y \in N \cap \mathbf{B}_{\mathscr{H}}\big\}$$

and

$$c(M,N) := c_0\big(M \cap (M \cap N)^\perp,\ N \cap (M \cap N)^\perp\big),$$

respectively. The following three key results in this linear setting were beautifully presented and proved in Deutsch's survey [6] to which we also refer the reader for interesting historical comments:

Fact 1.1. (Hundal's Lemma) (see [6, Lemma 14]) *Let M and N be closed linear subspaces of $\mathscr{H}$ such that $c_0(M,N) < 1$, and let X be a closed linear subspace such that $M + N \subseteq X \subseteq \mathscr{H}$. Then*

$$c_0(M,N) \leq c_0(M^\perp \cap X, N^\perp \cap X).$$

Fact 1.2. (Krein, Krasnoselskii, and Milman) (see [6, Theorem 15]) *Let M and N be closed linear subspaces of $\mathscr{H}$ such that $M \cap N = \{0\}$ and $M + N = \mathscr{H}$. Then*

$$c_0(M,N) = c_0(M^\perp, N^\perp).$$

Fact 1.3. (Solmon) (see [6, Theorem 16]) *Let M and N be closed linear subspaces of $\mathscr{H}$. Then*

$$c(M,N) = c(M^\perp, N^\perp).$$

We are now in a position to describe *the aim of this paper: We will study the possibility of generalizing Fact 1.1, Fact 1.2, and Fact 1.3 from linear subspaces to convex cones.* While we initially anticipated nice generalizations, it turned out that only Fact 1.1 appears to admit a natural and nice generalization. The situation is more complicated for Fact 1.2 and Fact 1.3; our results and examples show that there are ostensibly no "nice" conical variants. Our analysis relies on Deutsch's exposition [6] as well as our recent work [7].

The remainder of the paper is organized as follows. In Section 2, we collect facts useful in subsequent proofs. The (positive) results concerning Hundal's Lemma (Fact 1.1) are presented in Section 3. The (somewhat negative) results concerning Fact 1.2 and Fact 1.3 are provided in Section 4.

Finally, the notation we employ is standard and follows [8] and [3]. For instance, orthogonality of vectors and sets as well as orthogonal complements are indicated by "$\perp$" while "$+$"

denotes the sum (not necessarily orthogonal) of vectors and sets. If $S \subseteq \mathcal{H}$, then the smallest convex cone (resp. closed convex cone) containing S is denoted by $\text{cone}(S)$ (resp. $\overline{\text{cone}}(S)$).

2. Auxiliary Result

In this section, we simply list — for the reader's convenience — several known results that are used in proving our new results in Section 3 and Section 4.

Fact 2.1. [3, Theorem 4.5] *Let C be a nonempty subset of $\mathcal{H}$. Then the following statements hold:*

(i) $C^{\ominus}$ *is a closed convex cone and* $C^{\perp}$ *is a closed linear subspace.*
(ii) $C^{\ominus} = (\overline{C})^{\ominus} = (\text{cone}(C))^{\ominus} = (\overline{\text{cone}}(C))^{\ominus}$.
(iii) $C^{\ominus\ominus} = \overline{\text{cone}}(C)$.
(iv) *If C is a closed convex cone, then* $C^{\ominus\ominus} = C$.
(v) *If C is a linear subspace, then* $C^{\ominus} = C^{\perp}$*; if C is additionally closed, then* $C = C^{\ominus\ominus} = C^{\perp\perp}$.

Fact 2.2. [7, Lemma 2.5] *Let C be a nonempty subset of $\mathcal{H}$. Then the following statements hold:*

(i) $(-C)^{\ominus} = -C^{\ominus} = C^{\oplus}$.
(ii) $C^{\oplus\oplus} = \overline{\text{cone}}(C)$.
(iii) *If C is a linear subspace of $\mathcal{H}$, then* $C^{\perp} = C^{\ominus} = C^{\oplus}$.

Fact 2.3. [8, Propositions 6.3 and 6.4] *Let K be a nonempty convex cone in $\mathcal{H}$. Then $K+K = K$. If $-K \subseteq K$, then K is a linear subspace.*

Fact 2.4. [8, Propositions 6.28] *Let K be a nonempty closed convex cone in $\mathcal{H}$, let $x \in \mathcal{H}$, and let $p \in \mathcal{H}$. Then $p = \mathrm{P}_K x \Leftrightarrow [p \in K, x-p \perp p,$ and $x - p \in K^{\ominus}]$.*

Fact 2.5. [9, page 48] *Let C and D be nonempty convex subsets of $\mathcal{H}$ such that $C \neq \{0\}$ and $D \neq \{0\}$. Then*

$$c_0(C,D) = \max\Big\{0, \sup\big\{\langle x,y\rangle \;\big|\; x \in \overline{\text{cone}}(C) \cap \mathbf{S}_{\mathcal{H}},\ y \in \overline{\text{cone}}(D) \cap \mathbf{S}_{\mathcal{H}}\big\}\Big\},$$

where $\mathbf{S}_{\mathcal{H}}$ denotes the unit sphere in $\mathcal{H}$.

Fact 2.6. [1, Lemma 2.4, Theorem 2.5 and Proposition 3.3] *Let C and D be nonempty convex subsets of $\mathcal{H}$. Then the following hold:*

(i) $c_0(C,D) \in [0,1]$ *and* $c(C,D) \in [0,1]$.
(ii) $(\forall x \in \overline{\text{cone}}(C))\ (\forall y \in \overline{\text{cone}}(D))\ \langle x,y\rangle \leq c_0(C,D)\|x\|\|y\|$.
(iii) $c_0(C,D) = c_0(D,C) = c_0\left(\overline{C},\overline{D}\right) = c_0(\text{cone}(C),\text{cone}(D)) = c_0(\overline{\text{cone}}(C),\overline{\text{cone}}(D))$.

Fact 2.7. [7, Lemma 2.11] *Let C and D be nonempty convex subsets of $\mathcal{H}$. Then the following hold:*

(i) *If U and V are nonempty convex subsets of $\mathcal{H}$ such that $C \subseteq U$ and $D \subseteq V$, then* $c_0(C,D) \leq c_0(U,V)$.
(ii) $c_0(C,D) = c_0(-C,-D)$, $c_0(-C,D) = c_0(C,-D)$, $c(C,D) = c(-C,-D)$, *and* $c(-C,D) = c(C,-D)$.
(iii) *If* $(\overline{\text{cone}}(C) \cap \overline{\text{cone}}(D)) \smallsetminus \{0\} \neq \varnothing$, *then* $c_0(C,D) = 1$.

(iv) $0 \le c(C,D) \le c_0(C,D) \le 1$.

Fact 2.8. [3, Theorem 4.6] *Let K_1 and K_2 be nonempty closed convex cones in $\mathcal{H}$. Then*

$$(K_1 \cap K_2)^{\ominus} = \overline{K_1^{\ominus} + K_2^{\ominus}}.$$

Fact 2.9. [1, Propositions 3.3(4)] *Let K_1 and K_2 be nonempty closed convex cones in $\mathcal{H}$. Then*

$$c(K_1,K_2) = c_0\left(K_1 \cap (K_1 \cap K_2)^{\ominus}, K_2 \cap (K_1 \cap K_2)^{\ominus}\right).$$

Fact 2.10. [7, Lemma 4.1] *Let K_1 and K_2 be nonempty closed convex cones in $\mathcal{H}$. Then the following hold:*

(i) *If $K_1 \cap K_2 \neq \{0\}$, then $c_0(K_1,K_2) = 1$.*
(ii) *If $K_1 \cap K_2 = \{0\}$, then $c_0(K_1,K_2) = c(K_1,K_2)$.*
(iii) *$K_1 \cap K_2 = \{0\}$ if and only if $\overline{K_1^{\ominus} + K_2^{\ominus}} = \mathcal{H}$.*

Fact 2.11. [7, Proposition 4.4] *Let $\mathcal{K}$ be a finite-dimensional linear subspace of $\mathcal{H}$, and let K_1 and K_2 be nonempty closed convex cones in $\mathcal{H}$ such that $K_1 \subseteq \mathcal{K}$ or $K_2 \subseteq \mathcal{K}$. Then the following statements hold:*

(i) *$K_1 \cap K_2 \neq \{0\}$ if and only if $c_0(K_1,K_2) = 1$.*
(ii) *$K_1 \cap K_2 = \{0\}$ if and only if $c_0(K_1,K_2) = c(K_1,K_2)$.*
(iii) *$K_1 \cap K_2 = \{0\}$ if and only if $c_0(K_1,K_2) < 1$.*

Fact 2.12. [7, Theorem 4.7] *Let K_1 and K_2 be nonempty closed convex cones in $\mathcal{H}$. If $c_0(K_1,K_2) < 1$, then $K_1 - K_2$ is closed.*

Fact 2.13. [7, Theorem 4.11] *Let K_1 and K_2 be closed convex cones in $\mathcal{H}$ such that $K_1 \cap K_2 = \{0\}$ and K_1 is not linear. Furthermore, suppose that one of the following holds:*

(i) *There exists $u \in \mathcal{H}$ such that $K_2 = \{u\}^{\perp}$.*
(ii) *There exists $u \in \mathcal{H}$ such that $K_2 \subseteq \{u\}^{\perp}$ and $\{u\}^{\perp} \cap K_1 = \{0\}$.*
(iii) *There exists a finite-dimensional linear subspace $\mathcal{K}$ of $\mathcal{H}$ such that $K_1 \subseteq \mathcal{K}$ or $K_2 \subseteq \mathcal{K}$.*

Then

$$K_1^{\ominus} \cap K_2^{\oplus} \neq \{0\} \quad \text{and} \quad K_1^{\oplus} \cap K_2^{\ominus} \neq \{0\}.$$

3. Positive Results

We begin with a simple generalization of [6, Lemma 10.2] from two linear subspaces to one cone and one linear subspace.

Lemma 3.1. *Let K be a nonempty closed convex cone in $\mathcal{H}$ and let M be a closed linear subspace of $\mathcal{H}$. Then*

$$(\forall x \in K)(\forall y \in M) \quad |\langle x,y\rangle| \le c_0(K,M)\|x\|\|y\|.$$

Proof. Let $x \in K$ and $y \in M$. Then $-y \in M$ as well because M is linear. Applying Fact 2.6(ii) with $C = M$ and $D = K$ to obtain that $\pm\langle x,y\rangle = \langle x,\pm y\rangle \le c_0(K,M)\|x\|\|\pm y\| = c_0(K,M)\|x\|\|y\|$ and the result follows. ☐

When one cone is contained in another, then the following pleasing result holds:

Proposition 3.1. *Let K_1 and K_2 be nonempty closed convex cones in $\mathcal{H}$ such that $K_1 \subseteq K_2$. Then*

$$c(K_1,K_2) = 0 = c(K_1^{\ominus},K_2^{\ominus}).$$

Proof. Because $K_1 \subseteq K_2$, we have $K_1 \cap K_2 = K_1$ and $(K_1 \cap K_2)^{\ominus} = K_1^{\ominus}$. Now K_1 is a nonempty closed cone and so $0 \in K_1$. Hence, by Fact 2.3, we have that $K_2 = \{0\} + K_2 \subseteq K_1 + K_2 \subseteq K_2 + K_2 = K_2$, which implies that $K_1 + K_2 = K_2$. Combine this with Fact 2.8 and Fact 2.1(iv) to see that $(K_1^{\ominus} \cap K_2^{\ominus})^{\ominus} = \overline{K_1^{\ominus\ominus} + K_2^{\ominus\ominus}} = \overline{K_1 + K_2} = \overline{K_2} = K_2$. Moreover, using Fact 2.9 and Definition 1.1, we obtain

$$\begin{aligned} c(K_1,K_2) &= c_0\left(K_1 \cap (K_1 \cap K_2)^{\ominus}, K_2 \cap (K_1 \cap K_2)^{\ominus}\right) = c_0\left(K_1 \cap K_1^{\ominus}, K_2 \cap K_1^{\ominus}\right) \\ &= c_0\left(\{0\}, K_2 \cap K_1^{\ominus}\right) = 0 \end{aligned}$$

and

$$c(K_1^{\ominus},K_2^{\ominus}) = c_0\left(K_1^{\ominus} \cap (K_1^{\ominus} \cap K_2^{\ominus})^{\ominus}, K_2^{\ominus} \cap (K_1^{\ominus} \cap K_2^{\ominus})^{\ominus}\right) = c_0\left(K_1^{\ominus} \cap K_2, \{0\}\right) = 0,$$

which completes the proof. □

We are now ready for our first main result — the concial extension of Hundal's Lemma (Fact 1.1):

Theorem 3.1. (concial extension of Hundal's Lemma) *Let K_1 and K_2 be nonempty closed convex cones in $\mathcal{H}$. Suppose that $c_0(K_1,K_2) < 1$ and that X is a convex subset of $\mathcal{H}$ which contains $K_1 - K_2$. Then*

$$c_0(K_1,K_2) \le c_0(K_1^{\oplus} \cap X, K_2^{\ominus} \cap X) \le c_0(K_1^{\oplus}, K_2^{\ominus}).$$

Proof. Fact 2.7(i) yields

$$c_0(K_1^{\oplus} \cap (K_1 - K_2), K_2^{\ominus} \cap (K_1 - K_2)) \le c_0(K_1^{\oplus} \cap X, K_2^{\ominus} \cap X) \le c_0(K_1^{\oplus}, K_2^{\ominus}).$$

Hence, it suffices to prove that $c_0(K_1,K_2) \le c_0(K_1^{\oplus} \cap (K_1 - K_2), K_2^{\ominus} \cap (K_1 - K_2))$.

If $c_0(K_1,K_2) = 0$, then $c_0(K_1,K_2) \le c_0(K_1^{\oplus} \cap (K_1 - K_2), K_2^{\ominus} \cap (K_1 - K_2))$ is trivial.

Now assume that $c_0(K_1,K_2) \in \,]0,1[$. Then, by Fact 2.5, there exist sequences $(x_k)_{k\in\mathbb{N}}$ in $K_1 \cap \mathbf{S}_{\mathcal{H}}$ and $(y_k)_{k\in\mathbb{N}}$ in $K_2 \cap \mathbf{S}_{\mathcal{H}}$ such that $\langle x_k, y_k \rangle \to c_0(K_1,K_2)$.

Clearly, $K_1^{\oplus} = -K_1^{\ominus}$. Moreover, by Fact 2.4, $(\forall k \in \mathbb{N})$ $x_k - \mathrm{P}_{K_2} x_k \in K_2^{\ominus} \cap (K_1 - K_2)$ and $\mathrm{P}_{K_1}\mathrm{P}_{K_2} x_k - \mathrm{P}_{K_2} x_k \in K_1^{\oplus} \cap (K_1 - K_2)$. Set $c_0 := c_0(K_1,K_2)$ and $(\forall k \in \mathbb{N})$ $\alpha_k := \|\mathrm{P}_{K_2} x_k\|$. Hence, using Fact 2.7(i), Fact 2.4, and the same techniques employed in the proof of [6, Lemma 14], we obtain that $\alpha_k \to c_0$ and that for every $k \in \mathbb{N}$,

$$\begin{aligned} c_0(K_1^{\oplus} \cap (K_1 - K_2), K_2^{\ominus} \cap (K_1 - K_2)) &\ge \frac{\left\langle \mathrm{P}_{K_1}\mathrm{P}_{K_2} x_k - \mathrm{P}_{K_2} x_k, x_k - \mathrm{P}_{K_2} x_k \right\rangle}{\|\mathrm{P}_{K_1}\mathrm{P}_{K_2} x_k - \mathrm{P}_{K_2} x_k\| \|x_k - \mathrm{P}_{K_2} x_k\|} \\ &\ge \alpha_k - \sqrt{\frac{c_0^2 - \alpha_k^2}{1 - c_0^2}} \to c_0, \end{aligned}$$

which completes the proof. □

Remark 3.1. If K_1 and K_2 are linear in Theorem 3.1, then $K_1 - K_2 = K_1 + K_2$, $K_1^{\oplus} = K_1^{\perp}$, $K_2^{\ominus} = K_2^{\perp}$, and we recover Fact 1.1.

We conclude this section with further comments on Theorem 3.1 which are based on the following example.

Example 3.1. [7, Example 4.13] Suppose that $\mathcal{H} = \mathbb{R}^2$. Set $K_1 := \mathbb{R}^2_+$ and $K_2 := \{(x_1,x_2) \in \mathbb{R}^2 \mid -x_1 \geq x_2\}$. Then the following hold:

(i) $K_1^{\ominus} = \mathbb{R}^2_-$, $K_2^{\ominus} = \mathbb{R}_+(1,1)$, and $K_2^{\oplus} = \mathbb{R}_+(-1,-1)$.

(ii) $K_1 \cap K_2 = \{0\}$, $K_1^{\ominus} \cap K_2^{\ominus} = \{0\}$, $K_1^{\ominus} \cap K_2^{\oplus} = K_2^{\oplus}$, $K_1 + K_2 = \mathbb{R}^2$, and $K_1 - K_2 = -K_2 \neq \mathcal{H}$.

(iii) $c(K_1,K_2) = c_0(K_1,K_2) = \frac{1}{\sqrt{2}} > 0 = c_0(K_1^{\ominus},K_2^{\ominus}) = c_0(K_1^{\oplus},K_2^{\oplus}) = c(K_1^{\ominus},K_2^{\ominus}) = c(K_1^{\oplus},K_2^{\oplus})$.

(iv) $c_0(K_1^{\ominus},K_2^{\oplus}) = 1$, $c(K_1^{\ominus},K_2^{\oplus}) = 0$, $c_0(K_1,K_2) < c_0(K_1^{\ominus},K_2^{\oplus})$, and $c(K_1,K_2) > c(K_1^{\ominus},K_2^{\oplus})$.

Remark 3.2. (importance of using dual-polar pair) Let K_1 and K_2 be as in Example 3.1. Then $c_0(K_1,K_2) < 1$ yet

$$c_0(K_1,K_2) = \frac{1}{\sqrt{2}} > 0 = c_0(K_1^{\ominus},K_2^{\ominus}) = c_0(K_1^{\oplus},K_1^{\oplus}).$$

This illustrates that in the conical extension of Hundal's Lemma (Theorem 3.1), we indeed must work with a pair of dual-polar cones of the original pair — substituting with a polar-polar or dual-dual pair will not work!

4. NEGATIVE RESULTS

We start with characterizations for differences of convex cones to be linear subspaces.

Lemma 4.1. *Let K_1 and K_2 be nonempty closed convex cones in $\mathcal{H}$ such that $K_1 \cap K_2 = \{0\}$. Then the following are equivalent:*

(i) *$K_1 - K_2$ is a linear subspace of $\mathcal{H}$.*

(ii) *$(-K_1) \cup K_2 \subseteq K_1 - K_2$.*

(iii) *K_1 and K_2 are linear subspaces of $\mathcal{H}$.*

Proof. "(i)⇒(ii)": We have $K_1 = K_1 - \{0\} \subseteq K_1 - K_2$ and $-K_2 = \{0\} - K_2 \subseteq K_1 - K_2$. Hence $(-K_1) \cup K_2 = -(K_1 \cup (-K_2)) \subseteq -(K_1 - K_2) = K_1 - K_2$.

"(ii)⇒(iii)": Let $x \in -K_1$. Then there exist $y_1 \in K_1$ and $y_2 \in K_2$ such that $x = y_1 - y_2$. Then $y_2 = y_1 - x \in K_1 - (-K_1) = K_1 + K_1 = K_1$. Hence $y_2 \in K_1 \cap K_2 = \{0\}$. Thus $y_2 = 0$ and so $x = y_1 \in K_1$. This implies $-K_1 \subseteq K_1$ and therefore K_1 is linear by Fact 2.3. The argument for K_2 is similar.

"(iii)⇒(i)": Obvious. □

Here is a conical variant of Fact 1.2.

Theorem 4.1. *Let K_1 and K_2 be nonempty closed convex cones in $\mathcal{H}$ such that $K_1 \cap K_2 = \{0\}$ and $K_1 - K_2 = \mathcal{H}$. Then*

(i) *K_1 and K_2 are linear subspaces of $\mathcal{H}$.*

(ii) $c_0(K_1,K_2) = c_0(K_1^{\oplus},K_2^{\ominus}) = c_0(K_1^{\ominus},K_2^{\oplus}) = c_0(K_1^{\perp},K_2^{\perp})$.

Proof. (i): Clear from Lemma 4.1. (ii): In view of (i), we have $-K_2 = K_2$ and thus $K_1 + K_2 = K_1 - K_2 = \mathcal{H}$. Now apply Fact 1.2. □

Remark 4.1. (impossibility of extending Fact 1.2 to cones) The assumption on the two subspaces in Fact 1.2 is that their intersection is $\{0\}$ and their sum is the entire space. Now assume that K_2 is a linear subspace and $K_1 + K_2 = K_1 - K_2$. Then Lemma 4.1 implies that K_1 is a subspace as well. Hence Fact 1.2 does not appear to admit an extension to the case when one linear subspace is replaced by a convex cone!

Remark 4.2. (importance of the assumption that $K_1 - K_2 = \mathscr{H}$) Let K_1 and K_2 be as in Example 3.1. Then $K_1 \cap K_2 = \{0\}$ but neither K_1 nor K_2 is a linear subspace. This shows that the assumption that $\mathscr{H} = K_1 - K_2$ in Theorem 4.1 is critical.

Remark 4.3. (importance of the assumption that $K_1 \cap K_2 = \{0\}$) Let u_1, u_2 be unit vectors in $\mathbb{R}^3$ such that $0 < \langle u_1, u_2 \rangle < 1$. Set $K_1 := \{u_1\}^\perp$ and $K_2 = \{u_2\}^\perp$. Then $K_1^\perp = \mathbb{R}u_1$, $K_2^\perp = \mathbb{R}u_2$, $K_1 \cap K_2 \neq \{0\}$ and $K_1 - K_2 = \mathbb{R}^3$, but $c_0(K_1^\perp, K_2^\perp) = \langle u_1, u_2 \rangle < 1 = c_0(K_1, K_2)$.

We now turn to a conical variant of Fact 1.3. Our proof is an adaptation of Deutsch's proof of [6, Theorem 16].

Theorem 4.2. *Let K_1 and K_2 be nonempty closed convex cones in $\mathscr{H}$. Suppose that $c(K_1, K_2) < 1$, that $K_1 \cap K_2$ and $K_1^\oplus \cap K_2^\ominus$ are linear subspaces of $\mathscr{H}$, that $\overline{K_1^\oplus + (K_1 \cap K_2)} \cap (K_1 \cap K_2)^\perp = K_1^\oplus$, and that $\overline{K_2^\ominus + (K_1 \cap K_2)} \cap (K_1 \cap K_2)^\perp = K_2^\ominus$. Then*

$$c(K_1, K_2) \leq c(K_1^\oplus, K_2^\ominus).$$

If additionally $c(K_1^\oplus, K_2^\ominus) < 1$,

$$\overline{K_1 + (K_1^\oplus \cap K_2^\ominus)} \cap (K_1^\oplus \cap K_2^\ominus)^\perp = K_1, \quad \textit{and} \quad \overline{K_2 + (K_1^\oplus \cap K_2^\ominus)} \cap (K_1^\oplus \cap K_2^\ominus)^\perp = K_2,$$

then

$$c(K_1, K_2) = c(K_1^\oplus, K_2^\ominus).$$

Proof. Fact 2.9 and Fact 2.1(i) imply that

$$c_0\big(K_1 \cap (K_1 \cap K_2)^\perp, K_2 \cap (K_1 \cap K_2)^\perp\big) = c(K_1, K_2) < 1. \tag{4.1}$$

Because $K_1^\oplus \cap K_2^\ominus$ is a linear subspace of $\mathscr{H}$, Fact 2.8, Fact 2.2(iii) and Fact 2.1(i)&(v) imply that $(K_1^\oplus \cap K_2^\ominus)^\perp = \overline{K_1 - K_2}$ is a linear subspace of $\mathscr{H}$. Hence

$$X := (K_1 \cap K_2)^\perp \cap (K_1^\oplus \cap K_2^\ominus)^\perp = (K_1 \cap K_2)^\perp \cap \overline{K_1 - K_2} \tag{4.2}$$

is a closed linear subspace of $\mathscr{H}$. Because $\overline{K_1 - K_2}$ is a closed linear subspace of $\mathscr{H}$, we learn that $K_2 \subseteq \overline{K_1 - K_2}$ and that $K_1 \subseteq \overline{K_1 - K_2}$. Hence

$$K_1 \cap X = K_1 \cap (K_1 \cap K_2)^\perp \quad \text{and} \quad K_2 \cap X = K_2 \cap (K_1 \cap K_2)^\perp. \tag{4.3}$$

Moreover, $(K_1 \cap K_2)^\perp$ is a linear subspace and so $(K_1 \cap K_2)^\perp - (K_1 \cap K_2)^\perp = (K_1 \cap K_2)^\perp$. Combine this with (4.2) and (4.3) to see that

$$\big(K_1 \cap (K_1 \cap K_2)^\perp\big) - \big(K_2 \cap (K_1 \cap K_2)^\perp\big) = \big(K_1 \cap X\big) - \big(K_2 \cap X\big) \subseteq X. \tag{4.4}$$

Using Fact 2.8, Fact 2.2(i)&(ii) and the assumption that $\overline{K_1^\oplus + K_1 \cap K_2} \cap (K_1 \cap K_2)^\perp = K_1^\oplus$, we obtain

$$(K_1 \cap (K_1 \cap K_2)^\perp)^\oplus \cap X = \overline{K_1^\oplus + K_1 \cap K_2} \cap (K_1 \cap K_2)^\perp \cap (K_1^\oplus \cap K_2^\ominus)^\perp \tag{4.5a}$$

$$= K_1^\oplus \cap (K_1^\oplus \cap K_2^\ominus)^\perp. \tag{4.5b}$$

Similarly, using Fact 2.8, Fact 2.2(i), Fact 2.1(iv) and the assumption that $\overline{K_2^{\ominus}+K_1\cap K_2}\cap(K_1\cap K_2)^{\perp}=K_2^{\ominus}$, we see that

$$(K_2\cap(K_1\cap K_2)^{\perp})^{\ominus}\cap X=\overline{K_2^{\ominus}+K_1\cap K_2}\cap(K_1\cap K_2)^{\perp}\cap(K_1^{\oplus}\cap K_2^{\ominus})^{\perp} \tag{4.6a}$$

$$=K_2^{\ominus}\cap(K_1^{\oplus}\cap K_2^{\ominus})^{\perp}. \tag{4.6b}$$

We now use (4.1), (4.4), and Theorem 3.1 (with $K_1=K_1\cap X$, $K_2=K_2\cap X$, and X is as (4.2)) to deduce that $c_0(K_1\cap X,K_2\cap X)\le c_0((K_1\cap X)^{\oplus}\cap X,(K_2\cap X)^{\ominus}\cap X)$, which, recalling (4.3), (4.5), and (4.6), is equivalent to

$$c_0(K_1\cap(K_1\cap K_2)^{\perp},K_2\cap(K_1\cap K_2)^{\perp})\le c_0\left(K_1^{\oplus}\cap(K_1^{\oplus}\cap K_2^{\ominus})^{\perp},K_2^{\ominus}\cap(K_1^{\oplus}\cap K_2^{\ominus})^{\perp}\right).$$

This is — using Fact 2.9 and Fact 2.1(i) — the same as $c(K_1,K_2)\le c(K_1^{\oplus},K_2^{\ominus})$.

Finally, with the additional assumptions, apply the result just proved above to $K_1=K_1^{\oplus}$ and $K_2=K_2^{\ominus}$ to conclude that $c(K_1^{\oplus},K_2^{\ominus})\le c(K_1^{\oplus\oplus},K_2^{\ominus\ominus})=c(K_1,K_2)$, where the last equation follows from Fact 2.1(iv) and Fact 2.2(ii). Altogether, $c(K_1,K_2)=c(K_1^{\oplus},K_2^{\ominus})$ in this case. □

To derive some consequences of Theorem 4.2, we require the following result.

Lemma 4.2. *Let A and B be nonempty subsets of $\mathscr{H}$ such that $0\in B$ and $A\subseteq B^{\perp}$. Then*

$$(A+B)\cap B^{\ominus}=A.$$

Consequently, if B is a linear subspace of $\mathscr{H}$, then $(A+B)\cap B^{\perp}=A$.

Proof. Because $0\in B$ and $A\subseteq B^{\ominus}$, we have that $A\subseteq(A+B)\cap B^{\ominus}$. Conversely, let $x\in(A+B)\cap B^{\ominus}$. Then there exist $a\in A$ and $b\in B$ such that $x=a+b\in B^{\ominus}$. Because $x\in B^{\ominus}$, $b\in B$, and $a\in A\subseteq B^{\perp}=B^{\ominus}\cap B^{\oplus}$, we know that $\langle x,b\rangle\le 0$, and that $\langle a,b\rangle=0$. Hence $0\ge\langle x,b\rangle=\langle a+b,b\rangle=\langle a,b\rangle+\|b\|^2=\|b\|^2\ge 0$ so $\|b\|^2=0$, i.e., $b=0$. Thus $x=a\in A$ and so $(A+B)\cap B^{\ominus}\subseteq A$.

The "Consequently" part follows from what we just proved and Fact 2.2(iii) which states that $B^{\perp}=B^{\ominus}=B^{\oplus}$ provided that B is linear. □

We first reprove Solmon's results (Fact 1.3) from Theorem 4.2.

Corollary 4.1. (Solmon) *Let K_1 and K_2 be closed linear subspaces of $\mathscr{H}$. Then $c(K_1,K_2)=c(K_1^{\perp},K_2^{\perp})$.*

Proof. Because K_1 and K_2 are closed linear subspaces, by [6, Theorem 13],

$$c(K_1,K_2)=1\Leftrightarrow K_1+K_2\text{ is not closed}\Leftrightarrow K_1^{\perp}+K_2^{\perp}\text{ is not closed}\Leftrightarrow c(K_1^{\perp},K_2^{\perp})=1. \tag{4.7}$$

Assume that $c(K_1,K_2)<1$. By (4.7), $c(K_1^{\perp},K_2^{\perp})<1$. Note that $K_1^{\perp}=K_1^{\ominus}=K_1^{\oplus}$, and $K_2^{\perp}=K_2^{\ominus}=K_2^{\oplus}$. Moreover, because $K_1^{\perp}\perp(K_1\cap K_2)$, then by [8, Proposition 29.6], $\overline{K_1^{\perp}+(K_1\cap K_2)}=K_1^{\perp}+K_1\cap K_2$. Hence, by Lemma 4.2, $\overline{K_1^{\oplus}+K_1\cap K_2}\cap(K_1\cap K_2)^{\ominus}=(K_1^{\perp}+K_1\cap K_2)\cap(K_1\cap K_2)^{\ominus}=K_1^{\perp}=K_1^{\oplus}$. Similarly, $\overline{K_2^{\ominus}-K_1\cap K_2}\cap(K_1\cap K_2)^{\ominus}=K_2^{\ominus}$. Replacing (K_1,K_2) with $(K_1^{\perp},K_2^{\perp})$ in the above, we obtain $\overline{K_1+(K_1^{\oplus}\cap K_2^{\ominus})}\cap(K_1^{\oplus}\cap K_2^{\ominus})^{\perp}=K_1$ and $\overline{K_2-(K_1^{\oplus}\cap K_2^{\ominus})}\cap(K_1^{\oplus}\cap K_2^{\ominus})^{\perp}=K_2$. Hence, by Theorem 4.2, $c(K_1,K_2)=c(K_1^{\oplus},K_2^{\ominus})=c(K_1^{\perp},K_2^{\perp})$. □

Next we give some special cases of Theorem 4.2.

Corollary 4.2. *Let K_1 and K_2 be nonempty closed convex cones in $\mathcal{H}$ such that $c(K_1,K_2) < 1$, $K_1 \cap K_2$ and $K_1^{\oplus} \cap K_2^{\ominus}$ are linear subspaces of $\mathcal{H}$, and $K_1^{\oplus} + (K_1 \cap K_2)$ and $K_2^{\ominus} + (K_1 \cap K_2)$ are closed. Then*

$$c(K_1,K_2) \leq c(K_1^{\oplus},K_2^{\ominus}).$$

If $c(K_1^{\oplus},K_2^{\ominus}) < 1$, and $K_1 + (K_1^{\oplus} \cap K_2^{\ominus})$ and $K_2 + (K_1^{\oplus} \cap K_2^{\ominus})$ are closed, then

$$c(K_1,K_2) = c(K_1^{\oplus},K_2^{\ominus}).$$

Proof. Because $K_1 \cap K_2$ and $K_1^{\oplus} \cap K_2^{\ominus}$ are linear subspaces of $\mathcal{H}$, by Fact 2.1(v), we have that $K_1^{\oplus} \subseteq (K_1 \cap K_2)^{\perp}$, $K_2^{\ominus} \subseteq (K_1 \cap K_2)^{\perp}$, $K_1 \subseteq (K_1^{\oplus} \cap K_2^{\ominus})^{\perp}$ and $K_2 \subseteq (K_1^{\oplus} \cap K_2^{\ominus})^{\perp}$.

Hence, applying Lemma 4.2 with $A = K_1^{\oplus}$ and $B = K_1 \cap K_2$, with $A = K_2^{\ominus}$ and $B = K_1 \cap K_2$, with $A = K_1$ and $B = K_1^{\oplus} \cap K_2^{\ominus}$, and with $A = K_2$ and $B = K_1^{\oplus} \cap K_2^{\ominus}$, we obtain that $(K_1^{\oplus} + (K_1 \cap K_2)) \cap (K_1 \cap K_2)^{\perp} = K_1^{\oplus}$, $(K_2^{\ominus} + (K_1 \cap K_2)) \cap (K_1 \cap K_2)^{\perp} = K_2^{\ominus}$, $(K_1 + (K_1^{\oplus} \cap K_2^{\ominus})) \cap (K_1^{\oplus} \cap K_2^{\ominus})^{\perp} = K_1$, and $(K_2 + (K_1^{\oplus} \cap K_2^{\ominus})) \cap (K_1^{\oplus} \cap K_2^{\ominus})^{\perp} = K_2$, respectively. Therefore, the required results follow from the closedness assumptions and Theorem 4.2. □

Corollary 4.3. *Let K_1 and K_2 be nonempty closed convex cones in $\mathcal{H}$. Suppose that K_1 or K_2 is contained in a finite-dimensional linear subspace of $\mathcal{H}$, and that $K_1^{\oplus}$ or $K_2^{\ominus}$ is contained in a finite-dimensional linear subspace of $\mathcal{H}$. Suppose furthermore that $K_1 \cap K_2 = \{0\}$ and $K_1^{\oplus} \cap K_2^{\ominus} = \{0\}$. Then*

$$c_0(K_1,K_2) = c(K_1,K_2) = c(K_1^{\oplus},K_2^{\ominus}) = c_0(K_1^{\oplus},K_2^{\ominus}).$$

Proof. Because $K_1 \cap K_2 = \{0\}$, $K_1^{\oplus} \cap K_2^{\ominus} = \{0\}$, using the assumptions that K_1 or K_2 is contained in a finite-dimensional linear subspace of $\mathcal{H}$, and that $K_1^{\oplus}$ or $K_2^{\ominus}$ is contained in a finite-dimensional linear subspace of $\mathcal{H}$, by Fact 2.11, we have that

$$c(K_1,K_2) = c_0(K_1,K_2) < 1 \quad \text{and} \quad c(K_1^{\oplus},K_2^{\ominus}) = c_0(K_1^{\oplus},K_2^{\ominus}) < 1.$$

Hence, the desired identities are directly from Corollary 4.2. □

Here is a simple illustration of Corollary 4.3.

Example 4.1. Suppose that $\mathcal{H} = \mathbb{R}^2$, and set $K_1 := \mathbb{R}(1,0)$ and $K_2 := \mathbb{R}(1,1)$. Then

$$c_0(K_1,K_2) = c(K_1,K_2) = c(K_1^{\oplus},K_2^{\ominus}) = c_0(K_1^{\oplus},K_2^{\ominus}) = \tfrac{1}{\sqrt{2}}.$$

The next example follows easily from the definitions and Fact 2.7(ii). We will then comment on its relevance to previous results.

Example 4.2. Suppose that $\mathcal{H} = \mathbb{R}^2$, and set $K := \{(x_1,x_2) \in \mathbb{R}^2 \mid x_2 \geq x_1 \geq 0\}$ and $M := \mathbb{R}(1,0)$. Then $K \cap M = \{0\}$, $K^{\ominus} = \{(x_1,x_2) \in \mathbb{R}^2 \mid -x_1 \geq x_2 \text{ and } x_2 \leq 0\}$, $M^{\perp} = \mathbb{R}(0,1)$, $K^{\ominus} \cap M^{\perp} = \mathbb{R}_+(0,-1)$, $(K^{\ominus} \cap M^{\perp})^{\ominus} = \mathbb{R} \times \mathbb{R}_+$, and

$$\begin{aligned} 0 = c(K^{\oplus},M^{\perp}) &= c(K^{\ominus},M^{\perp}) \\ &< \tfrac{1}{\sqrt{2}} = c_0(K,M) = c(K,M) \\ &< 1 = c_0(K^{\oplus},M^{\perp}) = c_0(K^{\ominus},M^{\perp}). \end{aligned}$$

Remark 4.4. (impossiblility of a simple extension of Solmon's Fact) Example 4.2 very clearly shows the impossibility of a nice and simple generalization of Fact 1.3 from two subspaces to even just a subspace and a cone: Indeed, in Example 4.2, we have

$$c(K^{\ominus}, M^{\perp}) = c(K^{\oplus}, M^{\perp}) < c(K, M);$$

thus, neither the polar cone $K^{\ominus}$ nor the dual cone $K^{\oplus}$ will do the job!

Remark 4.5. (importance of linearity of $K_1^{\oplus} \cap K_2^{\ominus}$) Let us revisit Theorem 4.2 with K_1 and K_2 replaced by the M and K from Example 4.2, respectively. Then $c(K_1, K_2) = \frac{1}{\sqrt{2}} < 1$ and $K_1 \cap K_2 = \{0\}$. The latter clearly implies $\overline{K_1^{\oplus} + (K_1 \cap K_2)} \cap (K_1 \cap K_2)^{\perp} = K_1^{\oplus}$ and $\overline{K_2^{\ominus} + (K_1 \cap K_2)} \cap (K_1 \cap K_2)^{\perp} = K_2^{\ominus}$. However, $K_1^{\oplus} \cap K_2^{\ominus} = \mathbb{R}_+(0, -1)$ and so

$$K_1^{\oplus} \cap K_2^{\ominus} \text{ is } \textit{not} \text{ a linear subspace, and } c(K_1, K_2) > c(K_1^{\oplus}, K_2^{\ominus}).$$

This shows that the assumption that $K_1^{\oplus} \cap K_2^{\ominus}$ be a linear subspace is critical in Theorem 4.2. The very same example can be used to show the importance of the assumption that $K_1^{\oplus} \cap K_2^{\ominus}$ be a linear subspace in Corollary 4.2.

Remark 4.6. (assumptions in Theorem 4.2 are quite restrictive) Now suppose that $\mathscr{H} = \mathbb{R}^2$. Let us assume that K_1 and K_2 are two nonempty closed convex cones in $\mathbb{R}^2$ such that K_1 or K_2 is not a linear subspace. Furthermore, assume that $K_1 \cap K_2$ is a linear subspace. By Fact 2.13(iii), the intersections $K_1 \cap K_2$ and $K_1^{\oplus} \cap K_2^{\ominus}$ cannot be both equal to $\{(0,0)\} \subseteq \mathbb{R}^2$. Because $K_1^{\oplus\oplus} \cap K_2^{\ominus\ominus} = K_1 \cap K_2$, we assume without loss of generality that $K_1 \cap K_2 = \{0\} \times \mathbb{R}$. Without loss of generality, we can only have one of the following two cases: *Case 1*: $K_1 = \mathbb{R}_+ \times \mathbb{R}$ and $K_2 = \mathbb{R}_- \times \mathbb{R}$; *Case 2*: $K_1 = \{0\} \times \mathbb{R}$ and and $K_2 = \mathbb{R}_- \times \mathbb{R}$. But in either case, we have $K_1^{\oplus} \cap K_2^{\ominus} = \mathbb{R}_+ \times \{0\}$ which is *not* a linear subspace.

Therefore, in the Euclidean plane $\mathbb{R}^2$, there do not exist two nonlinear cones satisfying the assumptions in Theorem 4.2!

Remark 4.6 and Lemma 4.1 now prompt the following natural question with which we conclude this paper:

Question 4.1. Do there exist nonempty closed convex cones K_1 and K_2 that are *nonlinear* yet satisfy the assumptions in Theorem 4.2?

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