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SUPERIORIZATION WITH A PROJECTED SUBGRADIENT METHOD

ALEXANDER J. ZASLAVSKI

Department of Mathematics, The Technion – Israel Institute of Technology, 32000 Haifa, Israel

Abstract. In this paper, we study a constrained minimization problem with a convex objective function and a feasible region, which is the intersection of finitely many closed convex constraint sets. We use a projected subgradient method combined with a dynamic string-averaging projection method with variable strings and variable weights, as a feasibility-seeking algorithm. It is shown that any sequence, generated by the superiorized version of a dynamic string-averaging projection algorithm, not only converges to a feasible point but, additionally, also either its limit point solves the constrained minimization problem or the sequence is strictly Fejér monotone with respect to the solution set.

Keywords. Constrained minimization; Convex feasibility problem; Dynamic string-averaging projections; Subgradient.

1. Introduction

The theory of nonexpansive mappings, which are 1-Lipschitz continuous, finds various applications in pure and applied mathematics. There are numerous results on the fixed points of nonexpansive mappings; see, for example, [1]-[14] and the references cited therein. These results mainly stem from Banach's classical theorem [15] concerning the existence of a unique fixed point for a strict contraction. It also covers the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this field including, in particular, the studies of feasibility and common fixed point problems, which find important applications in engineering and medical sciences [13, 14, 16, 17, 18, 19, 20].

Let (X,ρ) be a complete metric space. In [2], it was studied the influence of errors on the convergence of orbits of nonexpansive mappings in metric spaces, and it was obtained the following result (see also [12, Theorem 2.72]).

Theorem 1.1. *Let $A : X \to X$ be a mapping satisfying*

$$\rho(Ax,Ay) \le \rho(x,y), \quad \forall x,y \in X. \tag{1.1}$$

Let $F(A)$ be the set of all fixed points of A, and let, for each $x \in X$, the sequence $\{A^n x\}_{n=1}^{\infty}$ converges in (X,ρ). Assume that $\{x_n\}_{n=0}^{\infty} \subset X$, $\{r_n\}_{n=0}^{\infty} \subset (0,\infty)$ satisfies $\sum_{n=0}^{\infty} r_n < \infty$ and $\rho(x_{n+1},Ax_n) \le r_n$, $n = 0,1,\dots$. Then $\{x_n\}_{n=1}^{\infty}$ converges to a fixed point of A in (X,ρ).

E-mail address: ajzasl@technion.ac.il.
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Theorem 1.1 found interesting applications and is an important ingredient in superiorization and perturbation resilience of algorithms; see, e.g., [21]-[28] and the references mentioned therein. The superiorization methodology works by taking an iterative algorithm, investigating its perturbation resilience, and then using proactively such perturbations in order to ”force” the perturbed algorithm to do in addition to its original task something useful. This methodology can be explained by the following result on convergence of inexact iterates.

Assume that $(X, \|\cdot\|)$ is a Banach space, $\rho(x,y) = \|x-y\|$ for all $x, y \in X$, a mapping $A : X \to X$ satisfies (1.1), and, for each $x \in X$, the sequence $\{A^n x\}_{n=1}^{\infty}$ converges in the norm topology. Let $x_0 \in X$. Let $\{\beta_k\}_{k=0}^{\infty}$ be a sequence of positive numbers satisfying $\sum_{k=0}^{\infty} \beta_k < \infty$, $\{v_k\}_{k=0}^{\infty} \subset X$ be a norm bounded sequence, and, for any integer $k \geq 0$,

$$x_{k+1} = A(x_k + \beta_k v_k). \tag{1.2}$$

Then it follows from Theorem 1.1 that $\{x_k\}_{k=0}^{\infty}$ converges in the norm topology of X and its limit is a fixed point of A. In this case, mapping A is called bounded perturbations resilient (see [23] and [25, Definition 10]). In other words, if the exact iterates of a nonexpansive mapping converge, then its inexact iterates with bounded summable perturbations converge too.

Now we assume that $x_0 \in X$ and the sequence $\{\beta_k\}_{k=0}^{\infty}$ satisfy $\sum_{k=0}^{\infty} \beta_k < \infty$. We need to find an approximate fixed point of A. In order to meet this goal, we construct a sequence $\{x_k\}_{k=1}^{\infty}$ defined by (1.2). Under an appropriate choice of the bounded sequence $\{v_k\}_{k=0}^{\infty}$, the sequence $\{x_k\}_{k=1}^{\infty}$ possesses some useful property. For example, the sequence $\{f(x_k)\}_{k=1}^{\infty}$ can be decreasing, where f is a given function. This superiorization methodology was used in [26] in order to study a constrained minimization problem with a convex objective function and a feasible region, which is the intersection of finitely many closed convex constraint sets. In [26], it was used a projected normalized subgradient method combined with a dynamic string-averaging projection method with variable strings and variable weights, as a feasibility-seeking algorithm, which was introduced in [29]. It is shown that any sequence, generated by the superiorized version of a dynamic string-averaging projection algorithm, not only converges to a feasible point but, additionally, also either its limit point solves the constrained minimization problem or the sequence is strictly Fejér monotone with respect to a subset of the solution set. It should be mentioned that, in [26], it was used a projected normalized subgradient method. It means that for any iteration a subgradient should be normalized. In the present paper, our goal is to show that the main result of [26] is also true without normalization of subgradients if the objective function is Lipschitz. It is obvious that this makes our computations much easier.

2. The Dynamic String-Averaging Projection Method

Let $(X, \langle \cdot, \cdot \rangle)$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, which induces a complete norm $\|\cdot\|$. For each $x \in X$ and each nonempty set $E \subset X$, put $d(x,E) = \inf\{\|x-y\| : \ y \in E\}$. For each $x \in X$ and each $r > 0$, set $B(x,r) = \{y \in X : \ \|x-y\| \leq r\}$.

It is well-known that the following proposition holds [25].

Proposition 2.1. *Let D be a nonempty closed convex subset of X. Then, for each $x \in X$, there is a unique point $P_D(x) \in D$ satisfying $\|x - P_D(x)\| = \inf\{\|x-y\| : \ y \in D\}$. Moreover, $\|P_D(x) - P_D(y)\| \leq \|x-y\|$ for all $x, y \in X$, and, for each $x \in X$ and each $z \in D$, $\langle z - P_D(x), x - P_D(x) \rangle \leq 0$.*

Corollary 2.1. *Assume that D is a nonempty convex closed subset of X. Then, for each $x \in X$ and each $z \in D$, $\|z - P_D(x)\|^2 + \|x - P_D(x)\|^2 \leq \|z - x\|^2$.*

Suppose that $C_1,\dots,C_m$ are nonempty closed convex subsets of X, where m is a natural number. Set $C=\cap_{i=1}^m C_i$. We suppose that $C\neq\emptyset$. For $i=1,\dots,m$ set $P_i=P_{C_i}$. By an index vector, we a mean a vector $t=(t_1,\dots,t_q)$ such that $t_i\in\{1,\dots,m\}$ for all $i=1,\dots,q$. For an index vector $t=(t_1,\dots,t_q)$ set $p(t)=q$ and $P[t]=P_{t_q}\cdots P_{t_1}$. A finite set Ω of index vectors is called fit if, for each $i\in\{1,\dots,m\}$, there exists $t=(t_1,\dots,t_q)\in\Omega$ such that $t_s=i$ for some $s\in\{1,\dots,q\}$. It is easy to see that, for each index vector t, $\|P[t](x)-P[t](y)\|\le\|x-y\|$ for all $x,y\in X$, and $P[t](x)=x$ for all $x\in C$. Denote by $\mathscr{M}$ the collection of all pairs (Ω,w), where Ω is a fit finite set of index vectors and $w:\Omega\to(0,\infty)$ is such that $\sum_{t\in\Omega}w(t)=1$.

Let $(\Omega,w)\in\mathscr{M}$. Define

$$P_{\Omega,w}(x)=\sum_{t\in\Omega}w(t)P[t](x),\ x\in X.$$

It is easy to see that

$$\|P_{\Omega,w}(x)-P_{\Omega,w}(y)\|\le\|x-y\|\text{ for all }x,y\in X,$$

and

$$P_{\Omega,w}(x)=x,\quad\forall x\in C.$$

We use the following assumption.

(A) For each $\varepsilon>0$ and each $M>0$, there exists $\delta=\delta(\varepsilon,M)>0$ such that, for each $x\in B(0,M)$ satisfying $d(x,C_i)\le\delta$, $i=1,\dots,m$, the inequality $d(x,C)\le\varepsilon$ holds.

Note that if the space X is finite-dimensional, then assumption (A) always holds [25]. It means that if a point is closed to any set C_i, $i=1,\dots,m$, then it is close to their intersection C.

In the sequel, we assume that Assumption (A) holds. Fix a number $\Delta\in(0,m^{-1})$ and an integer $\bar{q}\ge m$. Denote by $\mathscr{M}_*$ the set of all $(\Omega,w)\in\mathscr{M}$ such that $p(t)\le\bar{q}$ for all $t\in\Omega$ and $w(t)\ge\Delta$ for all $t\in\Omega$.

The following result was obtained in [25].

Theorem 2.1. *Let $\{\beta_k\}_{k=0}^\infty$ be a sequence of nonnegative numbers such that $\sum_{k=0}^\infty\beta_k<\infty$. Let $\{v_k\}_{k=0}^\infty\subset X$ be a norm bounded sequence, $\{(\Omega_i,w_i)\}_{i=1}^\infty\subset\mathscr{M}_*$, $x_0\in X$, and, for any integer $k\ge 0$, $x_{k+1}=P_{\Omega_{k+1},w_{k+1}}(x_k+\beta_k v_k)$. Then the sequence $\{x_k\}_{k=0}^\infty$ converges in the norm topology of X and its limit belongs to C.*

In the proof of this result assumption (A) was used.

3. The Main Result

In this paper, we study a constrained minimization problem with a convex objective function and with a feasible region, which is the intersection of finitely many closed convex constraint sets. We use a projected subgradient method combined with a dynamic string-averaging projection method, with variable strings and variable weights, as a feasibility-seeking algorithm. It is shown that any sequence, generated by the superiorized version of a dynamic string-averaging projection algorithm, not only converges to a feasible point but, additionally, also either its limit point solves the constrained minimization problem or the sequence is strictly Fejér monotone with respect to the solution set. It should be mentioned that the subgradient method was introduced by Shor in the early 60s [30, 31]. The projected subgradient method and its convergence properties were introduced and studied by Polyak [32] (see also [33]).

Assume that $U \subset X$ is an open convex set, $C_i \subset U$, $\forall i = 1,\dots,m$, $L \geq 1$, and that $f : U \to R^1$ is a convex function such that

$$|f(x) - f(y)| \leq L\|x-y\|, \quad \forall x,y \in U. \tag{3.1}$$

For each $x \in U$,

$$\partial f(x) = \{l \in X : \langle l, y-x\rangle \leq f(y) - f(x) \text{ for all } y \in U\}$$

is the subdifferential of the function f at the point x. Set

$$C_{\min} = \{x \in C : f(x) \leq f(y) \text{ for all } y \in C\}.$$

We suppose that $C_{min} \neq \emptyset$. Let us now describe our algorithm.

Suppose that

$$\{(\Omega_i, w_i)\}_{i=1}^{\infty} \subset \mathscr{M}_*,$$

N is a natural number,

$$N_i \in \{1,\dots,N\} \text{ for all integers } i \geq 0,$$

$$\alpha_{i,j} \in (0,1] \text{ for all integers } i \geq 0 \text{ and all } j \in \{1,\dots,N_i\}$$

and that

$$\sum_{i=0}^{\infty}\sum_{j=1}^{N_i} \alpha_{i,j} < \infty. \tag{3.2}$$

Let $x_0 \in U$. We define sequences $\{x_i\}_{i=0}^{\infty} \subset U$,

$$\{l_{i,j} : i = 0,1,\dots, j \in \{1,\dots,N_i\}\} \subset X$$

and

$$\{x_{i,j} : i = 0,1,\dots, j \in \{1,\dots,N_i\}\} \subset U$$

as follows: for each integer $i \geq 1$,

$$x_{i-1,0} = x_{i-1}, \tag{3.3}$$

for each integer $j \in \{1,\dots,N_i\}$,

$$l_{i-1,j} \in \partial f(x_{i-1,j-1}), \tag{3.4}$$

$$x_{i-1,j} = x_{i-1,j-1} - \alpha_{i-1,j} l_{i-1,j}, \tag{3.5}$$

and

$$x_i = P_{\Omega_i, w_i}(x_{i-1,N_i}). \tag{3.6}$$

In this paper, we prove the following result.

Theorem 3.1. *The sequence $\{x_k\}_{k=0}^{\infty}$ converges in the norm topology of X to $x_* \in C$ and one of the following cases holds:*

(a) $x_* \in C_{min}$;

(b) $x_* \notin C_{min}$ and there exist a natural number k_0 and $c_0 \in (0,1)$ such that for each $x \in C_{min}$ and each integer $k \geq k_0$,

$$\|x_k - x\|^2 \leq \|x_{k-1} - x\|^2 - c_0 \sum_{p=1}^{N_k} \alpha_{k,p}.$$

4. Auxiliary Results

Lemma 4.1. *Let $\bar{x} \in C_{min}$, $\alpha \in (0,1]$, $\Delta > 0$, $x \in U$ satisfy $f(x) \geq f(\bar{x}) + \Delta$, and $v \in \partial f(x)$. Then*

$$\|x - \alpha v - \bar{x}\|^2 \leq \|x - \bar{x}\|^2 + \alpha^2 L^2 - 2\alpha\Delta.$$

Proof. From (3.1), we have

$$\begin{aligned}\|x - \alpha v - \bar{x}\|^2 &= \|x - \bar{x}\|^2 + \alpha^2\|v\|^2 - 2\alpha\langle v, x - \bar{x}\rangle \\ &\leq \|x - \bar{x}\|^2 + \alpha^2 L^2 + 2\alpha(f(\bar{x}) - f(x)) \\ &\leq \|x - \bar{x}\|^2 + \alpha^2 L^2 - 2\alpha\Delta.\end{aligned}$$

This completes the proof of the lemma. □

Lemma 4.2. *Let $\bar{x} \in C_{min}$, $\alpha \in (0,1]$, $\Delta > 0$, $x \in U$ satisfy $f(x) \geq f(\bar{x}) + \Delta$, $v \in \partial f(x)$, and $(\Omega, w) \in \mathscr{M}_*$. Let $y = P_{\Omega,w}(x - \alpha v)$. Then*

$$\|y - \bar{x}\|^2 \leq \|x - \bar{x}\|^2 + \alpha^2 L^2 - 2\alpha\Delta.$$

Proof. In view of (3.1), we have $\|v\| \leq L$. Hence,

$$\langle v, \bar{x} - x\rangle \leq f(\bar{x}) - f(x). \tag{4.1}$$

It follows from (4.1) and the inclusion $\bar{x} \in C$ that

$$\begin{aligned}\|y - \bar{x}\|^2 &= \|P_{\Omega,w}(x - \alpha v) - \bar{x}\|^2 \\ &\leq \|x - \alpha v - \bar{x}\|^2 \\ &= \|x - \bar{x}\|^2 + \alpha^2\|v\|^2 - 2\alpha\langle v, x - \bar{x}\rangle \\ &\leq \|x - \bar{x}\|^2 + \alpha^2 L^2 + 2\alpha(f(\bar{x}) - f(x)) \\ &\leq \|x - \bar{x}\|^2 + \alpha^2 L^2 - 2\alpha\Delta.\end{aligned}$$

This lemma is proved. □

5. Proof of Theorem 3.1

In view of (3.6), for each integer $k \geq 1$, we have

$$x_k = P_{\Omega_k, w_k}(x_{k-1,N_k}). \tag{5.1}$$

By (3.3) and (3.5), for each integer $k \geq 1$ and each $p \in \{1, \ldots, N_k\}$, we have

$$\begin{aligned}\|x_{k-1,p} - x_{k-1}\| &= \|\sum_{j=1}^{p}(x_{k-1,j} - x_{k-1,j-1})\| \\ &\leq \sum_{j=1}^{N_k}\|x_{k-1,j} - x_{k-1,j-1}\| \\ &\leq \sum_{j=1}^{N_k}\alpha_{k-1,j}\|l_{k-1,j}\|.\end{aligned}$$

It follows from (3.1) and (3.4) that, for each integer $k \geq 1$ and each $p \in \{1, \dots, N_k\}$,

$$\|x_{k-1,p} - x_{k-1}\| \leq L \sum_{j=1}^{N_k} \alpha_{k-1,j}. \tag{5.2}$$

In particular,

$$\|x_{k-1,N_k} - x_{k-1}\| \leq L \sum_{j=1}^{N_k} \alpha_{k-1,j}. \tag{5.3}$$

In view of (3.2) and (5.3), we obtain

$$\sum_{k=1}^{\infty} \|x_{k-1,N_k} - x_{k-1}\| \leq L \sum_{k=1}^{\infty} (\sum_{j=1}^{N_k} \alpha_{k-1,j}) < \infty.$$

It follows from (5.1) and Theorem 2.1 that there exists $x_* = \lim_{k\to\infty} x_k \in C$ in the norm topology. Assume that case (a) does not hold. This implies that there exists $\Delta_0 > 0$ such that

$$f(x_*) > f(x) + 4\Delta_0 \text{ for all } x \in C_{min}. \tag{5.4}$$

By (3.1), (3.2), (5.2), and (5.4), there exists a natural number k_0 such that, for all integers $k \geq k_0$ and all $p \in \{0, 1, \dots, N_k\}$,

$$\sum_{j=1}^{N_k} \alpha_{k-1,j} < (16L^2)^{-1} \Delta_0 \tag{5.5}$$

and

$$\begin{aligned} f(x_{k-1,p}) &\geq f(x_{k-1}) - L\|x_{k-1} - x_{k-1,p}\| \\ &\geq f(x_{k-1}) - L \sum_{j=1}^{N_k} \alpha_{k-1,j} \\ &> f(x_{k-1}) - 16^{-1}\Delta_0 \\ &> f(x) + 2\Delta_0 \text{ for all } x \in C_{min}. \end{aligned} \tag{5.6}$$

(Note that (5.5) follows from (3.2), and (5.6) follows from (3.1), (5.2), (5.4), and (5.5)). Fix $\bar{x} \in C_{min}$. Let an integer $k \geq k_0$ and $p \in \{1, \dots, N_k\}$. By (3.3), (3.4), (3.5), (5.5), (5.6), and Lemma 4.1 applied with

$$\alpha = \alpha_{k-1,p},\ \Delta = 2\Delta_0,\ x = x_{k-1,p-1},\ v = l_{k-1,p},$$

we have

$$\begin{aligned} \|x_{k-1,p} - \bar{x}\|^2 &\leq \|x_{k-1,p-1} - \bar{x}\|^2 - 4\alpha_{k-1,p}\Delta_0 + 2\alpha_{k-1,p}^2 L^2 \\ &\leq \|x_{k-1,p-1} - \bar{x}\|^2 - \alpha_{k-1,p}\Delta_0. \end{aligned} \tag{5.7}$$

In view of (5.1), we have $\|x_k - \bar{x}\| \leq \|x_{k-1,N_k} - \bar{x}\|$, which together with (3.3) and (5.7) yields that

$$\begin{aligned} \|x_{k-1} - \bar{x}\|^2 - \|x_k - \bar{x}\|^2 &\geq \|x_{k-1,0} - \bar{x}\|^2 - \|x_{k-1,N_k} - \bar{x}\|^2 \\ &= \sum_{p=1}^{N_k} (\|x_{k-1,p-1} - \bar{x}\|^2 - \|x_{k-1,p} - \bar{x}\|^2) \\ &\geq \Delta_0 \sum_{p=1}^{N_k} \alpha_{k-1,p} \end{aligned}$$

and

$$\|x_k - \bar{x}\|^2 \le \|x_{k-1} - \bar{x}\|^2 - \Delta_0 \sum_{p=1}^{N_k} \alpha_{k-1,p}.$$

This completes the proof.

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