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SOLVABILITY OF CONVEX OPTIMIZATION PROBLEMS ON A $*$-CONTINUOUS CLOSED CONVEX SET

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Abstract. Given a closed convex set A in a Banach space X, motivated by the continuity of A [Gale and Klee, Math. Scand. 7 (1959), 379-391], this paper introduces and studies the $*$-continuity of A. Without the reflexivity assumption on X, we prove that the $*$-continuity of a closed convex set A implies that, for every continuous linear (or convex) function $f : X \to \mathbb{R}$ bounded below on A, the corresponding optimization problem $\inf_{x\in A} f(x)$ is weakly well-posed solvable, and that A has the attainable separation property if A is assumed to have a nonempty interior in addition.

Keywords. $*$-continuity of a closed convex set; Conjugate function; Well-posed solvability.

1. INTRODUCTION

In 1959, Gale and Klee [1] introduced the continuity of a closed convex set A in a Banach space X in the following sense:

$$\sigma_A(u^*) = \lim_{x^*\to u^*} \sigma_A(x^*) \quad \forall u^* \in X^* \setminus \{0\},$$

where $\sigma_A(u^*) := \sup_{a\in A}\langle x^*, a\rangle$ is the support functional of A. It is known and easy to verify that σ_A is sublinear and lower semicontinuous with respect to the weak* topology, and so $\lim_{x^*\to u^*} \sigma_A(x^*) = \sigma_A(u^*) = +\infty$ for all $u^* \in X^* \setminus \mathrm{dom}(\sigma_A)$. Thus A is continuous if and only if $\sigma_A(u^*) = \lim_{x^*\to u^*} \sigma_A(x^*)$ for all $u^* \in \mathrm{dom}(\sigma_A) \setminus \{0\}$ if and only if $\mathrm{dom}(\sigma_A) \setminus \{0\}$ is open. In the case that A is a closed convex set in $\mathbb{R}^n$, Gale and Klee [1] proved that A is continuous if and only if A has the strict separation property, that is, for any closed set B in $\mathbb{R}^n$ with $A \cap B = \emptyset$, there exists $x^* \in \mathbb{R}^n$ such that $\inf_{x\in A}\langle x^*, x\rangle > \sup_{x\in B}\langle x^*, x\rangle$. Note that a bounded closed convex set in a reflexive Banach space is trivially continuous and always has the strict separation property. Ernst, Théra, and Zălinescu [2], and Ernst and Théra [3] proved the following interesting result (see [2, Theorem 1] and [3, Proposition 3.1]).

Theorem 1.1. *An unbounded closed convex set A in a reflexive Banach space has the strict separation property if and only if A is continuous and* $\mathrm{int}(A)$ *is nonempty.*

Moreover, Ernst, Théra and Zălinescu [2] proved the following result on convex optimization.

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Theorem 1.2. *Let X be a reflexive Banach space and $f : X \to \mathbb{R}$ be a continuous convex function such that f attains its infimum on X. Then f attains its infimum on every nonempty closed convex set in X if and only if every nonempty level set of f is slice-continuous, that is, $\{x \in E : f(x) \leq \lambda\}$ is a continuous set in E for every closed linear subspace E of X and every $\lambda \in \mathbb{R}$.*

Recently, the author [4] considered the attainable strict separation property: *a closed convex set A in a Banach space X is said to have the attainable strict separation property if, for any closed convex set B in X with $A \cap B = \emptyset$, there exist $x^* \in X^*$, $a \in A$ and $b \in B$ such that*

$$\langle x^*, a\rangle = \min_{x\in A}\langle x^*, x\rangle > \max_{x\in B}\langle x^*, x\rangle = \langle x^*, b\rangle.$$

Still under the reflexivity assumption on space X, the author [4] proved the results on the attainable strict separation property:
(i) Every bounded closed convex set A in X has the attainable strict separation property.
(ii) An unbounded closed convex set A in X has the strict separation property if and only if A is continuous and $\mathrm{int}(A) \neq \emptyset$.
Moreover, motivated by Theorem 1.2, the author [5] proved the following theorem.

Theorem 1.3. *Let X be a reflexive Banach space and $f : X \to \mathbb{R}$ be a continuous convex function such that its conjugate function f^* is continuous at each point in $\mathrm{dom}(f^*)$. Then, for any closed convex set A in X with $\inf_{x\in A} f(x) > -\infty$, the corresponding convex optimization problem*

$$\mathscr{P}_A(f) \qquad\qquad \text{minimize } f(x) \quad \text{subject to } x \in A.$$

is $\mathscr{W}\mathscr{G}$-well-posed solvable, that is, every minimizing sequence $\{a_n\}$ of $\mathscr{P}_A(f)$ (i.e., $\{a_n\} \subset A$ and $f(a_n) \to \inf_{x\in A} f(x)$) has a subsequence converging with respect to the weak topology.

In contrast with Theorems 1.1, 1.2, and 1.3, in the case that X is a general Banach space, this paper will consider the corresponding issues.

2. Preliminaries

For convenience, this section first recall some notions and results in convex analysis, which will be used in our later analysis (see, e.g., [6, 7] for more details). For a Banach space X and a closed convex subset A of X, let σ_A denote the support functional of A, that is,

$$\sigma_A(x^*) := \sup_{x\in A}\langle x^*, x\rangle \quad \forall x^* \in X^*.$$

Clearly, the domain $\mathrm{dom}(\sigma_A)$ of σ_A is a convex cone (named as the barrier cone of A in the literature) and σ_A is sublinear and lower semicontinuous with respect to the weak* topology on the dual space X^*. Moreover, A is bounded if and only if $\mathrm{dom}(\sigma_A) = X^*$.

Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. The subdifferential $\partial\varphi(x)$ of φ at $x \in \mathrm{dom}(\varphi) := \{u \in X : \varphi(u) < +\infty\}$ is defined as

$$\partial\varphi(x) := \{x^* \in X^* : \langle x^*, y - x\rangle \leq \varphi(y) - \varphi(x) \quad \forall y \in X\}.$$

For $\varepsilon > 0$, let $\partial_\varepsilon\varphi(x)$ denote the ε-subdifferential of φ at x, that is,

$$\partial_\varepsilon\varphi(x) := \{x^* \in X^* : \langle x^*, y - x\rangle \leq \varphi(y) - \varphi(x) + \varepsilon \;\; \forall y \in X\}.$$

It is well-know that if x_0 is an interior point of $\mathrm{dom}(\varphi)$, then the convex function φ is locally Lipschitz at x_0 and $\partial\varphi(x_0)$ is a nonempty weak*-compact convex set in X^*.

The following known results on subdifferential and ε-subdifferential (see [7, Theorem 2.8.3] and [6, Theorem 3.18]) will play important roles in our later analysis.

Lemma 2.1. *Let X be a Banach space and $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then, for any $x \in \mathrm{int}(\mathrm{dom}(\varphi))$, $\partial\varphi$ is $\|\cdot\|$-w^* upper semicontinuous, that is, for any neighborhood W of 0 with respect to the weak* topology on X^*, there exists $\delta > 0$ such that $\partial\varphi(B(x,\delta)) \subset \partial\varphi(x) + W$.*

Lemma 2.2. *Let φ be a proper lower semicontinuous convex function on a Banach space X. Let $\varepsilon > 0$, $x \in \mathrm{dom}(\varphi)$ and $x^* \in \partial_\varepsilon\varphi(x)$. Then there exist $x_\varepsilon \in X$ and $x_\varepsilon^* \in X^*$ such that*

$$\|x_\varepsilon - x\| \leq \sqrt{\varepsilon},\ \|x_\varepsilon^* - x^*\| \leq \sqrt{\varepsilon} \text{ and } x_\varepsilon^* \in \partial\varphi(x_\varepsilon).$$

We also need the conjugate function $\varphi^* : X^* \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\varphi^*(x^*) := \sup_{x \in X}(\langle x^*, x\rangle - \varphi(x)) \quad \forall x^* \in X^*.$$

It is well known that the conjugate function φ^* is always lower semicontinuous with respect to the weak* topology on X^* and useful in convex analysis and duality theory of convex optimization (see, e.g., [7, 8]). It is also known that

$$x^* \in \partial\varphi(x) \Longleftrightarrow x \in \partial\varphi^*(x^*).$$

For a closed convex set A in a Banach space X, let δ_A denote the indicator function of A, that is, $\delta_A(x) = 0$ if $x \in A$ and $\delta_A(x) = +\infty$ if $x \in X \setminus A$. It is easy to verify that σ_A is just the conjugate function δ_A^* of δ_A.

For convenience, we also recall the following observation by Adly, Ernst and Théra [9]

$$x^* \in \mathrm{int}(\mathrm{dom}(f^*)) \Leftrightarrow (x^*, -1) \in \mathrm{int}(\mathrm{dom}(\sigma_{\mathrm{epi}(f)}))$$

and

$$\mathrm{dom}(f^*) \times \{-1\} = \mathrm{dom}(\sigma_{\mathrm{epi}(f)}) \cap (X^* \times \{-1\}) \tag{2.1}$$

(see (3.16) in [9]).

With the help of Lemmas 2.1 and 2.2, we have the following characterization for a closed convex set in a reflexive Banach space to be continuous.

Proposition 2.1. *Let A be a closed convex set in a reflexive Banach space X. Then A is continuous if and only if, for any $x^* \in \mathrm{dom}(\sigma_A) \setminus \{0\}$, every sequence $\{a_n\} \subset A$ with $\langle x^*, a_n\rangle \to \sigma_A(x^*)$ has a weakly convergent subsequence.*

Proof. First, we suppose that A is continuous. Then $\mathrm{dom}(\sigma_A) \setminus \{0\}$ is open. Let $x^* \in \mathrm{dom}(\sigma_A) \setminus \{0\}$ and take an arbitrary sequence $\{a_n\}$ in A such that $\langle x^*, a_n\rangle \to \sigma_A(x^*)$. Thus, one has

$$\langle y^* - x^*, a_n\rangle \leq \sigma_A(y^*) - \langle x^*, a_n\rangle \leq \sigma_A(y^*) - \sigma_A(x^*) + \varepsilon_n \quad \forall y^* \in X^*,$$

where $0 \leq \varepsilon_n := \sigma_A(x^*) - \langle x^*, a_n\rangle \to 0$. Hence $a_n \in \partial_{\varepsilon_n}\sigma_A(x^*)$ for all $n \in \mathbb{N}$. It follows from Lemma 2.2 that, for any $n \in \mathbb{N}$, there exist $x_n^* \in X^*$ and $x_n^{**} \in X^{**}$ such that

$$\|x_n^* - x^*\| \leq \sqrt{\varepsilon_n},\ \|x_n^{**} - a_n\| \leq \sqrt{\varepsilon_n} \text{ and } x_n^{**} \in \partial\sigma_A(x_n^*).$$

Noting that σ_A is locally Lipschitz at x^* and $\varepsilon_n \to 0$, $\{x_n^{**}\}$ is a bounded sequence in X^{**} and so $\{a_n\}$ is a bounded sequence in X. Since X is reflexive, the James theorem implies that $\{a_n\}$ has a weakly convergent subsequence. This shows that the necessity is valid.

To prove the sufficiency, let $x^* \in \mathrm{dom}(\sigma_A)\setminus\{0\}$ be such that every sequence $\{a_n\}\subset A$ with $\langle x^*, a_n\rangle \to \sigma_A(x^*)$ has a weakly convergent subsequence. We only need to show that $x^* \in \mathrm{int}(\mathrm{dom}(\sigma_A))$. We claim that

$$S(A,x^*,\varepsilon) = \{x\in A:\ \langle x^*,x\rangle \geq \sigma_A(x^*)-\varepsilon\}$$

is bounded for some $\varepsilon > 0$. Indeed, if this is not the case, $S(A,x^*,\frac{1}{n})$ is unbounded for all $n\in\mathbb{N}$. Then, for each $n\in\mathbb{N}$, there exists $a_n \in S(A,x^*,\frac{1}{n})$ such that $\|a_n\| > n$. On the other hand, $\sigma_A(x^*)-\frac{1}{n} \leq \langle x^*, a_n\rangle \leq \sigma_A(x^*)$ for all $n\in\mathbb{N}$. It follows that $\langle x^*,a_n\rangle \to \sigma_A(x^*)$, and so $\{a_n\}$ has a subsequence $\{a_{n_k}\}$ being weakly convergent. Hence, $\{a_{n_k}\}$ is a bounded sequence, contradicting $\|a_{n_k}\| \geq n_k$. This shows that $S(A,x^*,\varepsilon)$ is bounded for some $\varepsilon > 0$. Thus, by [5, Proposition 3.2], $x^* \in \mathrm{int}(\mathrm{dom}(\sigma_A)$. The proof is complete. □

Motivated by Proposition 2.1, we introduce the following notion.

Definition 2.1. A closed convex subset A of a Banach space X is said to be $*$-continuous if every sequence $\{a_n\}\subset A$ with $\langle x^*,a_n\rangle \to \sigma_A(x^*)$ has a weakly convergent subsequence for any $x^*\in\mathrm{dom}(\sigma_A)\setminus\{0\}$.

Remark 2.1. By the proof of the sufficiency part of Proposition 2.1, the $*$-continuity of A implies $\mathrm{dom}(\sigma_A)\setminus\{0\}\subset \mathrm{int}(\mathrm{dom}(\sigma_A))$, which means the continuity of A.

3. Main Results

Let A be a closed convex set in a Banach space X. Recall that A has the attainable separation property (resp. attainable strict separation property) if, for any closed convex set B with $A\cap\mathrm{int}(B)=\emptyset$ and $\mathrm{int}(B)\neq\emptyset$ (resp. with $A\cap B=\emptyset$), there exist $x^*\in X^*\setminus\{0\}$ and $(a,b)\in A\times B$ such that

$$\langle x^*,a\rangle = \min_{x\in A}\langle x^*,x\rangle,\ \ \langle x^*,b\rangle = \max_{x\in B}\langle x^*,x\rangle,$$

and

$$\langle x^*,a-b\rangle \geq 0\ \ (\text{reps. } \langle x^*,a-b\rangle > 0).$$

Without the reflexivity assumption on X but with the $*$-continuity of A replacing the continuity of A, this section first considers the attainable separation property and the attainable strict separation property of A.

Theorem 3.1. *Let A be a $*$-continuous closed convex set in a Banach space X. Then, for any closed convex set B in X with $\mathrm{int}(A-B)\neq\emptyset$, $A-B$ is closed.*

Proof. Let B be a closed convex set in X such that $\mathrm{int}(A-B)\neq\emptyset$ and let $z\in\mathrm{cl}(A-B)\setminus\mathrm{int}(A-B)$. Then, by the classical separation theorem, there exists $z^*\in X^*\setminus\{0\}$ such that

$$\langle z^*,z\rangle = \sup_{x\in\mathrm{int}(A-B)}\langle z^*,x\rangle = \sup_{x\in A-B}\langle z^*,x\rangle = \sigma_A(z^*) - \inf_{x\in B}\langle z^*,x\rangle. \tag{3.1}$$

Hence, $z^*\in\mathrm{dom}(\sigma_A)\setminus\{0\}$. Since $z\in\mathrm{cl}(A-B)$, there exists a sequence $\{(x_n,y_n)\}$ in $A\times B$ such that $x_n-y_n\to z$. Then, by (3.1), one has $\langle x^*,x_n\rangle\to\sigma_A(x^*)$. This and $*$-continuity of A imply that $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ weakly converging to some $a\in A$ (because A is a closed convex set and hence is weakly closed). Therefore, $\{y_{n_k}\}$ converges weakly to $b:=a-z\in B$, and so $z=a-b\in A-B$. This shows that $A-B$ is closed. The proof is complete. □

Theorem 3.2. *Let A be a $*$-continuous closed convex set in a Banach space X, and let B be a closed convex set B in X such that $A\cap B=\emptyset$ and $\operatorname{int}(A-B)\neq\emptyset$. Then there exist $x_0^*\in X^*\setminus\{0\}$, $a_0\in A$, and $b_0\in B$ such that*

$$\langle x_0^*,a_0\rangle=\inf_{x\in A}\langle x^*,x\rangle>\sup_{x\in B}\langle x_0^*,x\rangle=\langle x_0^*,b_0\rangle. \tag{3.2}$$

Proof. By Theorem 3.1, $A-B$ is closed. Let $f(x):=\|x\|+\delta_{B-A}(x)$ for all $x\in X$. Then f is a proper lower semicontinuous convex function and $\inf_{x\in X}f(x)=d(A,B)$. Let $\varepsilon\in(0,\ 1)$ and take $\bar{x}\in X$ such that $f(\bar{x})<\inf_{x\in X}f(x)+\varepsilon^2$. By the Ekeland variational principle, there exists $x_0\in X$ such that x_0 is a minimizer of the function $x\mapsto f(x)+\varepsilon\|x-\bar{x}\|$. It follows that $x_0\in\operatorname{dom}(f)=B-A$ and

$$0\in\partial(f+\varepsilon\|\cdot-x_0\|)\subset\partial f(x_0)+\varepsilon B_{X^*}.$$

Hence, $x_0\neq 0$ (thanks to $A\cap B=\emptyset$). It follows from the definition of f that

$$0\in\partial(\|\cdot\|+\delta_{B-A})(x_0)+\varepsilon B_{X^*}\subset\partial\|\cdot\|(x_0)+\partial\delta_{B-A}(x_0)+\varepsilon B_{X^*},$$

namely, there exist $e^*\in B_{X^*}$ and $x_0^*\in\partial\delta_{B-A}(x_0)$ such that $\varepsilon e^*-x_0^*\in\partial\|\cdot\|(x_0)$. Hence $\|\varepsilon e^*-x_0^*\|=1$ and $\langle\varepsilon e^*-x_0^*,x_0\rangle=\|x_0\|$ (thanks to $x_0\neq 0$), which implies that

$$\langle x_0^*,x_0\rangle\leq(\varepsilon-1)\|x_0\|<0.$$

Taking $(a_0,b_0)\in A\times B$ such that $x_0=b_0-a_0$, we have that $\langle x_0^*,a_0-b_0\rangle>0$. On the other hand, by $x_0^*\in\delta_{B-A}(x_0)$, we have

$$\langle x_0^*,b-a-(b_0-a_0)\rangle=\langle x_0^*,b-a-x_0\rangle\leq\delta_{B-A}(b-a)-\delta_{B-A}(b_0-a_0)=0$$

for all $(a,b)\in A\times B$. This implies that $\langle x_0^*,a_0\rangle=\inf_{a\in A}\langle x_0^*,a_0\rangle$ and $\langle x_0^*,b_0\rangle=\sup_{b\in B}\langle x_0^*,b\rangle$. Therefore, (3.2) holds. The proof is complete. □

Theorem 3.3. *Let A be a $*$-continuous closed convex set in a Banach space X. The following statements holds:*
(i) A has the attainable separation property.
(ii) If, in addition, $\operatorname{int}(A)\neq\emptyset$, A has the attainable strict separation property.

Proof. Let B is a closed convex set in X such that $\operatorname{int}(B)\neq\emptyset$ and $A\cap\operatorname{int}(B)=\emptyset$. Then, by the classical separation theorem, there exists $u^*\in X^*\setminus\{0\}$ such that $\inf_{x\in A}\langle u^*,x\rangle\geq\sup_{x\in B}\langle x^*,x\rangle$. It is trivial that if $A\cap B\neq\emptyset$, then

$$\langle u^*,u\rangle=\inf_{x\in A}\langle u^*,x\rangle=\sup_{x\in B}\langle x^*,x\rangle\quad\forall u\in A\cap B.$$

In the case that $A\cap B=\emptyset$, Theorem 3.2 implies that there exist $x_0^*\in X^*\setminus\{0\}$, $a_0\in A$, and $b_0\in B$ such that (3.2) holds. Hence both (i) and (ii) hold. □

Let f be a proper lower semicontinuous extended real-valued convex function on a Banach space X, and let A be closed set in X. Consider the following convex constrained optimization problem

$$\mathscr{P}_A(f)\qquad\qquad \text{minimize} f(x)\ \ \text{subject to}\ x\in A.$$

Recall that optimization problem $\mathscr{P}_A(f)$ is $\mathscr{GW}$-well-posed solvable if every sequence $\{a_n\}\subset A$ with $\lim_{n\to\infty}f(a_n)=\inf_{x\in A}f(x)$ has a subsequence being convergent with respect to the weak topology on X. It is known and easy to verify that if $\mathscr{P}_A(f)$ is $\mathscr{G}\mathscr{W}$-well-posed solvable, then its solution set $S(A,f)$ is a nonempty weak compact set.

In contrast with Theorems 1.2 and 1.3, without the reflexivity assumption on the concerned space, we will provide some sufficient conditions for the $\mathscr{G}\mathscr{W}$-well-posed solvability of $\mathscr{P}_A(f)$.

Theorem 3.4. *Let A be a closed convex set in a Banach space X. Then the following statements are equivalent:*
(i) A is $*$-continuous.
(ii) For each $u^* \in \mathfrak{L}(X|A)$, the corresponding linear optimization problem $\mathscr{P}_A(u^*)$ is $\mathscr{W}\mathscr{G}$-well-posed solvable.
(iii) For each $f \in \mathfrak{C}(X|A)$, the corresponding convex optimization problem $\mathscr{P}_A(f)$ is $\mathscr{W}\mathscr{G}$-well-posed solvable.

Proof. First, suppose that (i) holds, and let $u^* \in \mathfrak{L}(X|A)$. Take an arbitrary sequence $\{a_n\}$ in A such that $\langle u^*, a_n\rangle \to \inf_{x\in A}\langle u^*, x\rangle$. Then $-u^* \in \mathrm{dom}(\sigma_A)\setminus\{0\}$ and $\langle -u^*, a_n\rangle \to \sigma_A(-u^*)$. It follows from (i) that $\{a_n\}$ has a weakly convergent subsequence. This shows that (ii) holds. Hence, implication $\Rightarrow$(ii) is valid.

Next, suppose that (ii) holds. Let $f \in \mathfrak{C}(X|A)$ and let $\{a_n\}$ be a minimizing sequence of the corresponding convex optimization $\mathscr{P}_A(f)$. Then, by [5, Proposition 4.3], there exists $u_f^* \in \mathfrak{L}(X|A)$ such that $\{a_n\}$ is a minimizing sequence of linear optimization problem $\mathscr{P}_A(u_f^*)$. This and (ii) imply that $\{a_n\}$ has a weakly convergent sequence, and so (iii) holds.

Finally, suppose that (iii) holds. Let $x^* \in \mathrm{dom}(\sigma_A)$ and $\{a_n\}$ be a sequence in A such that $\langle x^*, a_n\rangle \to \sigma_A(x^*)$. Then $-x^* \in \mathfrak{C}(X|A)$ and $\langle -x^*, a_n\rangle \to \inf_{x\in A}\langle -x^*, x\rangle$. This and (iii) imply that $\{a_n\}$ has a weakly convergent subsequence. Therefore, A is $*$-continuous. This shows that (iii)$\Rightarrow$(i) holds. The proof is complete. □

In the case that the interior $\mathrm{int}(A)$ is nonempty, the objective f in (iii) of Theorem 3.4 can be relaxed to a proper lower semicontinuous convex extended-real function.

Theorem 3.5. *Let A be a $*$-continuous closed convex set in a Banach space X such that* $\mathrm{int}(A)$ *is nonempty. Then, for every proper lower semicontinuous convex function $f : X \to \mathbb{R}\cup\{+\infty\}$ with* $\inf_{x\in X} f(x) < \inf_{x\in A} f(x) < +\infty$*, the corresponding convex optimization problem $\mathscr{P}_A(f)$ is $\mathscr{W}\mathscr{G}$-well-posed solvable.*

Proof. Let $f : X \to \mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous convex function such that

$$+\infty > \lambda := \inf_{x\in A} f(x) > \inf_{x\in X} f(x),$$

and let $B := \{x \in X : \ f(x) \le \lambda\}$. We claim that $\mathrm{int}(A)\cap B = \emptyset$. Indeed, let a be an arbitrary point in $\mathrm{int}(A)$, and take $\bar{x} \in X$ such that $f(\bar{x}) < \lambda$. Then there exists $t \in (0,\ 1)$ sufficiently small such that $a + t(\bar{x} - a) \in A$. Thus, by the convexity of f, one has

$$\lambda \le f(a + t(\bar{x} - a)) \le (1-t)f(a) + tf(\bar{x}) < (1-t)f(a) + t\lambda.$$

This implies that $\lambda < f(a)$. Hence, $a \notin B$. This shows that $\mathrm{int}(A)\cap B = \emptyset$. It follows from the classical separation theorem that there exists $x_0^* \in X^*\setminus\{0\}$ such that

$$\sigma_A(x_0^*) = \sup_{x\in A}\langle x_0^*, x\rangle \le \inf_{x\in B}\langle x_0^*, x\rangle. \tag{3.3}$$

Hence, $x_0^* \in \mathrm{dom}(\sigma_A) \setminus \{0\}$. Let $\{a_n\}$ be an arbitrary minimizing sequence of $\mathscr{P}_A(f)$. Then $\{a_n\} \subset A$ and $f(a_n) \to \lambda$. Since $f(\bar{x}) < \lambda \le f(a_n)$ and the convex function $t \mapsto f(ta_n + (1-t)\bar{x})$ is continuous on the interval $[0,\ 1]$, there exists $t_n \in (0,\ 1]$ such that

$$\lambda = f(t_n a_n + (1-t_n)\bar{x}) \le t_n f(a_n) + (1-t_n) f(\bar{x}).$$

Hence, $t_n a_n + (1-t_n)\bar{x} \in B$ and $\lim_{n\to\infty} t_n = 1$. It follows from (3.3) that

$$\sigma_A(x_0^*) \le \liminf_{n\to\infty} \langle x_0^*, t_n a_n + (1-t_n)\bar{x}\rangle = \liminf_{n\to\infty} \langle x_0^*, a_n\rangle.$$

Since $\langle x_0^*, a_n\rangle \le \sigma_A(x_0^*)$ for all $n \in \mathbb{N}$, one has $\langle x_0^*, a_n\rangle \to \sigma_A(x_0^*)$. This and the $*$-continuity assumption on A imply that $\{a_n\}$ has a weakly convergent subsequence. This shows that $\mathscr{P}_A(f)$ is $\mathscr{W}\mathscr{G}$-well-posed solvable. The proof is complete. □

From a different angle than Theorems 3.4 and 3.5, we next consider a fixed continuous convex function $f : X \to \mathbb{R}$ such that, for any closed convex set A in X, the corresponding optimization problem $\mathscr{P}_A(f)$ is $\mathscr{G}\mathscr{W}$-well-posed solvable.

Theorem 3.6. *Let X be a Banach space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function such that the epigraph $\mathrm{epi}(f)$ is a $*$-continuous set in the product $X \times \mathbb{R}$. Then, for every closed convex subset A of X with $A \cap \mathrm{int}(\mathrm{dom}(f)) \neq \emptyset$ and $\inf_{x\in A} f(x) > -\infty$, the corresponding optimization problem $\mathscr{P}_A(f)$ is $\mathscr{G}\mathscr{W}$-well-posed solvable.*

Proof. Let A be a closed convex set in X such that $A \cap \mathrm{int}(\mathrm{dom}(f)) \neq \emptyset$ and $\lambda := \inf_{x\in A} f(x) > -\infty$, and let $\{a_n\}$ be a minimizing sequence of the corresponding optimization problem $\mathscr{P}_A(f)$. It suffices to prove that $\{a_n\}$ has a subsequence to be convergent with respect to the weak topology on X. Noting that

$$\mathrm{int}(\mathrm{epi}(f) = \{(x,t) :\ x \in \mathrm{int}(\mathrm{dom}(f)) \text{ and } f(x) < t\} \neq \emptyset$$

and $\mathrm{int}(\mathrm{epi}(f)) \cap (A \times \{\lambda\}) = \emptyset$, the separation theorem implies that there exists $(x^*, -\alpha) \in (X^* \times \mathbb{R}) \setminus \{(0,0)\}$ such that

$$\sigma_{\mathrm{epi}(f)}((x^*, -\alpha)) = \sup_{(x,t)\in\mathrm{epi}(f)} (\langle x^*, x\rangle - \alpha t) \le \inf_{x\in A} \langle x^*, x\rangle - \alpha\lambda. \tag{3.4}$$

It follows that $\alpha \ge 0$. We claim that $\alpha > 0$. Indeed, if this is not the case, then $\alpha = 0$. So,

$$\sup_{x\in\mathrm{dom}(f)} \langle x^*, x\rangle = \sup_{(x,t)\in\mathrm{epi}(f)} \langle x^*, x\rangle \le \inf_{x\in A} \langle x^*, x\rangle.$$

Since $\mathrm{int}(\mathrm{dom}(f)) \cap A \neq \emptyset$, there exist $a \in A$ and $r > 0$ such that $B(a,r) \subset \mathrm{dom}(f)$. Thus, $\sup_{x\in B(a,r)} \langle x^*, x\rangle \le \langle x^*, a\rangle$. This means $x^* = 0$, contradicting $(x^*, -\alpha) \neq (0,0)$. Hence $\alpha > 0$. Without loss of generality, we assume that $\alpha = 1$. Thus, by (3.4), one has $(x^*, -1) \in \mathrm{dom}(\sigma_{\mathrm{epi}(f)})$ and

$$\langle x^*, a_n\rangle - f(a_n) \le \sigma_{\mathrm{epi}(f)}(x^*, -1) \le \langle x^*, a_n\rangle - \lambda \quad \forall n \in \mathbb{N}.$$

Since $f(a_n) \to \lambda$, $\langle (x^*, -1), (a_n, f(a_n))\rangle \to \sigma_{\mathrm{epi}(f)}(x^*, -1)$. This and $*$-continuity of $\mathrm{epi}(f)$ imply that $\{(a_n, f(a_n))\}$ has a weakly convergent subsequence, and hence $\{a_n\}$ has a weakly convergent subsequence. The proof is complete. □

In the case when $\mathrm{dom}(f) = X$, $\mathrm{int}(\mathrm{dom}(f)) \cap A = A$. Thus, the following corollary is immediate from Theorem 3.6.

Corollary 3.1. *Let X be a Banach space, and let $f : X \to \mathbb{R}$ be a continuous convex function such that* $\mathrm{epi}(f)$ *is $*$-continuous. Then, for every closed convex subset A of X with* $\inf_{x\in A} f(x) > -\infty$*, the corresponding optimization problem $\mathscr{P}_A(f)$ is $\mathscr{GW}$-well-posed solvable.*

In the finite dimension case, we have the following sharp result.

Corollary 3.2. *Let X be a finite dimensional Banach space, and let $f : X \to \mathbb{R}$ be a continuous convex function such that* $\mathrm{epi}(f)$ *is $*$-continuous. Then, for every closed convex subset A of X with* $\inf_{x\in A} f(x) > -\infty$*, the solution set $S(A, f)$ of the corresponding optimization problem $\mathscr{P}_A(f)$ is a nonempty compact set and* $\lim_{n\to\infty} d(a_n, S(A, f)) = 0$ *for any sequence $\{a_n\}$ in A with* $\lim_{n\to\infty} f(a_n) = \inf\limits_{x\in A} f(x)$.

Proof. Let A be an arbitrary closed convex set A in X with $\inf_{x\in A} f(x) > -\infty$. Then, by Corollary 3.1, the corresponding optimization problem $\mathscr{P}_A(f)$ is $\mathscr{GW}$-well-posed solvable. Hence, every sequence $\{x_n\}$ in A with $f(x_n) = \inf\limits_{x\in A} f(x)$ has a subsequence weakly converging to some point $a \in A$; since X is finite dimensional, $\{x_n\}$ has a subsequence strongly converging to $a \in A$. It follows that $S(A, f)$ is a nonempty compact set. Next, we prove that $\lim_{n\to\infty} d(x_n, S(A, f)) = 0$. Indeed, if this is not the case, $\limsup_{n\to\infty} d(x_n, S(A, f)) > 0$. It follows that there exist a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $\varepsilon_0 > 0$ such that

$$d(x_{n_k}, S(A, f)) \geq \varepsilon_0 \quad \forall k \in \mathbb{N}. \tag{3.5}$$

On the other hand, by Corollary 3.1, we can assume without loss of generality that $\{x_{n_k}\}$ converges to some $a_0 \in S(A, f)$ (taking a subsequence of $\{x_{n_k}\}$ if necessary). This implies that

$$\lim_{k\to\infty} d(x_{n_k}, S(A, f)) \leq \lim_{k\to\infty} \|x_{n_k} - a_0\| = 0,$$

which contradicts (3.5). The proof is complete. □

The following proposition provides a sufficient condition for the epigraph $\mathrm{epi}(f)$ to be a $*$-continuous set.

Proposition 3.1. *Let X be a Banach space such that its dual space X^* is separable, and let $f : X \to \mathbb{R}$ be a continuous convex function such that its conjugate function f^* is continuous at each point of* $\mathrm{dom}(f^*)$ *and $\partial f^*(x^*) \subset X$ for all $x^* \in$* $\mathrm{dom}(f^*)$*. Then* $\mathrm{epi}(f)$ *is $*$-continuous.*

Proof. Let $(x^*, -\alpha)$ be an arbitrary element in $\mathrm{dom}(\sigma_{\mathrm{epi}(f)}) \setminus \{(0, 0)\}$. Then

$$\langle x^*, x\rangle - \alpha(f(x) + t) \leq \sigma_{\mathrm{epi}(f)}(x^*, -\alpha) < +\infty \quad \forall (x, t) \in X \times \mathbb{R}_+.$$

It follows that $\alpha \geq 0$. We claim that $\alpha > 0$. Indeed, if this is not the case, $\alpha = 0$. Then $\sup_{x\in X}\langle x^*, x\rangle < +\infty$, which means $x^* = 0$ (thanks to the linearity of X^*). This contradicts $(x^*, -\alpha) \neq (0, 0)$. Since $\mathrm{dom}(\sigma_{\mathrm{epi}(f)})$ is a cone, $(\frac{x^*}{\alpha}, -1) \in \mathrm{dom}(\sigma_{\mathrm{epi}(f)})$. This and (2.1) imply that $\frac{x^*}{\alpha} \in \mathrm{dom}(f^*)$. Since the convex function f^* is continuous at $\frac{x^*}{\alpha}$, $\frac{x^*}{\alpha}$ is an interior point of $\mathrm{dom}(f^*)$. Hence, there exist $r, M \in (0, +\infty)$ such that

$$B(\frac{x^*}{\alpha}, r) \subset \mathrm{dom}(f^*) \text{ and } \partial f^*(B(\frac{x^*}{\alpha}, r)) \subset MB_{X^{**}}. \tag{3.6}$$

By the separability of X^*, there exists a metric ρ_* on $2MB_{X^{**}}$, which is identical to the restriction topology to $2MB_{X^{**}}$ of the weak* topology on X^{**}. Take an arbitrary sequence $\{(x_n, t_n)\}$ in

$X\times\mathbb{R}_+$ such that $\langle x^*,x_n\rangle-\alpha(f(x_n)+t_n)\to\sigma_{\mathrm{epi}(f)}(x^*,-\alpha)$. Then $t_n\to 0$ (because $\alpha>0$), and hence

$$0\le\varepsilon_n:=\sigma_{\mathrm{epi}(f)}(\frac{x^*}{\alpha},-1)-\langle\frac{x^*}{\alpha},x_n\rangle-f(x_n)\longrightarrow 0. \tag{3.7}$$

We only need to show that $\{(x_n,f(x_n)+t_n)\}$ has a subsequence being convergent with respect to the weak topology on X. By the definition of $\sigma_{\mathrm{epi}(f)}$, one has

$$\varepsilon_n\ge\langle\frac{x^*}{\alpha},x\rangle-f(x)-(\langle\frac{x^*}{\alpha},x_n\rangle-f(x_n))\quad\forall x\in X.$$

This implies $\frac{x^*}{\alpha}\in\partial_{\varepsilon_n}f(x_n)$. It follows from Lemma 2.2 that there exist

$$y_n\in B(x_n,\sqrt{\varepsilon_n})\ \text{ and }\ x_n^*\in\partial f(y_n)\cap B(\frac{x^*}{\alpha},\sqrt{\varepsilon_n}). \tag{3.8}$$

Therefore,

$$\|y_n-x_n\|\to 0,\ \|x_n^*-\frac{x^*}{\alpha}\|\to 0\ \text{ and }\ y_n\in\partial f^*(x_n^*)\ \ \forall n\in\mathbb{N}.$$

Without loss of generality, we can assume that $x_n^*\in B(\frac{x^*}{\alpha},r)$ for all $n\in\mathbb{N}$. Noting that ∂f^* is $\|\cdot\|$-w^* upper semicontinuous at $\frac{x^*}{\alpha}$, it follows from (3.6) and (3.8) that $\rho_*(y_n,\partial f^*(\frac{x^*}{\alpha}))\to 0$ as $n\to\infty$. Hence, there exists a sequence $\{z_n\}$ in $\partial f^*(\frac{x^*}{\alpha})$ such that

$$\rho_*(y_n,z_n)\to 0 \text{ as } n\to\infty. \tag{3.9}$$

Since $\partial f^*(\frac{x^*}{\alpha})$ is a bounded and weak*-closed subset of X^{**}, the Alaoglu theorem implies that $\partial f^*(\frac{x^*}{\alpha})$ is weak*-compact. By the assumption, $\partial f^*(\frac{x^*}{\alpha})$ is contained in X. Hence $\partial f^*(\frac{x^*}{\alpha})$ is a weakly compact subset of X. This and the James theorem imply that $\{z_n\}$ has a subsequence $\{z_{n_k}\}$ converging to some $z_0\in\partial f^*(\frac{x^*}{\alpha})$ with respect to the weak topology on X. Thus, by (3.9), $\{y_{n_k}\}$ converges to z_0 with respect to the weak topology. This and (3.8) show that $\{x_{n_k}\}$ converges weakly to z_0. In particular,

$$\langle\frac{x^*}{\alpha},x_{n_k}\rangle\to\langle\frac{x^*}{\alpha},z_0\rangle. \tag{3.10}$$

Since the continuous convex function f is lower semicontinuous with respect to the weak topology, $f(z_0)\le\liminf_{k\to\infty}f(x_{n_k})$. Noting that $\langle\frac{x^*}{\alpha},x_{n_k}\rangle-f(x_{n_k})\le\sigma_{\mathrm{epi}(f)}(\frac{x^*}{\alpha})$ for all $k\in\mathbb{N}$, it follows from (3.7) and (3.10) that $f(x_{n_k})\to f(z_0)$ as $k\to\infty$. Hence $\{(x_{n_k},f(x_{n_k})+t_{n_k}))\}$ converges weakly to $(z_0,f(z_0))$. The proof is complete. □

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