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OPTIMALITY AND DUALITY FOR SEMIDEFINITE MULTIOBJECTIVE PROGRAMMING PROBLEMS USING CONVEXIFICATORS

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Abstract. In this paper, we establish Fritz John optimality conditions for nonsmooth nonlinear semidefinite multiobjective programming in terms of convexificators, and introduce generalized Cottle type and generalized Guignard type constraint qualification to achieve strong Karush-Kuhn-Tucker optimality conditions from Fritz John optimality. Strong Karush-Kuhn-Tucker necessary and sufficient optimality conditions also established independently. Furthermore, we formulate Wolfe and Mond-Weir type dual model, and establish usual duality results for the problems. Some examples are provided in the support of the main results.

Keywords. Convexificators; Duality; Multiobjective programming; Nonsmooth analysis; Semidefinite programming.

1. Introduction

In this paper, we consider the nonsmooth semidefinite multiobjective programming

$$\text{(NSD-MOP)} \qquad \begin{aligned} &\min f(X) = (f_1(X),\dots,f_p(X)) \\ &\text{subject to } g_i(X) \le 0,\ i \in I = \{1,\dots,m\},\ X \in \mathbb{S}^n_+, \end{aligned}$$

where $\mathbb{S}^n_+$ is set of $n \times n$ positive semidefinite matrix, $f_i : \mathbb{S}^n_+ \to \mathbb{R} \cup \{+\infty\}$ $(i = 1,\dots p)$, and $g_i : \mathbb{S}^n_+ \to \mathbb{R} \cup \{+\infty\}$ are extended real-valued locally Lipschitz functions.

Nonlinear semidefinite programming problems include several classes of optimization problems, such as linear programming, quadratic programming, second order cone programming, and semidefinite programming; see, e.g., [1, 2]. The nonlinear semidefinite programming problem has broad applications in system control [3], truss topology optimization [4], and so on. It has become a center point in optimization research from last two decades. For instance, in the release of library COMPleib [5], where 168 test examples on nonlinear semidefinite programs from various fields like control system design, academia and many real-life based problems were collected.

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Nonlinear semidefinite programming problems were studied mostly in two different forms (see, [6] and [7]), where their themes are same, and several useful results and applications were obtained. In [6], first and second order necessary and sufficient optimality conditions were established under convexity assumptions, while in [8], convexity was not a necessarily requirement. In [7, 10], algorithmic approach to solve the nonlinear semidefinite optimization problems were studied. In [11], a survey on numerical methods for solving nonlinear SDP problems was described. In 2015, Golestani and Nobakhtian [12] introduced generalized Abadie constraint qualification $(GACQ)$, and established necessary and sufficient optimality conditions for nonlinear semidefinite programming by using convexificators.

Multiobjective optimization problems (MOP) like situations produces in science, technology, business, economics, and many others field of daily demand, where optimal decisions need to be taken among many conflicting objectives and all objective functions to be optimized simultaneously. Effect of conflict on objectives leads some change in the solution of (MOP) compared to optimal solutions of single-objective optimization problems. Therefore, weak efficient points (weak Pareto optimal solutions), efficient points (Pareto optimal solutions) are coined for the solutions of (MOP). Initially, the concept of Pareto optimal was given by Italian civil engineer and economist Vilfredo Pareto, who used it in the studies of economic efficiency and income distribution. Basic concepts and literatures on the solutions of multiobjective optimization problems can be found in [13] and [14]. Constraint qualification and strong Karush-Kuhn-Tucker optimality have been discussed for differentiable [15], semidifferentiable [16], and nonsmooth [17] cases, respectively.

In optimization theory corresponding to any minimization (maximization) problem, there is another formulated maximization (minimization) problem, which is called dual (see [18]). Duality is the principle in which the same optimization problem may be viewed from two different views. The first is a primal problem, and the second is a dual problem. The solution of the dual problem provides a lower bound to the solution of the primal problem. Thus, from the viewpoint of applications, the duality play a vital role in optimization. However, in several cases, it is observed that the optimal values of both objectives need not appear to be equal. Their difference is so called duality gap. In case of convexity assumptions, under a suitable constraint qualification, the duality gap becomes zero. Some results on the duality of multiobjective optimization can be found [19, 20, 21].

Since nonsmoothness in optimization is naturally generated from the mathematical formulations of real-world problems, it is interesting to study effective ways for solving these problems. Indeed, the solutions of many smooth problems still require the use of nonsmooth optimization techniques in order to either make them easy or simplify their forms. Thus, the field of nonsmooth optimization is an important branch of mathematical programming which is based on classical concepts of variational analysis and generalized derivatives. In recent years, the research in nonsmooth analysis has focused on the growth of generalized subdifferentials that give sharp results and good calculus rules for nonsmooth functions. Convexificators [22] was used to extend, unify, and sharpen the results in various aspects of optimization. In [23], Jeyakumar and Luc gave a more sophisticated version of convexificators by introducing the new notion of convexificators, which is a closed set but not necessarily bounded or convex. From the viewpoint of application, this new version of convexificators has an advantage that one may have convexificators consisting of only finitely many points. That is why we are using this new notion of convexificators in our study.

Motivated by the above results, in this paper, we propose some new constraint qualification to establish necessary and sufficient optimality conditions for nonsmooth nonlinear semidefinite multiobjective optimization problems. Moreover, we extend the dual model by using the notion of convexificators, and establish duality results. Organization of this paper is as follows. In Section 2, we recall some necessary preliminaries and fundamental results. In Section 3, we establish Fritz John necessary optimality conditions, and propose generalized Cottle and generalized Guignard type constraint qualifications to establish strong Karush-Kuhn-Tucker necessary optimality conditions. Further, sufficient optimality conditions are also established under generalized convexity. In Section 4, we formulate Wolfe [18] and Mond-Weir [24] type duality and establish fundamental duality results. Some examples are also given in this section. Section 5 presents the conclusion of the paper as well as some possible views towards future work.

2. PRELIMINARIES

This section recalls some necessary notations, definitions, and preliminaries, that will be used later. $\mathbb{S}^n$ is denoted as the space of $n\times n$ symmetric matrices. The notation $X\succeq 0 (X\succ 0)$ means that X is a positive semidefinite matrix (positive definite matrix), and we denote by $\mathbb{S}^n_+$ ($\mathbb{S}^n_{++}$) the set of all positive semidefinite matrices (positive definite matrices). The inner product of the symmetric matrices $A,B\in\mathbb{S}^n$ is denoted by $\langle A,B\rangle$, and defined by $\langle A,B\rangle = tr(AB)$, where $tr(.)$ denotes the summation of the diagonal elements of a square matrix. The inner product of $x = (x_1,\ldots,x_n)$, $y=(y_1,\ldots,y_n)\in\mathbb{R}^n$ is defined and denoted by $x^Ty=\sum_{i=1}^n x_iy_i$. The norm associated with matrix inner product is called the Frobenius norm $||A||_F = tr(AA)^{\frac{1}{2}} = (\sum_{i,\ j=1}^n a_{ij}^2)^{\frac{1}{2}}$. The vector space $\mathbb{S}^n$ with this norm is a Hilbert space, and $\mathbb{S}^n_+$ is a closed convex cone in $\mathbb{S}^n$. The interior of the positive semidefinite matrices is the positive definite matrices. For more basics on matrices, we refer to [9, 25]. For $y,z\in\mathbb{R}^n$, $y\leq z \iff y\leqq z, y\neq z$, $y\leqq z \iff y_i\leqq z_i$, and $y<z \iff y_i<z_i$, $i=1,\ldots,n$. Some sets are as follows

$$S=\{X\in\mathbb{S}^n_+ :\ g_i(X)\leqq 0\ (i\in I)\},$$
$$I_f=\{1,\ldots,p\},\ I_f^k=\{1,\ldots,p\}\setminus\{k\},\ I(\bar{X})=\{i: g_i(\bar{X})=0\},$$
$$Q=\{X\in\mathbb{S}^n_+ : f_i(X)\leqq f_i(\bar{X})\ (i\in I_f),\ g_i(X)\leqq 0\ (i\in I)\},$$
$$Q^k=\{X\in\mathbb{S}^n_+ : f_i(X)\leqq f_i(\bar{X})\ (i\in I_f^k),\ g_i(X)\leqq 0\ (i\in I)\},\ \text{where } \bar{X}\in S,$$
$$\mathbb{R}^n_+=\{x\in\mathbb{R}^n : x\geqq 0\},\ \mathbb{R}^n_{++}=\{x\in\mathbb{R}^n : x>0\}.$$

The motivation for the solution of (NSD-MOP), we refer to [14].

Definition 2.1. A feasible point $\bar{X}$ is said to be a weak efficient solution of (NSD-MOP) if there is no other $X\in S$ such that $f_i(X)<f_i(\bar{X})$, $\forall i\in I_f$.

Definition 2.2. A feasible point $\bar{X}$ is said to be a local weak efficient solution of (NSD-MOP) if there exists a neighborhood $N(\bar{X})$ of $\bar{X}$ such that there is no other $X\in S\cap N(\bar{X})$ such that $f_i(X)<f_i(\bar{X})$, $\forall\ i\in I_f$.

Given a nonempty subset S of $\mathbb{S}^n$, the closure, the convex hull, and the convex cone (including the origin) generated by S are denoted by clS, coS, and $coneS$, respectively. The negative and the strictly negative polar cone of S are defined, respectively, by

$$S^{-} := \{V \in \mathbb{S}^n : \langle V, W\rangle \leq 0,\ \forall\, W \in S\},\ S^{s} := \{V \in \mathbb{S}^n : \langle V, W\rangle < 0,\ \forall\, W \in S\}.$$

Contingent cone $T(S,X)$ to S at point $X \in clS$ is defined by

$$T(S,X) := \{V \in \mathbb{S}^n : \exists\, t_n \downarrow 0,\ V_n \to V \text{ such that } X + t_n V_n \in S\, \forall\, n\}.$$

The notion of semi-regular convexificators [23] will be used. It is observed that for locally Lipschitz functions many generalized subdifferential like Clarke subdifferential [26], Michel-Penot subdifferential [27], Mordukhovich subdifferential [28], and Treiman subdifferential [29] are examples of upper semi-regular convexificators.

Let $f : \mathbb{S}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function, and let $X \in \mathbb{S}^n$ at which f is finite. The lower and upper Dini derivatives of f at X in the direction $V \in \mathbb{S}^n$ are defined, respectively, by $f^{-}(X;V) := \liminf_{t\downarrow 0} \frac{f(X+tV)-f(X)}{t}$, and $f^{+}(X;V) := \limsup_{t\downarrow 0} \frac{f(X+tV)-f(X)}{t}$. Now, we recall the definition of upper and lower semi-regular convexificators from [22, 23].

Definition 2.3. Let $f : \mathbb{S}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function, and let $X \in \mathbb{S}^n$ at which f is finite. The function f is said to admit an upper semi-regular convexificator $\partial^* f(X) \subset \mathbb{S}^n$ at X if $\partial^* f(X)$ is closed and for each $V \in \mathbb{S}^n$, $f^{+}(X;V) \leq \sup_{\xi \in \partial^* f(X)} \langle \xi, V\rangle$. f is said to admit a lower semi-regular convexificator $\partial_* f(X) \subset \mathbb{S}^n$ at X if $\partial_* f(X)$ is closed and, for each $V \in \mathbb{S}^n$, $f^{-}(X;V) \geq \inf_{\xi \in \partial_* f(X)} \langle \xi, V\rangle$.

Definition 2.4. Set $\partial f(X)$ is said to be semi-regular convexificators if it satisfies both upper semi-regular convexificators and lower semi-regular convexificators.

Definition 2.5. Let $f : \mathbb{S}^n \to \mathbb{R} \cup \{+\infty\}$ be an extended real-valued function. Suppose that $X \in \mathbb{S}^n, f(X)$ is finite and admits a convexificator $\partial^* f(X)$ at X.
– f is said to be ∂^*-convex at X if and only if, for all $Y \in \mathbb{S}^n$,

$$f(Y) - f(X) \geq \langle \xi, Y - X\rangle,\ \forall\, \xi \in \partial^* f(X).$$

– f is said to be strictly ∂^*-convex at X if and only if, for all $Y(\neq X) \in \mathbb{S}^n$,

$$f(Y) - f(X) > \langle \xi, Y - X\rangle,\ \forall\, \xi \in \partial^* f(X).$$

– f is said to be ∂^*-pseudoconvex at X if and only if, for all $Y \in \mathbb{S}^n$,

$$f(Y) < f(X) \implies \langle \xi, Y - X\rangle < 0,\ \forall\, \xi \in \partial^* f(X).$$

– f is said to be strictly ∂^*-pseudoconvex at X if and only if, for all $Y(\neq X) \in \mathbb{S}^n$,

$$\langle \xi, Y - X\rangle \geq 0 \implies f(Y) > f(X)\ \forall\, \xi \in \partial^* f(X).$$

–f is said to be ∂^*-quasiconvex at X if and only if, for all $Y \in \mathbb{S}^n$,

$$f(Y) \leq f(X) \implies \langle \xi, Y - X\rangle \leq 0,\ \forall\, \xi \in \partial^* f(X).$$

Now, we recall generalized version of Farkas' lemma [30], which plays a vital role in this paper.

Lemma 2.1. *(Farkas' Lemma) Let $h : \mathbb{S}^n \to \mathbb{R}^m$ be convex functions. Then the following system:*

$$\begin{cases} h(X) < 0, \\ X \in S^n_{++}. \end{cases}$$

has no solution if and only if there exists $(\lambda, U) \in \mathbb{R}^m \times \mathbb{S}^n$ *with* $\lambda \geqq 0$, $U \preceq 0$ *and* $(\lambda, U) \neq (0,0)$ *such that* $\lambda^T h(X) + \langle U, X\rangle \geqq 0$, $\forall X \in \mathbb{S}^n$.

3. Optimality Conditions

Now, we present necessary and sufficient optimality conditions for a local weak efficient solution.

Theorem 3.1. *(Fritz-John necessary optimality conditions) Let* $\bar{X}$ *be a local weak efficient solution to (NSD-MOP). Assume that* f_i $(i \in I_f)$, g_i $(i \in I(\bar{X}))$ *admit bounded upper semi-regular convexificators, and each* g_i $(i \in I \setminus I(\bar{X}))$ *is continuous. Then there exist* $\bar{\lambda}_i \geqq 0$ $(i \in I_f)$, $\bar{\mu}_i \geqq 0$ $(i \in I)$, *and* $\bar{U} \in \mathbb{S}^n_+$, *not all zero, such that*

$$0 \in \sum_{i=1}^{p} \bar{\lambda}_i co\partial^* f_i(\bar{X}) + \sum_{i=1}^{m} \bar{\mu}_i \partial^* g_i(\bar{X}) - \bar{U}$$
$$\langle \bar{X}, \bar{U}\rangle = 0,\ \bar{\mu}_i g_i(\bar{X}) = 0,\ \forall\, i \in I.$$

Proof. We claim that

$$\left(\left(\bigcup_{i\in I_f} \partial^* f_i(\bar{X})\right)^s + \bar{X}\right) \bigcap \left(\left(\bigcup_{i\in I(\bar{X})} \partial^* g_i(\bar{X})\right)^s + \bar{X}\right) \bigcap S^n_{++} = \emptyset. \tag{3.1}$$

On the contrary, we suppose that

$$X \in \left(\left(\bigcup_{i\in I_f} \partial^* f_i(\bar{X})\right)^s + \bar{X}\right) \bigcap \left(\left(\bigcup_{i\in I(\bar{X})} \partial^* g_i(\bar{X})\right)^s + \bar{X}\right) \bigcap S^n_{++}.$$

Since f_i and g_i, $i \in I(\bar{X})$ admit bounded upper semi-regular convexificators, we deduce that $f_i^+(\bar{X}, X-\bar{X}) < 0$, $\forall i \in I_f$, and $g_i^+(\bar{X}, X-\bar{X}) < 0$, $i \in I(\bar{X})$. Hence, there exists $\delta > 0$ such that

$$f_i(\bar{X} + t(X-\bar{X})) < f_i(\bar{X}),\ i \in I_f,\ g_i(\bar{X} + t(X-\bar{X})) < 0,\ i \in I(\bar{X}), \forall\, t \in (0,\delta). \tag{3.2}$$

By the continuity of g_i, $i \notin I(\bar{X})$, there exists $\delta > 0$ such that

$$g_i(\bar{X} + t(X-\bar{X})) < 0,\ i \notin I(\bar{X}),\ \forall\, t \in (0,\delta). \tag{3.3}$$

From (3.2), (3.3), and the convexity of S^n_+, we obtain a contradiction with the local weak efficient point of $\bar{X}$. Now, we define $\varphi_i(X) = \sup_{\xi_i \in \partial^* f_i(\bar{X})} \langle \xi_i, X-\bar{X}\rangle$, $\forall i \in I_f$, and $\psi_i(X) = \sup_{\eta_i \in \partial^* g_i(\bar{X})} \langle \eta_i, X-\bar{X}\rangle$, $\forall i \in I(\bar{X})$. It is easy to see that φ_i and ψ_i are convex functions. From (3.1), it follows that the following system has no solution

$$W = \begin{cases} \varphi_i(X) < 0 & \text{if } i \in I_f, \\ \psi_i(X) < 0 & \text{if } i \in I(\bar{X}), \\ S^n_{++}. \end{cases}$$

Therefore, by the Farkas' Lemma 2.1, there exist $\bar{\lambda}_i \geqq 0$ $(i \in I_f)$, $\bar{\mu}_i \geqq 0$ $(i \in I(\bar{X}))$, and $\bar{U} \in S^n_+$, not all zero, such that $\sum_{i\in I_f} \bar{\lambda}_i \varphi_i(X) + \sum_{i\in I(\bar{X})} \bar{\mu}_i \psi_i(X) - \langle \bar{U}, X\rangle \geqq 0$, $\forall X \in \mathbb{S}^n$. This implies $\langle \bar{U}, \bar{X}\rangle \leqq 0$. On the other hand, $\bar{U}$ and $\bar{X}$ are two elements in $\mathbb{S}^n_+$. Hence $\langle \bar{U}, \bar{X}\rangle = 0$. Consequently, $\omega(X) = \sum_{i\in I_f} \bar{\lambda}_i \varphi_i(X) + \sum_{i\in I(\bar{X})} \bar{\mu}_i \psi_i(X) - \langle \bar{U}, X\rangle$, which is a convex function and $\omega(\bar{X}) = 0$. Hence $0 \in \partial\omega(\bar{X})$, where ∂ is the symbol of the subdifferential in the sense of convex analysis. Thus $0 \in \sum_{i\in I_f} \bar{\lambda}_i \partial\varphi_i(\bar{X}) + \sum_{i\in I(\bar{X})} \bar{\mu}_i \partial\psi_i(\bar{X}) - \bar{U}$. This shows that

$0 \in \sum_{i=1}^{p} \bar{\lambda}_i co\partial^* f_i(\bar{X}) + \sum_{i=1}^{m} \bar{\mu}_i co\partial^* g_i(\bar{X}) - \bar{U}$. Setting $\bar{\mu}_i = 0$ for $i \notin I(\bar{X})$, we conclude the desired results immediately. □

Definition 3.1. We say that the generalized Cottle constraint qualification (GCCQ) is satisfied at $\bar{X}$ if

$$\left(\bigcup_{i\in I_f^k} co\partial^* f_i(\bar{X})\right)^s \bigcap \left(\bigcup_{i\in I(\bar{X})} co\partial^* g_i(\bar{X})\right)^s \bigcap S_+^n \neq \emptyset, \ \forall\, k \in I_f.$$

Theorem 3.2. *Let $\bar{X}$ be a local weak efficient solution to $(NSD-MOP)$. Suppose that f_i and g_i $(i \in I(\bar{X}))$ admits bounded upper semi-regular convexificator at $\bar{X}$. If (GCCQ) holds at $\bar{X}$, then there exists $\bar{\lambda} > 0$, $\bar{\mu}_i \geq 0$ $(i \in I)$ and $\bar{U} \in \mathbb{S}_+^n$ such that*

$$0 \in \sum_{i=1}^{p} \bar{\lambda}_i co\partial^* f(\bar{X}) + \sum_{i=1}^{m} \bar{\mu}_i co\partial^* g_i(\bar{X}) - \bar{U}$$
$$\langle \bar{X}, \bar{U} \rangle = 0,\ \bar{\mu}_i g_i(\bar{X}) = 0,\ \forall\, i \in I.$$

Proof. Since $\bar{X}$ is a local weak efficient solution, it follows from Theorem 3.1 that there exist $\bar{\lambda}_i \geqq 0$ $(i \in I_f)$, $\bar{\mu}_i \geqq 0$ $(i \in I)$ and $\bar{U} \in \mathbb{S}_+^n$ such that

$$0 \in \sum_{i=1}^{p} \bar{\lambda}_i co\partial^* f(\bar{X}) + \sum_{i=1}^{m} \bar{\mu}_i co\partial^* g_i(\bar{X}) - \bar{U}$$
$$\langle \bar{X}, \bar{U} \rangle = 0,\ \bar{\mu}_i g_i(\bar{X}) = 0,\ \forall\, i \in I.$$

Without loss of generality, we may assume that $\lambda_1 = 0$. Then, there exist $\xi_i \in co\partial f_i(\bar{X})$ $(i \in I_f^1)$, $\eta_i \in co\partial g_i(\bar{X})$ $(i \in I(\bar{X}))$ such that $0 = \sum_{i\in I_f^1} \bar{\lambda}_i \xi_i + \sum_{i=1}^{m} \bar{\mu}_i \eta_i - \bar{U}$. By (GCCQ), there exists $X \in \mathbb{S}_+^n$ such that $0 > \sum_{i\in I_f^1} \bar{\lambda}_i \langle \xi_i, X\rangle + \sum_{i=1}^{m} \bar{\mu}_i \langle \eta_i, X\rangle - \langle \bar{U}, X\rangle = \langle \sum_{i\in I_f^1} \bar{\lambda}_i \xi_i + \sum_{i=1}^{m} \bar{\mu}_i \eta_i - \bar{U}, X\rangle = 0$, which is a contradiction. Thus, $\lambda_1 > 0$. By the continuation of the above process for each $k \in I_f$, we conclude the desired result. □

Definition 3.2. We say that the generalized Guignard constraint qualification (GGCQ) holds at $\bar{X}$ if

$$D = cone\ co\left(\bigcup_{i\in I(\bar{X})} co\partial^* g_i(\bar{X})\right) - \mathbb{S}_+^n \text{ is closed set and}$$

$$\left(\bigcup_{i\in I_f} co\partial^* f_i(\bar{X})\right)^- \bigcap \left(\bigcup_{i\in I(\bar{X})} co\partial^* g_i(\bar{X})\right)^- \bigcap S_+^n \subset \bigcap_{i=1}^{p} coT(Q^i, \bar{X}).$$

Lemma 3.1. *Let $\bar{X}$ be any feasible solution to problem (NSD-MOP). Suppose that f_i $(i \in I_f)$ and g_i $(i \in I(\bar{X}))$ admit bounded upper semi-regular convexificators, and each g_i $(i \in I \setminus \{I(\bar{X})\})$ is a continuous function. If D is closed and (GCCQ) holds at $\bar{X}$, then (GGCQ) holds at $\bar{X}$.*

Proof. Without loss of generality, we assume that X satisfies (GCCQ) for $k = 1$.

$$X \in \left(\bigcup_{i\in I_f^1} co\partial^* f_i(\bar{X})\right)^- \bigcap \left(\bigcup_{i\in I(\bar{X})} co\partial^* g_i(\bar{X})\right)^- \bigcap S_+^n \neq \emptyset,$$

Since all f_i $(i \in I_f)$ and g_i $(i \in I(\bar{X}))$, admit bounded upper semi-regular convexificators, we have $f_i^+(\bar{X};X) < 0, \forall i \in I_f^1$ and $g_i^+(\bar{X};X) < 0, \forall i \in I(\bar{X})$. Since $\mathbb{S}_+^n$ is a convex cone, there exists $\delta > 0$ such that

$$f_i(\bar{X}+tX) < f_i(\bar{X}) \ (i \in I_f^1),\ g_i(\bar{X}+tX) < 0, \forall\ i \in I(\bar{X}),\ \bar{X}+tX \in \mathbb{S}_+^n,\ \forall\ t \in (0,\delta).$$

On the other hand, each g_i, $(i \in I \setminus I(\bar{X}))$ is a continuous function. Therefore, there exists $\delta > 0$ such that $g_i(\bar{X}+tX) < 0$, $\forall i \in I$, and $\bar{X}+tX \in \mathbb{S}_+^n$, $t \in (0,\delta)$. Thus, $X \in T(Q^1,\bar{X})$. Therefore,

$$\begin{aligned} A &= \Big(\bigcup_{i\in I_f} co\partial^* f_i(\bar{X})\Big)^- \bigcap \Big(\bigcup_{i\in I(\bar{X})} co\partial^* g_i(\bar{X})\Big)^- \bigcap \mathbb{S}_+^n \\ &= cl\Big(\Big(\bigcup_{i\in I_f} co\partial^* f_i(\bar{X})\Big)^s \bigcap \Big(\bigcup_{i\in I(\bar{X})} co\partial^* g_i(\bar{X})\Big)^s \bigcap \mathbb{S}_{++}^n\Big) \\ &\subset cl\Big(\Big(\bigcup_{i\in I_f^1} co\partial^* f_i(\bar{X})\Big)^s \bigcap \Big(\bigcup_{i\in I(\bar{X})} co\partial^* g_i(\bar{X})\Big)^s \bigcap \mathbb{S}_{++}^n\Big) \\ &\subset clcoT(Q^1,\bar{X}) = coT(Q^1,\bar{X}). \end{aligned}$$

Similarly, it can be proved that $A \subset coT(Q^i,\bar{X})$, $\forall i \in I_f$. Therefore

$$\Big(\bigcup_{i\in I_f} co\partial^* f_i(\bar{X})\Big)^- \bigcap \Big(\bigcup_{i\in I(\bar{X})} co\partial^* g_i(\bar{X})\Big)^- \bigcap \mathbb{S}_+^n \subset \bigcap_{i=1}^p coT(Q^i,\bar{X}).$$

□

In what follows, by applying the generalized Guignard constraint qualification, we derive the Karush-Kuhn-Tucker type necessary optimality conditions for (NSD-MOP).

Theorem 3.3. *Let $\bar{X}$ be a local weak efficient solution of $(NSD-MOP)$. Suppose that f_i and g_i admit bounded upper semi-regular convexificator $\partial^* f_i(\bar{X})$ $(i \in I_f)$, $\partial^* g_i(\bar{X})$ $(i \in I(\bar{X}))$, respectively at $\bar{X}$. If (GGCQ) holds at $\bar{X}$, then there exists $\bar{\lambda}_i > 0$ $(i \in I_f)$, $\bar{\mu}_i \geq 0$ $(i \in I)$ and $\bar{U} \in \mathbb{S}_+^n$ such that $0 \in \sum_{i=1}^{p} \lambda_i co\partial^* f_i(\bar{X}) + \sum_{i=1}^m \bar{\mu}_i co\partial^* g_i(\bar{X}) - \bar{U}$, $\langle \bar{X},\bar{U}\rangle = 0$, $\bar{\mu}_i g_i(\bar{X}) = 0, \forall i \in I$.*

Proof. It is sufficient to show that

$$0 \in \sum_{i=1}^p \lambda_i co\partial^* f_i(\bar{X}) + D,\ \lambda > 0. \tag{3.4}$$

On the contrary, we assume that $0 \notin \sum_{i=1}^p \lambda_i co\partial^* f_i(\bar{X}) + D$, $\lambda > 0$. Since f_i $(i \in I_f)$ admits an upper semi-regular convexificator, we have that the right side in (3.4) is a closed convex set in $\mathbb{S}^n$. By the classical separation theorem, there exists $X \in \mathbb{S}^n$ such that $\langle \tau, X\rangle < 0$, $\forall \tau \in \sum_{i=1}^p \lambda_i co\partial^* f_i(\bar{X}) + D$, $\lambda > 0$. Consequently, $\langle \xi_i, X\rangle < 0$, $\forall \xi_i \in co\partial^* f_i(\bar{X})$ $(i \in I_f)$, $\langle \eta_i, X\rangle \leqq 0$, $\forall \eta_i \in co\partial^* g_i(\bar{X})$ $(i \in I(\bar{X}))$, and $-\langle \bar{U}, X\rangle \leqq 0$, $\forall \bar{U} \in \mathbb{S}_+^n$. It follows from the inequalities above and (GGCQ) that

$$X \in \Big(\bigcup_{i\in I_f} co\partial^* f_i(\bar{X})\Big)^- \bigcap \Big(\bigcup_{i\in I(\bar{X})} co\partial^* g_i(\bar{X})\Big)^- \bigcap \mathbb{S}_+^n \subset \bigcap_{i=1}^p cl\ coT(Q^i,\bar{X}).$$

Thus, $X \in \bigcap_{i=1}^{p} coT(Q^i, \bar{X})$, which implies that there exist $t_n \downarrow 0$ such that $\bar{X} + t_n X \in S$. It follows that $f_i(\bar{X} + tX) < f_i(\bar{X}), \forall i \in I_f$. Thus, we obtain the contradiction that the feasible point $\bar{X}$ is a local weak efficient solution to (NSD-MOP). □

Example 3.1. Consider following optimization problem

$$min\ (f_1(X),\ f_2(X)),\ \text{subject to } g(X) = -2x_2 \leqq 0,\ X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \in \mathbb{S}^2_+,$$
$$\text{where } f_1(X) = |x_1 - 1|,\ f_2(X) = |x_3|.$$

Feasible set $S = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \in \mathbb{S}^2_+ : x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0 \right\}$. $\bar{X} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is a weak efficient solution to the problem. Now, we can find the upper semi-regular convexificator of each functions at point $\bar{X}$ easily as follows:

$$\partial^* f_1(\bar{X}) = \left\{ \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\},\ \partial^* f_2(\bar{X}) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},\ \partial^* g(\bar{X}) = \left\{ \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right\}.$$

$$Q^1 = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_2 & 0 \end{bmatrix} \in \mathbb{S}^2_+ : x_1 \geqq 0,\ x_2 \geqq 0 \right\},\ Q^2 = \left\{ \begin{bmatrix} 1 & x_2 \\ x_2 & x_3 \end{bmatrix} \in \mathbb{S}^2_+ : x_2 \geqq 0,\ x_3 \geqq 0 \right\}.$$

So, we conclude that

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \bigcap_{i=1}^{2} coT(Q^i, \bar{X}) \text{ and } \bigcup_{i=1}^{2} co\partial^* f_i(\bar{X}) = \left\{ \begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix} : t,\ s \in [-1, 1] \right\}.$$

It follows that

$$\left(\bigcup_{i=1}^{2} co\partial^* f_i(\bar{X}) \right)^- = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} : x_1 = 0,\ x_2 = 0,\ x_3 = 0 \right\}.$$

Since,

$$co\partial^* g(\bar{X}) = \left\{ \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right\},\ \text{then } \left(co\partial^* g(\bar{X}) \right)^- = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} : x_i \geqq 0\ \forall\ i \right\},$$

we have

$$\left(\bigcup_{i=1}^{2} co\partial^* f_i(\bar{X}) \right)^- \bigcap \left(co\partial^* g(\bar{X}) \right)^- \bigcap \mathbb{S}^2_+ = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \subset \bigcap_{i=1}^{2} coT(Q^i, \bar{X}).$$

Obviously, $D = cone\ co\partial^* g(\bar{X}) - \mathbb{S}^2_+$ is closed set. Hence (GGCQ) satisfied at $\bar{X}$. Now, for $\lambda_1 = 1,\ \lambda_2 = 1,\ \mu = 0,\ \bar{U} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ \xi_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in co\partial^* f_1(\bar{X}),\ \xi_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in co\partial^* f_2(\bar{X})$, and $\eta = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \in co\partial^* g(\bar{X})$, we have

$$0 = \lambda_1 \xi_1 + \lambda_2 \xi_2 + \mu \eta - \bar{U} = 1 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + 1 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + 0 \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$\in \lambda_1 co\partial^* f_1(\bar{X}) + \lambda_2 co\partial^* f_2(\bar{X}) + \mu co\partial^* g(\bar{X}) - \bar{U},$$

and $\langle \bar{X}, \bar{U} \rangle = Tr\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0$. Hence, KKT conditions are satisfied at weak efficient point $\bar{X}$.

Corollary 3.1. *Let $\bar{X}$ be a local weak efficient solution to $(NSD-MOP)$. Suppose that f_i admits a bounded upper semi-regular convexificator $\partial^* f_i(\bar{X})$ $(i \in I_f)$ at $\bar{X}$. If (GGCQ) holds at $\bar{X}$, then there exists $\bar{\lambda}_i > 0$ $(i \in I_f)$, $\bar{\mu}_i \geq 0$ $(i \in I)$, and $\bar{U} \in \mathbb{S}^n_+$ such that*

$$0 \in \sum_{i=1}^{p} \bar{\lambda}_i co\partial^* f_i(\bar{X}) + \sum_{i=1}^{m} \bar{\mu}_i co\partial^* g_i(\bar{X}) - \bar{U},$$

$$\sum_{i=1}^{p} \bar{\lambda}_i = 1,\ \langle \bar{X}, \bar{U} \rangle = 0,\ \bar{\mu}_i g_i(\bar{X}) = 0,\ \forall\, i \in I.$$

Proof. Since all the conditions of Theorem 3.3 are satisfied for some $\lambda > 0$ and $\mu \geqq 0$, and

$$\begin{aligned} 0 &\in \sum_{i=1}^{p} \lambda_i co\partial^* f_i(\bar{X}) + \sum_{i=1}^{m} \mu_i co\partial^* g_i(\bar{X}) - U \\ \lambda_i &> 0,\ \langle \bar{X}, U \rangle = 0,\ \mu_i g_i(\bar{X}) = 0,\ \forall\, i \in I. \end{aligned} \tag{3.5}$$

Now, dividing (3.5) by $\sum_{i=1}^{p} \lambda_i$ and taking

$$\bar{\lambda}_i = \frac{\lambda_i}{\sum_{i=1}^{p} \lambda_i},\ \bar{\mu}_i = \frac{\mu_i}{\sum_{i=1}^{p} \lambda_i},\ \bar{U} = \frac{U}{\sum_{i=1}^{p} \lambda_i},$$

and $\bar{\lambda}_i > 0$, $\langle \bar{X}, \bar{U} \rangle = 0$ and $\bar{\mu}_i g_i(\bar{X}) = 0$, $\forall i \in I$. Hence, we obtain the desired result. □

Theorem 3.4. *(Sufficient Karush-Kuhn-Tucker optimality conditions) Let f_i $(i \in I_f)$, g_i $(i \in I)$ admit bounded upper semi-regular convexificators at $\bar{X}$. Suppose that $(\bar{X}, \bar{\lambda}, \bar{\mu}, \bar{U})$ satisfy the KKT optimality conditions. If f_i $(i \in I_f)$, g_i $(i \in I)$ are ∂^*-convex, then $\bar{X}$ is a weak efficient solution to the (NSD-MOP).*

Proof. Suppose that $\bar{X}$ is not a weak efficient solution to (NSD-MOP). Then there exists a feasible point Y such that $f_i(Y) < f_i(\bar{X})$, $\forall i \in I_f$. On the other hand, $(\bar{X}, \bar{\lambda}, \bar{\mu}, \bar{U})$ satisfies the KKT conditions. Thus, there exist $\xi_i \in co\partial f_i(\bar{X})$ $(i \in I_f)$, $\eta_i \in co\partial g_i(\bar{X})$ $(i \in I(\bar{X}))$ such that

$$\sum_{i \in I_f} \lambda_i \xi_i + \sum_{i \in I(\bar{X})} \mu_i \eta_i - \bar{U} = 0. \tag{3.6}$$

By the ∂-convexity of functions, we obtain $0 > \bar{\lambda}_i f_i(Y) - \bar{\lambda}_i f_i(\bar{X}) \geqq \langle \bar{\lambda}_i \xi_i, Y - \bar{X} \rangle$, $\forall i \in I_f$, $0 \geqq \bar{\mu}_i g_i(Y) - \bar{\mu}_i g_i(\bar{X}) \geqq \langle \bar{\mu}_i \eta_i, Y - \bar{X} \rangle$, $\forall i \in I$, and $0 \geqq -\langle \bar{U}, Y \rangle + \langle \bar{U}, \bar{X} \rangle \geqq -\langle \bar{U}, Y - \bar{X} \rangle$. From above relations, we reach a contradictions of (3.6). This completes the proof. □

Theorem 3.5. *(Sufficient Karush-Kuhn-Tucker optimality conditions) Let f_i $(i \in I_f)$ and g_i $(i \in I)$, admit bounded upper semi-regular convexificators at $\bar{X}$. Suppose that $(\bar{X}, \lambda, \mu, U)$ satisfies the KKT optimality conditions. If f_i $(i \in I_f)$, are ∂^*-pseudoconvex and g_i $(i \in I)$, are ∂^*-quasiconvex, then $\bar{X}$ is a weak efficient solution to (NSD-MOP).*

Proof. Suppose that $\bar{X}$ is not a weak efficient solution to (NSD-MOP). Then there exists a feasible point Y such that $f_i(Y) < f_i(\bar{X})$, $\forall i \in I_f$. On the other hand, $(\bar{X}, \lambda, \mu, U)$ satisfies the KKT conditions thus there exist $\xi_i \in co\partial f_i(\bar{X})$ $(i \in I_f)$ and $\eta_i \in co\partial g_i(\bar{X})$ $(i \in I(\bar{X}))$ such that $\sum_{i \in I_f} \lambda_i \xi_i + \sum_{i \in I(\bar{X})} \mu_i \eta_i - U = 0$. By the ∂-pseudoconvexity of f_i $(i \in I_f)$, we obtain $\langle \xi_i, Y - \bar{X} \rangle < 0, \forall i \in I_f$. By the ∂-quasi-convexity of g_i $(i \in I)$, we obtain $\langle \eta_i, Y - \bar{X} \rangle \leqq 0, \forall\, i \in I$. Since $\bar{U},\ Y \in \mathbb{S}^n_+$, we have $-\langle U, Y \rangle + \langle U, \bar{X} \rangle = -\langle U, Y - \bar{X} \rangle \leqq 0$. From the inequalities above, we obtain a contradictions. Hence, the result follows. □

4. Duality

In this section, we formulate Wolfe and Mond-Weir dual type models for the NSD-MOP and establish fundamental duality results.

4.1. **Wolfe dual-type model.** Following the lines of [18], we define Wolfe dual type model, and give some duality results

$$\text{(WD-NSD-MOP)} \max \varphi(Y, \lambda, \mu, U), \text{ subject to } (Y, \lambda, \mu, U) \in F,$$

where $\varphi(Y, \lambda, \mu, U) = f(Y) + \mu^T g(Y)\ e - \langle U, Y \rangle\ e,\ e = (1, 1, ..., 1) \in \mathbb{R}^p$, and

$$F = \{(Y, \lambda, \mu, U) \in \mathbb{S}^n_+ \times \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{S}^n_+ : 0 \in \sum_{i=1}^m \lambda_i co\partial^* f_i(Y) + \sum_{i=1}^m \mu_i co\partial^* g_i(Y) - U,$$
$$\lambda > 0,\ \lambda^T e = 1,\ \mu \geqq 0,\ U,\ Y \in \mathbb{S}^n_+\},\ F^1 = \{Y \in \mathbb{S}^n_+ : (Y, \lambda, \mu, U) \in F\}.$$

The following result shows how the feasible points of problems NSD-MOP and corresponding WD-NSD-MOP are connected.

Theorem 4.1. *(Weak duality) Let X and (Y, λ, μ, U) be the feasible point of the NSD-MOP and the WD-NSD-MOP, respectively. If f_i and g_i admit bounded upper semi-regular convexificators and all functions are ∂^*-convex at feasible point Y, then $f(X) \nless \varphi(Y, \lambda, \mu, U)$.*

Proof. Suppose on contrary that $f(X) < f(Y) + \mu^T g(Y)\ e - \langle U, Y \rangle\ e$. Multiplying by $\lambda \in \mathbb{R}^p, \lambda^T e = 1$, we obtain

$$\lambda^T f(X) - \lambda^T f(Y) - \mu^T g(Y) + \langle U, Y \rangle < 0. \tag{4.1}$$

Since all functions are ∂^*-convex at Y, we have $f_i(X) - f_i(Y) \geqq \langle \xi_i^f, X - Y \rangle, \forall \xi_i^f \in co\partial^* f_i(Y)$ $(i \in I_f)$, and $g_i(X) - g_i(Y) \geqq \langle \xi_i^g, X - Y \rangle, \forall \xi_i^g \in co\partial^* g_i(Y)$ $(i \in I)$. These imply that

$$\sum_{i=1}^p \lambda_i f_i(X) + \sum_{i=1}^m \mu_i g_i(X) \geqq \sum_{i=1}^p \lambda_i f_i(Y) + \sum_{i=1}^m \mu_i g_i(Y) + \langle \sum_{i=1}^p \lambda_i \xi_i^f + \sum_{i=1}^m \mu_i \xi_i^g, X - Y \rangle,$$

for all $\xi_i^f \in co\partial^* f_i(Y)$ and $\xi_i^g \in co\partial^* g_i(Y)$, which gives

$$\sum_{i=1}^p \lambda_i f_i(X) + \sum_{i=1}^m \mu_i g_i(X) \geqq \sum_{i=1}^p \lambda_i f_i(Y) + \sum_{i=1}^m \mu_i g_i(Y) + \langle U, X \rangle - \langle U, Y \rangle$$
$$+ \langle \sum_{i=1}^p \lambda_i \xi_i^f + \sum_{i=1}^m \mu_i \xi_i^g - U, X - Y \rangle,$$

for all $\xi_i^f \in co\partial^* f_i(Y)$, $\xi_i^g \in co\partial^* g_i(Y)$, and $U \in \mathbb{S}_+^n$. Since, $g_i(X) \leqq 0$, $\mu_i \geqq 0$, and $\langle U, X\rangle \geqq 0\ \forall\, X, U \in S_+^n$, then we can rewrite as:

$$\sum_{i=1}^{p} \lambda_i f_i(X) \geqq \sum_{i=1}^{p} \lambda_i f_i(Y) + \sum_{i=1}^{m} \mu_i g_i(Y) - \langle U, Y\rangle + \langle \sum_{i=1}^{p} \lambda_i \xi_i^f + \sum_{i=1}^{m} \mu_i \xi_i^g - U,\ X - Y\rangle, \quad (4.2)$$

for all $\xi_i^f \in co\partial^* f_i(Y)$, $\xi_i^g \in co\partial^* g_i(Y)$, and $U \in \mathbb{S}_+^n$. Now, we choose $\bar{\xi}_i^f \in co\partial^* f_i(Y)$, $\bar{\xi}_i^g \in co\partial^* g_i(Y)$, and $\bar{U}$ such that

$$\sum_{i=1}^{p} \lambda_i \bar{\xi}^f + \sum_{i=1}^{m} \mu_j \bar{\xi}_i^g - \bar{U} = 0, \text{ as } 0 \in \sum_{i=1}^{p} \lambda_i co\partial^* f_i(Y) + \sum_{i=1}^{m} \mu_i co\partial^* g_i(Y) - U. \quad (4.3)$$

Substituting (4.3) into (4.2), we obtain $\lambda^T f(X) - \lambda^T f(Y) - \mu^T g(Y) + \langle U, Y\rangle \geqq 0$, which is contradiction to (4.1). Hence, we obtain the desired conclusion. $\square$

The following result provides the sufficient conditions for the existence of the optimal solution of the WD-NSD-MOP with zero duality gap.

Theorem 4.2. *(Strong duality) Let f_i and g_i admit bounded upper semi-regular convexificators, and be ∂^*-convex on $\mathbb{S}^n$. Suppose that $\bar{X}$ is an optimal solution to the NSD-MOP, and satisfies generalized Guignard constraint qualifications at $\bar{X}$. Then there exist $\bar{\lambda} > 0$, $\bar{\mu} \in \mathbb{R}_+^m$, and $\bar{U} \in \mathbb{S}_+^n$ such that $(\bar{X}, \bar{\lambda}, \bar{\mu}, \bar{U})$ solves the WD-NSD-MOP and $f(\bar{X}) = \varphi(\bar{X}, \bar{\lambda}, \bar{\mu}, \bar{U})$.*

Proof. Since, f_i, g_i are ∂^*-convex at $\bar{X}$ and satisfying generalized Guignard constraints qualifications at $\bar{X}$, then from Theorem 3.2, there exist $\bar{\lambda} > 0$, $\bar{\lambda}^T e = 1$, $\bar{\mu} \in \mathbb{R}_+^m$, $\bar{U} \in \mathbb{S}_+^n$, such that $0 \in \sum_{i=1}^{p} \bar{\lambda}_i co\partial^* f_i(\bar{X}) + \sum_{i=1}^{m} \bar{\mu}_i co\partial^* g_i(\bar{X}) - \bar{U}$, $\langle \bar{X}, \bar{U}\rangle = 0$, $\bar{\mu}_i g_i(\bar{X}) = 0, \forall i \in I$. So it is obvious $(\bar{X}, \bar{\lambda}, \bar{\mu}, \bar{U})$ is a feasible point of the WD-NSD-MOP. Let (X, λ, μ, U) be an arbitrary feasible point of the WD-NSD-MOP. From Theorem 4.1, we have $f(\bar{X}) = f(\bar{X}) + \bar{\mu}^T g(\bar{X}) e - \langle \bar{U}, \bar{X}\rangle e \nless f(X) + \mu^T g(X) e - \langle U, X\rangle e$ and $f(\bar{X}) = \varphi(\bar{X}, \bar{\lambda}, \bar{\mu}, \bar{U})$. Hence, we achieve the desired result. $\square$

The following result is important from the viewpoint of applications, as it gives optimal solutions to the NSD-MOP provided that one has a feasible point of the corresponding WD-NSD-MOP with few additional conditions.

Theorem 4.3. *(Converse duality) Let $(\bar{Y}, \bar{\lambda}, \bar{\mu}, \bar{U})$ be a feasible solution to the WD-NSD-MOP such that $\bar{\mu}_i g_i(\bar{Y}) \geqq 0$, $\langle \bar{U}, \bar{Y}\rangle \leqq 0$. If f_i and g_i admit bounded upper semi-regular convexificators, and f_i is ∂^*-convex and g_i is ∂^*-quasiconvex at $\bar{Y}$, then $\bar{Y}$ is a weak efficient solutions to the NSD-MOP.*

Proof. We prove the result by contradiction. On contrary, we assume $\bar{Y}$ is not a weak efficient solution to the NSD-MOP. Therefore, there exist another feasible point Y^*, which is a weak efficient solution, that is, $f_i(Y^*) < f_i(\bar{Y})$. From ∂^*-convexity, we have

$$\langle \xi_i^f, Y^* - \bar{Y}\rangle \leqq f_i(Y^*) - f_i(\bar{Y}) < 0,\ \forall\, \xi_i^f \in co\partial^* f_i(\bar{Y}). \quad (4.4)$$

$$\bar{\mu}_i g_i(Y^*) \leqq 0 \leqq \bar{\mu}_i g_i(\bar{Y}) \implies \bar{\mu}_i \langle \xi_i^g, Y^* - \bar{Y}\rangle \leqq 0,\ \forall\, \xi_i^g \in co\partial^* g_i(\bar{Y}), \quad (4.5)$$

and

$$\langle -\bar{U}, Y^* - \bar{Y}\rangle = -\langle \bar{U}, Y^*\rangle + \langle \bar{U}, \bar{Y}\rangle \leqq 0. \quad (4.6)$$

Now, multiplying (4.4) by $\bar{\lambda}_i > 0$ and adding (4.5) and (4.6) yield that

$$\langle \sum_{i=1}^{p} \bar{\lambda}_i \xi_i^f + \sum_{j=1}^{m} \bar{\mu}_j \xi_j^g - \bar{U}, Y^* - \bar{Y} \rangle < 0, \ \forall\ \xi_i^f \in co\partial^* f_i(\bar{Y}),\ \forall\ \xi_i^g \in co\partial^* g_i(\bar{Y}),$$

which contradicts the fact that $(\bar{Y}, \bar{\lambda}, \bar{\mu}, \bar{U})$ is a feasible point of the WD-NSD-MOP. Thus, $\bar{Y}$ is a weak efficient solution to the NSD-MOP. This completes the proof. □

The beauty of the following result is, without being an optimal solution to the NSD-MOP, one can not obtain the zero duality gap.

Theorem 4.4. *(Restricted converse duality) Let $\bar{X}$, $(\bar{Y}, \bar{\lambda}, \bar{\mu}, \bar{U})$ be feasible solutions of the NSD-MOP and the WD-NSD-MOP, respectively, and $f(\bar{X}) = \varphi(\bar{Y}, \bar{\lambda}, \bar{\mu}, \bar{U})$. If f_i and g_i admit bounded upper semi-regular convexificators and are ∂^*-convex at $\bar{Y}$, then $\bar{X}$ is a weak efficient solution to the NSD-MOP.*

Proof. We suppose that $\bar{X}$ is not a weak efficient solution of the NSD-MOP. Then, there exists a feasible point X^* of $NSD-MOP$ such that $f(X^*) < f(\bar{X})$, which implies that $f(X^*) < f(\bar{X}) = \varphi(\bar{Y}, \bar{\lambda}, \bar{\mu}, \bar{U})$. Hence, the weak duality theorem is contradicted. Thus, we obtain the desired result. □

The following result provides sufficient conditions under which, an optimal solution of the NSD-MOP coincides with first coordinate of an optimal solution of the corresponding WD-NSD-MOP.

Theorem 4.5. *(Strict converse duality) Let $\bar{X}$ be a local weak efficient solution of the NSD-MOP such that the GGCQ be satisfied at $\bar{X}$. Let $(\bar{Y}, \bar{\lambda}, \bar{\mu}, \bar{U})$ be a global weak efficient solution of the WD-NSD-MOP. If f_i and g_i admit bounded upper semi-regular convexificators and f_i is strictly ∂^*-convex and g_i is ∂^*-convex, then $\bar{X} = \bar{Y}$.*

Proof. Suppose that $\bar{X} \neq \bar{Y}$. Since $\bar{X}$ is a local weak efficient solution to the NSD-MOP, and satisfies the GGCQ, we find from thestrong duality result that there exist λ^*, μ^*, U^* such that $f(\bar{X}) = \varphi(\bar{X}, \lambda^*, \mu^*, U^*) = \varphi(\bar{Y}, \bar{\lambda}, \bar{\mu}, \bar{U})$, $f(\bar{X}) = f(\bar{Y}) + \bar{\mu}^T g(\bar{Y})\ e - \langle \bar{U}, \bar{Y} \rangle\ e$, and $\bar{\lambda}^T f(\bar{X}) = \bar{\lambda}^T f(\bar{Y}) + \bar{\mu}^T g(\bar{Y}) - \langle \bar{U}, \bar{Y} \rangle$. In view of the following facts $f_i(\bar{X}) - f_i(\bar{Y}) > \langle \xi^f, \bar{X} - \bar{Y} \rangle, \forall \xi_i^f \in co\partial^* f_i(\bar{Y}), \forall i \in I_f$, $g_i(\bar{X}) - g_i(\bar{Y}) \geqq \langle \xi_i^g, \bar{X} - \bar{Y} \rangle, \forall \xi_i^g \in co\partial^* g_i(\bar{Y}), i \in I$, and $-\langle \bar{U}, \bar{X} \rangle + \langle \bar{U}, \bar{Y} \rangle = \langle -\bar{U}, \bar{X} - \bar{Y} \rangle$, we conclude

$$\begin{aligned}\sum_{i=1}^{p} \bar{\lambda}_i f_i(\bar{X}) + \sum_{i=1}^{m} \bar{\mu}_i g_i(\bar{X}) - \langle \bar{U}, \bar{X} \rangle > & \sum_{i=1}^{p} \bar{\lambda}_i f_i(\bar{Y}) + \sum_{i=1}^{m} \bar{\mu}_i g_i(\bar{Y}) - \langle \bar{U}, \bar{Y} \rangle \\ & + \langle \sum_{i=1}^{p} \bar{\lambda}_i \xi_i^f + \sum_{i=1}^{m} \bar{\mu}_i \xi_i^g - \bar{U}, \bar{X} - \bar{Y} \rangle,\end{aligned}$$

for all $\xi_i^f \in co\partial^* f_i(\bar{Y})$ and $\xi_i^g \in co\partial^* g_i(\bar{Y})$. Since, $\bar{\mu}_i g_i(\bar{X}) \leq 0$, $-\langle \bar{U}, \bar{X} \rangle \leq 0$, and $(\bar{Y}, \bar{\lambda}, \bar{\mu}, \bar{U})$ is a feasible solutions, we obtain $\bar{\lambda}^T f(\bar{X}) > \bar{\lambda}^T f(\bar{Y}) + \bar{\mu}^T g(\bar{Y}) - \langle \bar{U}, \bar{Y} \rangle$, which is a contradiction. Thus, we have the desired result. □

4.2. **Mond-Weir dual-type model.** In this subsection, following the idea of [24], we introduce the Mond-Weir dual type model, and present various duality results.

$(MWD-NSD-MOP)$ max $f(Y)$ subject to $(Y,\lambda,\mu,U) \in F_{MWD}$, where

$$F_{MWD} = \{(Y,\lambda,\mu,U) \in \mathbb{S}^n_+ \times \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{S}^n_+ : 0 \in \sum_{i=1}^{p} \lambda_i co\partial^* f_i(Y) + \sum_{i=1}^{m} \mu_i co\partial^* g_i(Y) - U,$$

$$\lambda > 0,\ \mu \geqq 0,\ \mu^T g(Y) \geqq 0,\ \langle U,Y\rangle = 0\},\ F^1_{MWD} = \{Y : Y \in F_{MWD}\}.$$

The following result connects a feasible point of the NSD-MOP and a feasible point of the MWD-NSD-MOP.

Theorem 4.6. *(Weak duality) Let f_i and g_i admit bounded upper semi-regular convexificators at Y. Let $X, (Y,\lambda,\mu,U)$ be feasible points of the NSD-MOP and the MWD-NSD-MOP, respectively. If f_i is ∂^*-pseudoconvex and g_i is ∂^*-quasiconvex at feasible point Y, then $f(X) \not< f(Y)$.*

Proof. Suppose that $f(X) < f(Y) \implies f_i(X) < f_i(Y),\ i \in I_f$. Since f_i are ∂^*-pseudoconvex at Y, then $\langle \xi_i^f, X-Y\rangle < 0,\ \forall \xi_i^f \in co\partial^* f_i(Y),\ i \in I_f$. From the ∂^*-quasiconvexity of g_i at Y, we have that $\mu^T g(X) \leqq \mu^T g(Y)$ implies $\langle \sum_{i=1}^m \mu_i \xi_i^g, X-Y\rangle \leqq 0,\ \forall \xi_i^g \in co\partial^* g_i(Y)$. Observe that $\langle -U, X-Y\rangle = -\langle U,X\rangle + \langle U,Y\rangle \leqq 0,\ \forall (Y,\mu,U) \in F_{MWD}$ and $\forall X \in \mathbb{S}^n_+$. Hence, $\langle \sum_{i=1}^p \lambda_i \xi_i^f + \sum_{i=1}^m \mu_i \xi_i^g - U, X-Y\rangle < 0,\ \forall \xi_i^f \in co\partial^* f_i(Y),\ \forall \xi_i^g \in co\partial^* g_i(Y),\ \forall (Y,\lambda,\mu,U) \in F_{MWD}$, and $\forall X \in \mathbb{S}^n_+$, which contradicts the feasibility of $(Y,\lambda,\mu,U) \in F_{MWD}$. Thus, we obtain the desired result. □

The following result is known as the strong duality theorem, which gives the sufficient conditions for the existence of optimal solutions of the MWD-NSDP with zero duality gaps.

Theorem 4.7. *(Strong duality) Let f_i and g_i admit bounded upper semi-regular convexificators at $\bar{X}$. If f_i and g_i are ∂^*-pseudoconvex and ∂^*-quasiconvex, respectively, at $\bar{X}$, which solves the NSD-MOP, and let the (GGCQ) be satisfied at $\bar{X}$. Then there exist $\bar{\lambda} \in \mathbb{R}^p_{++}$, $\bar{\mu} \in \mathbb{R}^m_+$ and $\bar{U} \in \mathbb{S}^n_+$ such that $(\bar{X},\bar{\lambda},\bar{\mu},\bar{U})$ is a solution of the MWD-NSD-MOP and the values of both objective functions are equal.*

Proof. Since the generalized Guignard constraint qualification is satisfied at $\bar{X}$, then from Theorem 4.6 there exist $\bar{\lambda} \in \mathbb{R}^m_{++}$, $\bar{\mu} \in \mathbb{R}^m_+$, and $\bar{U} \in \mathbb{S}^n_+$ such that $0 \in \sum_{i=1}^p \bar{\lambda}_i co\partial^* f_i(\bar{X}) + \sum_{i=1}^m \bar{\mu}_i$ $co\partial^* g_i(\bar{X}) - \bar{U}$, $\langle \bar{X},\bar{U}\rangle = 0$, $\bar{\mu}_i g_i(\bar{X}) = 0,\ \forall i \in I$. That is, $(\bar{X},\bar{\lambda},\bar{\mu},\bar{U})$ is the feasible solution of the MWD-NSD-MOP. So, from weak duality Theorem 4.6, $(\bar{X},\bar{\lambda},\bar{\mu},\bar{U})$ is a weak efficient solution to the MWD-NSD-MOP, and both objectives have same value. Hence, we obtain the desired conclusion. □

The following result provides sufficient conditions for the first component of the feasible solution of the MWD-NSD-MOP to be an optimal solution of the corresponding NSD-MOP.

Theorem 4.8. *(Converse duality) Let f_i and g_i admit bounded upper semi-regular convexificators at $\bar{Y}$, and let $(\bar{Y},\bar{\lambda},\bar{\mu},\bar{U})$ be a feasible solution of the MWD-NSD-MOP with conditions $\bar{\mu}_i g_i(\bar{Y}) \geqq 0$, $\langle \bar{U},\bar{Y}\rangle \leqq 0$. If f_i is ∂^*-pseudoconvex and g_i is ∂-quasiconvex, then $\bar{Y}$ is a weak efficient solutions to the NSD-MOP.*

Proof. Assume that $\bar{Y}$ is not an optimal solution of the NSD-MOP. Therefore, there exist another feasible point Y^*, which is an optimal solutions, that is, $f(Y^*) < f(\bar{Y}) \implies f_i(Y^*) < f_i(Y),\ i \in$

I_f. From the ∂^*-pseudoconvexity and the ∂^*-quasiconvexity, we have $\langle \xi_i^f, Y^* - \bar{Y} \rangle < 0, \forall \xi_i^f \in co\partial^* f_i(\bar{Y}), i \in I_f$, $\bar{\mu}_i g_i(Y^*) \leqq 0 \leqq \bar{\mu}_i g_i(\bar{Y}) \implies \bar{\mu}_i \langle \xi_i^g, Y^* - \bar{Y} \rangle \leqq 0, \forall \xi_i^g \in co\partial^* g_i(\bar{Y}), i \in I$, and $\langle -\bar{U}, Y^* - \bar{Y} \rangle = -\langle \bar{U}, Y^* \rangle + \langle \bar{U}, \bar{Y} \rangle \leqq 0$. Hence, we have $\langle \sum_{i=1}^p \bar{\lambda}_i \xi_i^f + \sum_{i=1}^m \bar{\mu}_i \xi_i^g - \bar{U}, Y^* - \bar{Y} \rangle < 0, \forall \xi_i^f \in co\partial^* f_i(\bar{Y}), \forall\ \xi_i^g \in co\partial^* g_i(\bar{Y})$, which contradicts that $(\bar{Y}, \bar{\lambda}, \bar{\mu}, \bar{U})$ is a feasible point of the MWD-NSD-MOP. Hence, $\bar{Y}$ is a weak efficient solution of the NSD-MOP. □

The following result gives conditions for a feasible solution of the NSD-MOP to be the optimal solution.

Theorem 4.9. *(Restricted converse duality) Let f_i, g_i admit bounded upper semi-regular convexificators at $\bar{Y}$. Let $\bar{X}$, $(\bar{Y}, \bar{\lambda}, \bar{\mu}, \bar{U})$ be feasible solutions of the NSD-MOP, the MWD-NSD-MOP, respectively with $f(\bar{X}) = f(\bar{Y})$. If f_i is ∂^*-pseudoconvex and g_i is ∂^*-quasiconvex at $\bar{Y}$, then $\bar{X}$ is a weak efficient solution to the NSD-MOP.*

Proof. Suppose that $\bar{X}$ is not a weak efficient solution of the NSD-MOP. Then, there exist a feasible point X^* of the NSD-MOP such that $f(X^*) < f(\bar{X})$, which implies that $f(X^*) < f(\bar{X}) = f(\bar{Y})$. Hence, the weak duality theorem is contradicted. Thus, we have the desired result. □

Theorem 4.10. *(Strict converse duality) Let f_i, g_i admit bounded upper semi-regular convexificators at $\bar{Y}$. Let $\bar{X}$ be a local weak efficient solution of the NSD-MOP such that the GGCQ is satisfies at $\bar{X}$. Let $(\bar{Y}, \bar{\lambda}, \bar{\mu}, \bar{U})$ be a global weak efficient solution of the MWD-NSD-MOP. If f_i is strictly ∂^*-pseudoconvex and g_i is ∂^*-quasiconvex, then $\bar{X} = \bar{Y}$.*

Proof. Suppose that $\bar{X} \neq \bar{Y}$. Since $\bar{X}$ is a local weak efficient solution of the NSD-MOP and satisfies the GGCQ, we find from the strong duality result that there exist λ^*, μ^*, and U^* such that $(\bar{X}, \lambda^*, \mu^*, U^*)$ is a feasible solution of the MWD-NSD-MOP, but it is given that $(\bar{Y}, \bar{\lambda}, \bar{\mu}, \bar{U})$ is a global weak efficient solution of the MWD-NSD-MOP. Hence, using the weak duality theorem, we conclude that $f(\bar{X}) = f(\bar{Y})$. Observe that $g_i(\bar{X}) \leqq 0 \leqq g_i(\bar{Y}) \implies \langle \xi_i^g, \bar{X} - \bar{Y} \rangle \leqq 0, \forall \xi_i^g \in co\partial^* g_i(\bar{Y})$, and $\langle -\bar{U}, \bar{X} - \bar{Y} \rangle = -\langle \bar{U}, \bar{X} \rangle + \langle \bar{U}, \bar{Y} \rangle \leqq 0$. Hence, $\langle \sum_{i=1}^m \bar{\mu}_i \xi_i^g - \bar{U}, \bar{X} - \bar{Y} \rangle \leqq 0, \forall \xi_i^g \in co\partial^* g_i(\bar{Y})$. Since $(\bar{Y}, \bar{\lambda}, \bar{\mu}, \bar{U})$ is a feasible solutions of the MWD-NSD-MOP, then $\langle \xi_i^f, \bar{X} - \bar{Y} \rangle \geqq 0, \forall \xi_i^f \in co\partial^* f_i(\bar{Y}),\ i \in I_f$, which shows that $f_i(\bar{X}) > f_i(\bar{Y}), i \in I_f \implies f(\bar{X}) > f(\bar{Y})$. Thus, we obtain a contradiction. Hence, the result follows. □

The following semidefinite programming problem supports the result that we proved in Theorem 4.3.

Example 4.1. Consider the following problem

$$min\ (f_1(X) = x_1,\ f_2(X) = x_3),\ \text{subject to}\ g(X) = -2x_2\ \leqq 0,\ X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \in \mathbb{S}_+^2.$$

We have the feasible set $S = \left\{ \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} : x_1 \geqq 0,\ x_2 \geqq 0,\ x_3 \geqq 0,\ x_1 x_3 - x_2^2 \geqq 0 \right\}$ and

$$\partial^* f_1(Y) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\},\ \partial^* f_2(Y) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\},\ \partial^* g(Y) = \left\{ \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right\},\ \forall\ Y \in \mathbb{S}_+^2.$$

We observe that all functions are ∂^*-convex for all $Y \in \mathbb{S}_+^n$. For $\lambda_1 = \frac{1}{2},\ \lambda_2 = \frac{1}{2},\ \mu = 0,\ Y = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{S}_+^2,\ U = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \in \mathbb{S}_+^2$, we have $\mu g(Y) \geqq 0$ and $\langle U, Y \rangle \leqq 0$. Now, it is easy to see that $0 \in \sum_{i=1}^{2} \lambda_i co\partial^* f_i(Y) + \mu co\partial^* g(Y) - U$. Thus, we conclude that (Y, λ, μ, U) is a feasible point of the Wolfe dual model. Therefore, from Theorem 4.3, $Y = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a weak efficient solution of the problem.

5. Conclusions

In this paper, we established necessary and sufficient optimality conditions under new proposed constraints qualifications, and formulated Wolfe and Mond-Weir type dual models to the primal nonlinear semidefinite multiobjective programming problems. We also established weak, strong, converse, restricted converse, and strict converse duality results under the assumptions of generalized convexity. Furthermore, we included examples to verify our results.

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