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STRATEGIC DECISION IN A TWO-PERIOD GAME USING A MULTI-LEADER-FOLLOWER APPROACH. PART 2 - DECISION-MAKING FOR THE NEW PLAYER

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Abstract. In Part 1 of this couple of papers, we introduced the notion of *weighted generalized Nash problem* and proved the uniqueness of the solution. As a result, multi-leader-follower games and the player-expanded generalized Nash equilibrium problem were rigorously established and well-posed. In this part, we combine our studies of those equilibria and the decision concepts to find the most advantageous strategy for player $n+1$, who enters in the game already played by n players. Under different hypotheses on the exchange volume, consumption bounds, and cost functions, we determine the favourable strategies for player $n+1$, which builds a decision-making policy. Numerical simulations are also provided.

Keywords. Concavity; Multi-Leader-Single-Follower game; Nash Equilibrium Problem; Single-Leader-Multi-Follower game; Two-period game.

1. INTRODUCTION

As in the first part of this couple of papers [1], the general setup of this work has been described as follows: while n players interact with each other in a non-cooperative Nash game, namely in a GNEP$_n$, a new player $n+1$ appears and wants to join the game. Obviously, this is not a symmetric circumstance as it leads to different possible interactions between a group of n players and a single player. The simplier case involving only two players has been studied by B. von Stengel [2].

The main purpose is to provide the player $n+1$ a "good way" to access the game more productively. In [1], we conducted the basic comparison of the different possible payoffs for player $n+1$ and we introduced the concepts of weighted Nash equilibrium and of "safe/optimal/neutral/beneficial" decisions.

In this part, using the results of the first part, we focus now on determining the strategic decisions of player $n+1$ when he decides to take part in the game. Assuming that the utility function of each player i is a diagonally strictly concave function, we investigate under which conditions we can guarantee "good" payoffs to player $n+1$ depending on the decisions to play

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in period 1 or 2 and on the information he can obtain on the cost structure of the opponent group that has already proceeded a GNEP$_n$ between n players. Note that in this second part, we will assume, for technical reasons, that there is only one commodity in the market. Notations of [1] are thus adapted accordingly.

Since it is hard to know the way the opponents will react, the goal is to drive player $n+1$ to a suitable/favourable plan even without knowing the intentions of other n players, more specifically at which period they want to play (period 1 or 2). Finally, in case player $n+1$ has perfect information about the time period in which the n players will enter the game, the best strategies for player $n+1$ are also exploited.

The paper is organized as follows. In Section 2, a brief modification for the three games (SLMF, GNEP, and MLSF) is given while recalling fundamental settings and hypotheses. By using the preliminary analysis of [1], in Section 3, the strategic decisions of player $n+1$ are fully discussed for all possible scenarios. Finally, in Section 4, some illustrations and numerical results are provided for a specific model by varying strategic parameters such as production costs, exchange volume, price, etc.

2. Two-Period Games: A Multi-Leader-Follower Approach

This section treats only with a single-commodity market, that is with $(q=1)$. Then, the notations are simplified by eliminating the index of commodity l. To prepare for the next section, let us recall some adaptive notions from previous sections, the three different models that raise from this two-period game context:

- if both players $\{1,\dots,n\}$ and player $n+1$ decide to play the same period (1 or 2), then they interact through a generalized Nash game, GNEP$_{n+1}$;
- if player $n+1$ decides to play period 1 while the group $\{1,\dots,n\}$ opts for period 2, then a single-leader-multi-follower game will be played;
- if the group $\{1,\dots,n\}$ decides to play period 1 while player $n+1$ plays period 2 then it will be a multi-leader-single-follower game.

In each of these three cases, a generalized Nash game -possibly parametrized- will be considered, either with n or $n+1$ players. As explained in [1], we propose to consider a selection process for the resulting generalized Nash equilibrium. This selection process is based on the concept of *weighted Nash equilibrium* which we recall below:

Definition 2.1. Consider that $n+1$ players are interacting on a market with a maximum exchange volume $\beta \in \mathbb{R}_+^*$ such that $\sum_{i=1}^{n+1} \bar{X}_i \neq \beta$. Let $\mathscr{W} = \{w_1,\dots,w_{n+1}\}$ be a given family of weights of the players satisfying the following conditions

$$\begin{cases} \text{for any } i,\ w_i \in \left[0, \dfrac{\bar{X}_i}{\left|\sum_{i=1}^{n+1} \bar{X}_i - \beta\right|}\right[\\ \text{and } \sum_{i=1}^{n} w_i = 1. \end{cases} \tag{2.1}$$

Then, for $p = n$ or $p = n+1$ and any pre-booking vector $\delta \in [0,\beta]$, the *weighted generalized Nash equilibrium problem* $\mathrm{GNEP}_p^w(\beta-\delta)$ consists in:

$$\begin{array}{l}\text{Finding } \bar{x} = (\bar{x}_1,\dots,\bar{x}_p) \in \mathbb{R}^p \text{ such that } ,\forall i = 1,\dots,p,\ \bar{x}_i \text{ is a solution}\\ \text{of the problem of player } i\\ \qquad \left(P_i^w(\bar{x}_{-i})\right) \qquad \max\limits_{x_i}\ \theta_i(x_i,\bar{x}_{-i})\\ \qquad\qquad\qquad\qquad \text{s.t.}\quad x_i \in \left[0,\overline{X}_i^w(p,\beta-\delta)\right],\end{array}$$

where the weighted consumption bounds $\overline{X}_i^w$ are defined as follows

- For a generalized Nash game between players $\{1,\dots,n\}$,

$$\begin{aligned}\overline{X}_i^w(n,\delta) &= \overline{X}_i - w_i \max\left\{0, \left[\sum_{j=1}^n \overline{X}_j - (\beta-\delta)\right]\right\}\\ &= \begin{cases}\overline{X}_i & \text{if } \sum_{j=1}^n \overline{X}_j \le \beta-\delta,\\ \overline{X}_i - w_i\left[\sum_{j=1}^n \overline{X}_j - (\beta-\delta)\right] & \text{otherwise;}\end{cases}\end{aligned}$$

- For a generalized Nash game between players $\{1,\dots,n+1\}$,

$$\begin{aligned}\overline{X}_i^w(n+1,\delta) &= \overline{X}_i - w_i \max\left\{0, \left[\sum_{j=1}^{n+1} \overline{X}_j - (\beta-\delta)\right]\right\}\\ &= \begin{cases}\overline{X}_i & \text{if } \sum_{j=1}^{n+1} \overline{X}_j \le \beta-\delta,\\ \overline{X}_i - w_i\left[\sum_{j=1}^{n+1} \overline{X}_j - (\beta-\delta)\right] & \text{otherwise.}\end{cases}\end{aligned}$$

The equilibria $\bar{x} = (\bar{x}_1,\dots,\bar{x}_p)$ are called weighted generalized Nash equilibria and their set will be denoted by $\mathrm{GNE}_p^w(\beta-\delta)$. Then according to [1, Proposition 3.4 and Proposition 3.7], player $n+1$ will thus face one of the three following models $\mathrm{SLMF}_{n+1}^w(\beta)$, $\mathrm{GNEP}_{n+1}^w(\beta)$, and $\mathrm{MLSF}_{n+1}^w(\beta)$:

a) **The weighted single-leader-multi-follower game $\mathbf{SLMF}_{n+1}^w(\beta)$**:
An $(n+1)$-player single-leader-multi-follower game is defined as

$$\left(\bar{P}^w(\beta)\right) \qquad \max_{x_{n+1}}\ \theta_{n+1}(x_{n+1},x_{-(n+1)})$$
$$\text{s.t.}\quad \begin{cases} 0 \le x_{n+1} \le \overline{X}_{n+1},\\ (x_1^w,\dots x_n^w) = Eq^w(\beta - x_{n+1}),\end{cases}$$

where $Eq^w(\beta - x_{n+1})$ is the unique weighted Nash equilibrium of $\mathrm{GNEP}_n^w(\beta - x_{n+1})$, defined by

$$\forall i = 1,\dots,n,\ \left(\bar{P}_i^w(\beta - x_{n+1})\right) \qquad \max_{x_i}\ \theta_i(x_i,x_{-i})$$
$$\text{s.t.}\quad x_i \in \left[0,\overline{X}_i^w(n,x_{n+1})\right].$$

b) **The weighted generalized Nash equilibrium problem $\mathbf{GNEP}_{n+1}^w(\beta)$**:
An $(n+1)$-player generalized Nash game is defined as

$$\forall i = 1,\dots,n+1,\ \left(\widetilde{P}_i^w(\beta)\right) \qquad \max_{x_i}\ \theta_i(x_i,x_{-i})$$
$$\text{s.t.}\quad x_i \in \left[0,\overline{X}_i^w(n+1,0)\right].$$

c) **The weighted multi-leader-single-follower game $\mathrm{MLSF}^w_{n+1}(\beta)$**:
An $(n+1)$-player multi-leader-single-follower game is defined as

$$\forall i=1,\dots,n,\ \big(\widehat{P}^w_i(\beta)\big) \qquad \max_{x_i}\ \theta_i(x_i,x_{-i})$$
$$\text{s.t.} \begin{cases} 0\le x_i\le \bar{X}^w_i(n,0), \\ x_{n+1} \text{ solves } \big(\widehat{P}(\beta-\sum_{j=1}^n x_j)\big), \end{cases}$$

where

$$\big(\widehat{P}^w(\beta-\sum_{j=1}^n x_j)\big) \qquad \max_{x_{n+1}}\ \theta_{n+1}(x_{n+1},x_{-(n+1)})$$
$$\text{s.t.} \begin{cases} 0\le x_{n+1}\le \bar{X}_{n+1}, \\ \sum_{i=1}^{n+1} x_i\le \beta. \end{cases}$$

Let us adopt the well-posedness and uniqueness assumptions to weighted models as the following assumptions 2.1 and 2.2.

Assumption 2.1 (Well-posedness)**.** For the considered maximum exchange volume $\beta\in\mathbb{R}^*_+$, each of the three problems $\mathrm{SLMF}^w_{n+1}(\beta)$, $\mathrm{GNEP}^w_{n+1}(\beta)$, and $\mathrm{MLSF}^w_{n+1}(\beta)$ are assumed to be well-posed, that is,

a) for each possible value of x_{n+1} the equilibrium problem $\mathrm{GNEP}^w_n(\beta-x_{n+1})$ admits at least an equilibrium;
b) $\mathrm{GNEP}^w_{n+1}(\beta)$ admits at least a generalized Nash equilibrium;
c) for any $(x_1,\dots,x_n)\in\prod_{i=1}^n[0,\bar{X}_i]$, the lower-level problem $\big(\widehat{P}^w(\beta,x_1,\dots,x_n)\big)$ admits a unique solution.

Assumption 2.2 (Uniqueness)**.** For the considered maximum exchange volume $\beta\in\mathbb{R}^*_+$, each of the three problems $\mathrm{SLMF}^w_{n+1}(\beta)$, $\mathrm{GNEP}^w_{n+1}(\beta)$ and $\mathrm{MLSF}^w_{n+1}(\beta)$ admits at most a solution $\bar{x}=(\bar{x}_1,\dots,\bar{x}_{n+1})$.

The uniqueness in Assumption 2.2 for $\mathrm{SLMF}^w_{n+1}(\beta)$ can be satisfied by combining a diagonally strict quasi-concavity hypothesis of function $\theta_{n+1}(\cdot,x_{-(n+1)})$ with [1, Proposition 3.4]. The unique solution of $\mathrm{SLMF}^w_{n+1}(\beta)$ will be denoted by $(\bar{x}^L_{n+1},\bar{x}^F_1,\dots,\bar{x}^F_n)$ while the payoff of player $n+1$ will thus be $P^L_{n+1}(\beta)$.

Similarly, for $\mathrm{GNEP}^w_{n+1}(\beta)$, the uniqueness can be satisfied simply through [1, Proposition 3.4]. It will be denoted by $(\bar{x}^G_1,\dots,\bar{x}^G_n,\bar{x}^G_{n+1})$ the unique solution of $\mathrm{GNEP}^w_{n+1}(\beta)$, while the payoff of player $n+1$ will thus be $P^G_{n+1}(\beta)$.

Finally, the uniqueness of *best response* x_{n+1} of player $n+1$ in game $\mathrm{MLSF}^w_{n+1}(\beta)$ can be obtained by a diagonally strict quasi-concavity hypothesis of function $\theta_{n+1}(\cdot,x_{-(n+1)})$ while the uniqueness of the equilibrium of the upper level generalized Nash game can be inferred from [1, Proposition 3.4]. Hence, $\mathrm{MLSF}^w_{n+1}(\beta)$ admits a unique solution, which is denoted by $(\bar{x}^L_1,\dots,\bar{x}^L_n,\bar{x}^F_{n+1})$, and the payoff of player $n+1$ is $P^F_{n+1}(\beta)$.

3. Strategic Decision of the Two-Period Game

Rebounding on the preliminary analysis done in [1], in particular, [1, Proposition 3.7], [1, Proposition 2.5] and [1, Proposition 2.8], our aim in this section is to analyse the "favourable" strategies for player $n+1$, only basing our decision-making on the given constants of the problem. See [1, definitions 2.3 and 2.6] for the classification/terminology of player $(n+1)$'s strategies.

Theorem 3.1. *Consider that $n+1$ players are interacting on a market with a maximum exchange volume $\beta \in \mathbb{R}_+^*$ such that $\sum_{i=1}^{n+1} \overline{X}_i \neq \beta$. Let $\mathscr{W} = \{w_1, \ldots, w_{n+1}\}$ be a given family of weights of the players satisfying conditions (2.1). Assume that, for any $i = 1, \ldots, n+1$,*

- *the function $\theta_i(\cdot, x_{-i})$ is continuous in both variables x_i and x_{-i};*
- *for any $x_{-i} \in \prod_{\substack{k=1\\k\neq i}}^{n+1} [0, \overline{X}_k]$, the function $\theta_i(\cdot, x_{-i})$ is diagonally strictly concave and $\operatorname{argmax}_{x_i \in \mathbb{R}_+} \theta_i(x_i, x_{-i})$ is a singleton (denoted by $\{x_i^\star\}$).*

Then, by setting $\chi^F = \beta - \sum_{i=1}^{n} \min\left\{x_i^\star, \overline{X}_i^w(n,0)\right\}$, the following assertions hold:

i) If $\chi^F < \min\left\{x_{n+1}^\star, \overline{X}_{n+1}\right\} \leq \overline{X}_{n+1}^w(n+1,0)$, then an optimal strategy for player $n+1$ is to play in period 1.
ii) If $\chi^F < \overline{X}_{n+1}^w(n+1,0) < \min\left\{x_{n+1}^\star, \overline{X}_{n+1}\right\}$, then a safe strategy for player $n+1$ is to play in period 1.
iii) If $\overline{X}_{n+1}^w(n+1,0) < \min\left\{x_{n+1}^\star, \overline{X}_{n+1}\right\} \leq \chi^F$, then two most beneficial strategies for player $n+1$ are to be in $\mathrm{SLMF}_{n+1}^w(\beta)$ or to be in $\mathrm{MLSF}_{n+1}^w(\beta)$.
iv) If $\overline{X}_{n+1}^w(n+1,0) = \chi^F < \min\left\{x_{n+1}^\star, \overline{X}_{n+1}\right\}$, then a most beneficial strategy for player $n+1$ is to be in $\mathrm{SLMF}_{n+1}^w(\beta)$.
v) If $\overline{X}_{n+1}^w(n+1,0) < \chi^F < \min\left\{x_{n+1}^\star, \overline{X}_{n+1}\right\}$, then a most beneficial strategy for player $n+1$ is to be in $\mathrm{SLMF}_{n+1}^w(\beta)$ and a least beneficial strategy is to be in $\mathrm{GNEP}_{n+1}^w(\beta)$.
vi) Otherwise, any decision of player $n+1$ (that is to play period 1 or period 2) is a neutral strategy.

Regarding cases (iii)-(v), the conclusion is less precise since one cannot advise an optimal or safe or neutral strategy for player $n+1$. It could be understood in the sense that player $n+1$ would need some additional information from the group of players $\{1, \ldots, n\}$ to be able to elaborate a more favourable strategy. For example in case (v), player $n+1$ could choose the most favourable by playing period 1 if he knows that the group will avoid to play a GNEP_{n+1}^w game.

As quoted above, the proof of Theorem 3.1 is an essential consequence of [1, Proposition 3.7], [1, Proposition 2.5], and [1, Proposition 2.8].

Proof. From [1, Proposition 3.7], one obtains that

$$\begin{aligned}
\bar{x}_{n+1}^L &= \min\left\{x_{n+1}^\star, \overline{X}_{n+1}\right\},\\
\bar{x}_{n+1}^G &= \min\left\{x_{n+1}^\star, \overline{X}_{n+1}^w(n+1,0)\right\},\\
\bar{x}_{n+1}^F &= \min\left\{x_{n+1}^\star, \overline{X}_{n+1}, \chi^F\right\}.
\end{aligned}$$

From Definition 2.1, one always has $\bar{x}_{n+1}^G \leq \bar{x}_{n+1}^L$. Let us now consider the different possible inequalities between the data and deduce, when possible, the favourable strategy for player $n+1$.

i) From the left inequality, one can immediately deduce that $\bar{x}_{n+1}^F = \chi^F < \bar{x}_{n+1}^L \leq x_{n+1}^\star$. Since $\theta_i(\cdot, x_{-i})$ is diagonally strictly concave and $\operatorname{argmax}_{x_i \in \mathbb{R}_+} \theta_{n+1}(x_{n+1}, x_{-i}) = \{x_{n+1}^\star\}$,

$\theta_{n+1}(\cdot,x_{-i})$ is increasing on $[0,x^{\star}_{n+1}]$ and thus $P^F_{n+1} < P^L_{n+1}$. On the other hand, from the right side inequality, one can easily deduce that $\bar{x}^L_{n+1} = \bar{x}^G_{n+1}$. Thus $P^F_{n+1} < P^L_{n+1} = P^G_{n+1}$. And the conclusion follows from [1, Proposition 2.5 (i)];

$ii)$ In this case, the two strict inequalities show us $\bar{x}^F_{n+1} < \bar{x}^G_{n+1} < \bar{x}^L_{n+1} \leq x^{\star}_{n+1}$. Using the fact that $\theta_{n+1}(\cdot,x_{-i})$ is increasing on $[0,x^{\star}_{n+1}]$, one deduces that $P^F_{n+1} < P^G_{n+1} < P^L_{n+1}$ and, according to [1, Proposition 2.5 (i)] that a safe strategy for the player $n+1$ is to play in period 1;

$iii)$ From $\min\{x^{\star}_{n+1},\bar{X}_{n+1}\} \leq \chi^F$, one has $\bar{x}^F_{n+1} = \bar{x}^L_{n+1}$. Moreover, $\bar{X}^w_{n+1}(n+1,0)$ is strictly less than $\min\{x^{\star}_{n+1},\bar{X}_{n+1}\}$. Therefore $\bar{x}^G_{n+1} < \bar{x}^F_{n+1} = \bar{x}^L_{n+1} \leq x^{\star}_{n+1}$. By the same argument as in the previous case, $P^G_{n+1} < P^L_{n+1} = P^F_{n+1}$. Combining (i) and (ii) of [1, Proposition 2.8], one obtains that there exists two most beneficial strategies for player $n+1$ which are $\text{SLMF}^w_{n+1}(\beta)$ as a leader or $\text{MLSF}^w_{n+1}(\beta)$ as a follower;

(iv) Here, the expression $\bar{X}^w_{n+1}(n+1,0) = \chi^F < \min\{x^{\star}_{n+1},\bar{X}_{n+1}\}$ directly implies that $\bar{x}^G_{n+1} = \bar{x}^F_{n+1} < \bar{x}^L_{n+1} \leq x^{\star}_{n+1}$ and thus again by the same arguments that $P^G_{n+1} = P^F_{n+1} < P^L_{n+1}$. The conclusion follows from [1, Proposition 2.8 (i)];

$v)$ Similarly to case (iv), the condition $\bar{X}^w_{n+1}(n+1,0) < \chi^F < \min\{x^{\star}_{n+1},\bar{X}_{n+1}\}$ implies that $\bar{x}^G_{n+1} < \bar{x}^F_{n+1} < \bar{x}^L_{n+1} \leq x^{\star}_{n+1}$ and again, using the increasing property of function $\theta_{n+1}(\cdot,x_{-i})$, one has $P^G_{n+1} < P^F_{n+1} < P^L_{n+1}$. Conclusion then follows [1, Proposition 2.8 (i)];

$vi)$ If none of the previous case occurs, it means that the relations between the constants is given by one of these three sub-cases:

$a)$ $\bar{X}^w_{n+1}(n+1,0) = \min\left\{x^{\star}_{n+1},\bar{X}_{n+1}\right\} \leq \chi^F$;

$b)$ $\chi^F = \min\left\{x^{\star}_{n+1},\bar{X}_{n+1}\right\} < \bar{X}^w_{n+1}(n+1,0)$;

$c)$ $\min\{x^{\star}_{n+1},\bar{X}_{n+1}\} \leq \min\left\{\chi^F,\bar{X}^w_{n+1}(n+1,0)\right\}$.

With these three sub-cases, playing period 1 or period 2 leads to the same payoff for player $n+1$ since one can show that $\bar{x}^F_{n+1} = \bar{x}^L_{n+1} = \bar{x}^G_{n+1}$.

In sub-case (a), one immediately obtains that $\bar{x}^L_{n+1} = \bar{x}^G_{n+1} = \bar{x}^F_{n+1} = \min\{x^{\star}_{n+1},\bar{X}_{n+1}\}$. Now, in the sub-case (b), the equality shows that $\bar{x}^F_{n+1} = \bar{x}^L_{n+1} = \min\{x^{\star}_{n+1},\bar{X}_{n+1}\}$. Since one always has $\bar{X}^w_{n+1}(n+1,0) \leq \bar{X}_{n+1}$, it can be deduced that $\bar{x}^L_{n+1} = \bar{x}^G_{n+1} = \bar{x}^F_{n+1} = x^{\star}_{n+1} < \bar{X}^w_{n+1}(n+1,0)$ and thus that $\bar{x}^G_{n+1} = x^{\star}_{n+1}$.

Finally in the last sub-case (c), when $\min\{x^{\star}_{n+1},\bar{X}_{n+1}\} = x^{\star}_{n+1}$, the desired double equality directly holds since $x^{\star}_{n+1} = \bar{x}^F_{n+1} = \bar{x}^L_{n+1} = \bar{x}^G_{n+1}$. Otherwise, $\min\{x^{\star}_{n+1},\bar{X}_{n+1}\} = \bar{X}_{n+1}$. From condition (c), we have $\bar{X}^w_{n+1}(n+1,0) = \bar{X}_{n+1} \leq x^{\star}_{n+1} \leq \chi^F$, and therefore $\bar{x}^F_{n+1} = \bar{x}^L_{n+1} = \bar{x}^G_{n+1} = \bar{X}_{n+1}$. For these three sub-cases, no matter the period which the player $n+1$ will choose, his payoff will be the same for the three games, that is, $P^L_{n+1} = P^G_{n+1} = P^F_{n+1}$. Any decision of this player is a neutral strategy.

□

Let us examine more in details cases (iii), (iv), and (v). In all these cases, a (one of the) best/most beneficial strategy for the player $n+1$ is to be a leader. In other words, he can prioritize to play in period 1 in order to maximize his payoff. But, there is still a risk, which can occur, there will be a possibility for him to be in the GNEP^w_{n+1}. This Nash game can bring the worst payoff among three types (leader, follower, Nash player) and drives him to have no idea for the playing decision. And the same situation happens even if player $n+1$ decides to play in MLSF^w_{n+1}, there is a bad chance to be in GNEP^w_{n+1} too. However, at least, he knows precisely the circumstance that he has to face, which games should be avoided and which games should be played although he has no information about the strategies of the n players.

Remark 3.1. It is interesting to emphasize that if $\chi^F < \bar{X}^w_{n+1}(n+1,0)$, then only case (i), (ii) and (vi) can occur which means that, by playing period 1, player $(n+1)$'s strategy will be optimal, safe or neutral.

Provided "ignorance" on which period the n players will play, there is no safe or optimal choice outside of period 1 for the player $n+1$. In the corollaries below, a strategy will appear safe and optimal in period 2 if and only if the player $n+1$ has the "full knowledge" of which exact period the opponent group will play inevitably. These cases appear because items (iii)-(v) in Theorem 3.1 gain extra information.

Corollary 3.1. *Let us use the same hypotheses as in Theorem 3.1 and assume player $n+1$ knows that the group of n players would like to play in period 1. Then the following assertions hold.*

- $i)$ *If $\chi^F < \min\{x^\star_{n+1}, \bar{X}_{n+1}\} \leq \bar{X}^w_{n+1}(n+1,0)$, then an optimal strategy for player $n+1$ is to play in period 1.*
- $ii)$ *If $\chi^F < \bar{X}^w_{n+1}(n+1,0) < \min\{x^\star_{n+1}, \bar{X}_{n+1}\}$, then a safe strategy for player $n+1$ is to play in period 1.*
- $iii)$ *If $\bar{X}^w_{n+1}(n+1,0) < \min\{x^\star_{n+1}, \bar{X}_{n+1}\} \leq \chi^F$, then an optimal strategy for player $n+1$ is to play in period 2.*
- $iv)$ *If $\bar{X}^w_{n+1}(n+1,0) = \chi^F < \min\{x^\star_{n+1}, \bar{X}_{n+1}\}$, then any decision of player $n+1$ is a neutral strategy.*
- $v)$ *If $\bar{X}^w_{n+1}(n+1,0) < \chi^F < \min\{x^\star_{n+1}, \bar{X}_{n+1}\}$, then a safe strategy for player $n+1$ is to play in period 2.*
- $vi)$ *Otherwise, any decision of player $n+1$ is a neutral strategy.*

Corollary 3.2. *Let us use the same hypotheses as in Theorem 3.1 and assume player $n+1$ knows that the group of n players would like to play in period 2. Then the following assertions hold.*

- $i)$ *If $\chi^F < \min\{x^\star_{n+1}, \bar{X}_{n+1}\} \leq \bar{X}^w_{n+1}(n+1,0)$, then any decision of player $n+1$ is an optimal solution.*
- $ii)$ *If $\chi^F < \bar{X}^w_{n+1}(n+1,0) < \min\{x^\star_{n+1}, \bar{X}_{n+1}\}$, then an optimal strategy for player $n+1$ is to play in period 1, and safe strategy is to play in period 2.*
- $iii)$ *If $\bar{X}^w_{n+1}(n+1,0) < \min\{x^\star_{n+1}, \bar{X}_{n+1}\} \leq \chi^F$, then an optimal strategy for player $n+1$ is to play in period 1.*
- $iv)$ *If $\bar{X}^w_{n+1}(n+1,0) = \chi^F < \min\{x^\star_{n+1}, \bar{X}_{n+1}\}$, then an optimal strategy for player $n+1$ is to play in period 1.*
- $v)$ *If $\bar{X}^w_{n+1}(n+1,0) < \chi^F < \min\{x^\star_{n+1}, \bar{X}_{n+1}\}$, then an optimal strategy for player $n+1$ is to play in period 1.*

vi) Otherwise, any decision of player $n+1$ is a neutral strategy.

Corollary 3.1 (respectively Corollary 3.2) can be proved by adapting the proof of Theorem 3.1 to the fact that player $n+1$ cannot have P^L_{n+1} (respectively P^M_{n+1}) as payoff.

Remark 3.2. Assume that player $n+1$ knows certainly that the group of n players would like to play in period 1 (respectively 2), then he never can be a leader (resp. follower) because player $n+1$ loses the chance to be in SLMF^w_{n+1} (resp. MLSF^w_{n+1}). Now, the neutral strategy is just considered among the two left possible games between GNEP^w_{n+1} and MLSF^w_{n+1} (resp. between GNEP^w_{n+1} and SLMF^w_{n+1}).

4. Data Estimation in a Specific Model

As already discussed in the previous sections, player $n+1$ takes his decision to play period 1 or 2 from an assumed knowledge of collected data $\left(\beta, \{\bar{X}_i\}_{i=1,\dots,n}, \{x^\star_i\}_{i=1,\dots,n}\right)$ of $\text{GNEP}^w_n(\beta)$. The most important one is the global maximum $x^\star_i$ of the payoff function $\theta_i(\cdot, x_{-i})$, which is assumed to be independent of x_{-i}, for any $i = 1,\dots,n$. This value can be quite tricky to determine for player $n+1$. Throughout this section, let us assume that the objective function of each player is given by the following quadratic form:

$$\theta_i(x_i, x_{-i}) = \alpha x_i - c_i x_i^2. \tag{4.1}$$

In the context of electricity market, the constants can be interpreted as follows: let us assume that players = producers all use a certain rough material to produce electricity (coal, oil,...) and that x_i stands for the amount of rough material (in tons) that player i uses to produce his quantity of electricity. If the total amount of available rough product (in tons) is limited by $\beta > 0$ and the price of electricity is $\alpha \geq 0$ (in euro/tons of rough material, thus assuming that all the producers' plans have the same "efficiency") then function θ_i represents the revenue αx_i player i obtains minus the cost of production $c_i x_i^2$. Moreover, it is clear that θ_i is diagonally strictly concave.

Given this specific formula of the payoff functions θ_i, one clearly has, for any $i = 1,\dots,n+1$, the maxima $x^\star_i = \alpha/(2c_i)$. But, from the perspective of player $n+1$, each C_i, $i = 1,\dots,n$ is, a priori, only known by player i. That means player $n+1$ is just able to know c_{n+1}, not $\{c_i\}_{i=1,\dots,n}$.

Let us assume that before entering into the game with the group of n players, player $n+1$ can observe a certain number of iterations of the Nash game between the n players. Our aim in forthcoming subsection is to describe one context in which player $n+1$ can deduce the family of cost coefficients c_i from the observation of some GNEP_n and how this knowledge can be used.

4.1. **Observation phase.** Let us assume now that player $n+1$ has observed a finite number $k \in \{1,\dots,K\}$ of iterations of $\text{GNEP}^{w,k}_n$ played between the n players, each one with a different value of the price α^k and of the maximal exchange volume β^k. From this observation, player $n+1$ will deduce the values of cost coefficients $\{c_i\}_{i=1,\dots,n}$ of the group of n players.

By using the same arguments about weighted generalized Nash game, let us introduce here a minor modification of $\text{GNEP}^w_n(\alpha,\beta)$, which is a weighted generalized Nash game for n players corresponding to a given couple (α,β), to obtain some results in the observation phase. Let us recall that the man idea of the weighted notion is to put some weights on the constraint sets of the optimization problems. In Definition 2.1, we assume $\sum_{i=1}^n w_i = 1$ in condition (2.1). These parameters appear to guarantee that the sum of bounds of all players cannot exceed the current market volume. But during this observation phase, only n players are involved into $\text{GNEP}_n^w(\alpha,\beta)$.

So let us consider that n players are interacting on a market with a maximum exchange volume $\beta \in \mathbb{R}_+^*$ such that $\sum_{i=1}^{n} \bar{X}_i \neq \beta$. Let $\mathscr{W} = \{w_1,\ldots,w_n\}$ be a given family of weights of the players satisfying the following conditions

$$\begin{cases} \text{for any } i,\ w_i \in \left[0, \dfrac{\bar{X}_i}{\left|\sum_{i=1}^{n+1} \bar{X}_i - \beta\right|}\right[\\ \text{and } \sum_{i=1}^{n} w_i = 1. \end{cases} \tag{4.2}$$

Thus, given the values (α,β), the corresponding weighted generalized Nash equilibrium problem $\text{GNEP}_n^w(\alpha,\beta)$ can be defined here as

Finding $\bar{x}^w = (\bar{x}_1^w,\ldots,\bar{x}_n^w) \in \mathbb{R}^n$ such that $\forall i = 1,\ldots,n,\ \bar{x}_i^w$ is a solution of the player's i problem

$$\left(P_i^w(\bar{x}_{-i})\right) \qquad \max_{x_i}\ \alpha x_i - c_i x_i^2$$
$$\text{s.t.}\quad x_i^w \in [0, \bar{X}_i^w(n,0)],$$

where $\bar{X}_i^w(n,0) = \bar{X}_i - w_i \max\left\{0, \sum_{j=1}^{n} \bar{X}_j - \beta\right\}$.

Let us recall from [1, Proposition 3.3 (i)] that the inequality $x_i + \sum_{j\neq i}^{n} \bar{x}_j \leq \beta$ is implicitly considered. Let us also observe that due to the positiveness of the cost coefficients c_i, the well-posedness assumption (Assumption 2.1) and the uniqueness assumption (Assumption 2.2) are automatically fulfilled.

Now, in this context, let us specify the characterization of the solution of $\text{GNEP}_n^w(\alpha,\beta)$ in the following lemma with θ_i as just defined in (4.1). This lemma directly follows [1, Proposition 3.4].

Lemma 4.1. *Consider that n players are interacting on a market with a price $\alpha \in \mathbb{R}_+$, a maximum exchange volume $\beta \in \mathbb{R}_+^*$ such that $\sum_{i=1}^{n} \bar{X}_i \neq \beta$. Assume condition (4.2) is satisfied. Then,*

i) $\text{GNEP}_n^w(\alpha,\beta)$ admits a unique weighted Nash equilibrium $\bar{x}^w = \left(\bar{x}_1^w,\ldots,\bar{x}_n^w\right)$;

ii) for any $i = 1,\ldots,n$, one has

$$\bar{x}_i^w = \begin{cases} \bar{X}_i^w(n,0) & \text{if } \bar{X}_i^w(n,0) < \frac{\alpha}{2c_i}, \\ \frac{\alpha}{2c_i} & \text{otherwise.} \end{cases}$$

So let us go back now to the observation phase and assume that player $n+1$ has observed a finite sequence of n-player generalized Nash game $\text{GNEP}_n^{w,k}(\alpha^k,\beta^k)$, with $k = 1,\ldots,K$, where the optimization problem of player i at iteration k is defined as

$$\forall i = 1,\ldots,n,\ \left(P_i^w(\alpha^k,\beta^k)\right) \qquad \max_{x_i}\ \alpha^k x_i - c_i x_i^2,$$
$$\text{s.t.}\quad 0 \leq x_i \leq \bar{X}_i^{w,k}(n,0,\beta^k),$$

where $\bar{X}_i^{w,k}(n,0,\beta^k)$ is the weighted bound with regarding to β^k at the iteration k.

By "observing the $\text{GNEP}_n^{w,k}(\alpha^k,\beta^k)$", we mean that player $n+1$ has accessed to the knowledge of the corresponding equilibrium $\text{GNE}_n^{w,k} = (\bar{x}_1^{w,k},\ldots,\bar{x}_n^{w,k})_k$.

Let us consider the following assumption.

Assumption 4.1. For any $i = 1,\ldots,n$, there exists $k(i) \in \{1,\ldots,K\}$ such that $x_i^{w,k(i)} < \bar{X}_i^w(n,0,\beta^{k(i)})$.

The interpretation is that, for a finite number K, there are some cycles/iterations of Nash games, which n players took part in. Player $n+1$ inspects and collects data during the process. In fact, each time one cycle ends, this player has more information about the strategy value that each player i in the group opted for. By selecting a subset of cycles $k(i)$ corresponding to player $i = 1,\ldots,n$ such that the values of his strategies are strictly less than its maximal consumption bounds. For each player i, one can obtain the cost coefficients in this specific circumstance. As a direct consequence of Lemma 4.1, the forthcoming corollary allows then to deduce the values $\{c_i\}_{i=1,\ldots,n}$.

Corollary 4.1. *Consider that n players are interacting on a market with finite families of price $\{\alpha^k\}_{k=1,\ldots,K} \in \mathbb{R}_+$ and maximum exchange volume $\{\beta^k\}_{k=1,\ldots,K} \in \mathbb{R}_+^*$ such that, for any k, $\sum_{i=1}^{n} \bar{X}_i \neq \beta^k$. Assume Assumption 4.1 hold. Then,*

$$\text{for each } i = 1,\ldots,n, \quad c_i = \frac{\alpha^{k(i)}}{2\bar{x}_i^{w,k(i)}}.$$

Assumption 4.1 can be consequence of specific structures of the finite sequence $(\alpha^k,\beta^k)_k$ or obtained after a “sufficiently large” number of observations. From that, player $n+1$ picks up the public outputs $\{\bar{x}_i^{w,k(i)}\}_{k(i)}$ for each player i, one by one, among the n players and detects their cost coefficients. The explanation is quite simple, since the optimal value is equal to either $\frac{\alpha}{2c_i^{k(i)}}$ or the individual maximal consumption $\bar{X}_i^{w,k(i)}$, if $\bar{x}_i^{w,k(i)} \neq \bar{X}_i^{w,k(i)}$, surely the optimal solution is the one relating to cost coefficient $c_i^{k(i)}$. With this achievement, player $n+1$ can decide how to act in the two-period game by using the results of Theorem 3.1.

4.2. **Sensitivity analysis: a numerical illustration.** In order to illustrate the above results and particularly their sensitivity to the exogenous parameters α and β, we develop here some numerical simulations. It motives here to show how the situation of player $n+1$ changes when these parameter evolve or, in other words, how the status of the “favourable strategy” (optimal, safe, neutral, most beneficial...) is sensible to the changes of parameters α and β.

The tests have been conducted by using MATLAB (version R2020b). For each pair of values (α,β), there will be a point in two-dimensional plane $O_{\alpha\beta}$, where the point will be coloured to represent the corresponding strategic property of player $n+1$. For instance,

- a green point ■ (OPTIMAL 1, resp. 2) represents a value (α,β) such that decision of player $n+1$ is an optimal strategy when he plays in period 1 (resp. 2);
- red point ■ (M.SLMF) implies the situation at which the most beneficial strategy for player $n+1$ is to be in SLMF game.

All possibilities and corresponding colours are described in Table 1.

OPTIMAL 1	M.SLMF, M.MLSF
SAFE 1	M.SLMF
NEUTRAL	M.SLMF, L.GNEP
OPTIMAL 2	OPTIMAL 1 or OPTIMAL 2
SAFE 2	OPTIMAL 1 or SAFE 2
*Abbreviations M. and L. stand for *most beneficial* and *lowest beneficial* strategies respectively.	

TABLE 1. Properties of strategic decition for player $n+1$.

The "status/colour" of the favourable strategy has been determined thanks to Theorem 3.1 and Corollaries 3.1 and 3.2.

The examples are built for three players which means that the role of player $n+1$ will be played by the third player ($n+1=3$). We will consider the four scenarios below described by the following family of four input data $\mathscr{I}=\{\mathscr{I}_k\}_{k=1,\dots,4}$ as follows:

$\mathscr{I}_1$) $\beta\in]20,60]$, $\bar{X}=(45,35,20)$, $c=(3,5,5/2)$ and $w=(9/16,7/16,1/10)$;
$\mathscr{I}_2$) $\beta\in]20,60]$, $\bar{X}=(45,35,20)$, $c=(2,2,4)$ and $w=(9/16,7/16,1/10)$;
$\mathscr{I}_3$) $\beta\in]20,90]$, $\bar{X}=(45,35,20)$, $c=(3,5,5/2)$ and $w=(9/16,7/16,1/10)$;
$\mathscr{I}_4$) $\beta\in]8,60]$, $\bar{X}=(18,24,8)$, $c=(3,5,1)$ and $w=(3/7,4/7,1/10)$,

In the four cases, the value of α varies in the interval $]0,120]$, β is the maximum exchange volume of the market, $c=(c_1,c_2,c_3)$ is the vector of cost coefficients and $\bar{X}=(\bar{X}_1,\bar{X}_2,\bar{X}_3)$ stands for the vector of consumption bounds, and weight vector $w=(w_1,w_2,w_3)$, which is used to represents the rate cut in bargain (negotiated ratio).

We take $\mathscr{I}_2$ as a benchmark and compare the other three cases with it. Input set $\mathscr{I}_1$ differs from $\mathscr{I}_2$ only by a change of c, whereas with $\mathscr{I}_3$ the difference is in the range of β. Lastly, in $\mathscr{I}_4$, there is a modification in c, and in $\bar{X}$ which entails the change of β and w as a consequence.

Simulation 4.1. The results of Theorem 3.1 are illustrated in figures 1-4 for input $\mathscr{I}_1$ and in figures 5, 6 for input $\mathscr{I}_2$.

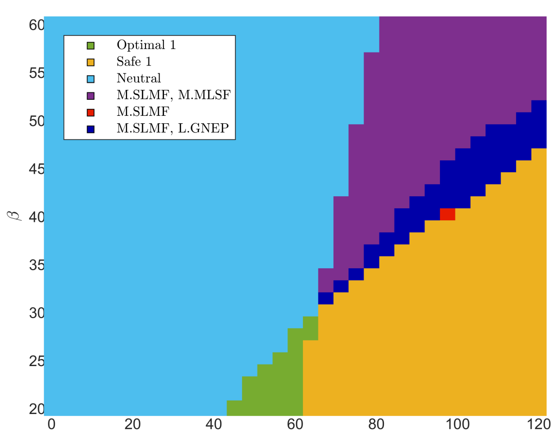

FIGURE 1. Illustrated result of Theorem 3.1 for input $\mathscr{I}_1$.

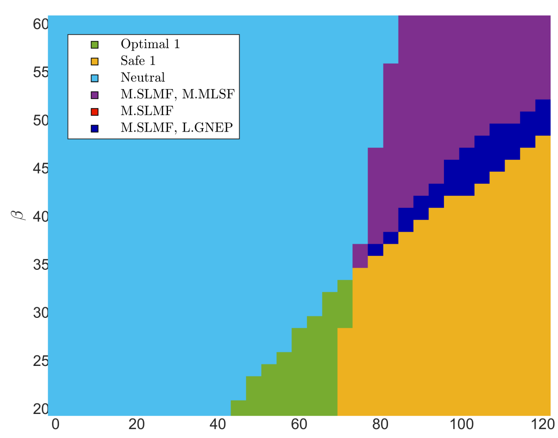

FIGURE 2. Illustrated result of Theorem 3.1 for input $\mathscr{I}_1$ with replacing $w_3=0.08$.

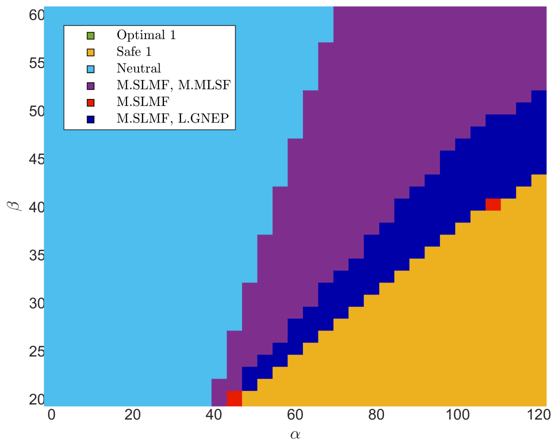

FIGURE 3. Illustrated result of Theorem 3.1 for input $\mathscr{I}_1$ with replacing $w_3 = 0.15$.

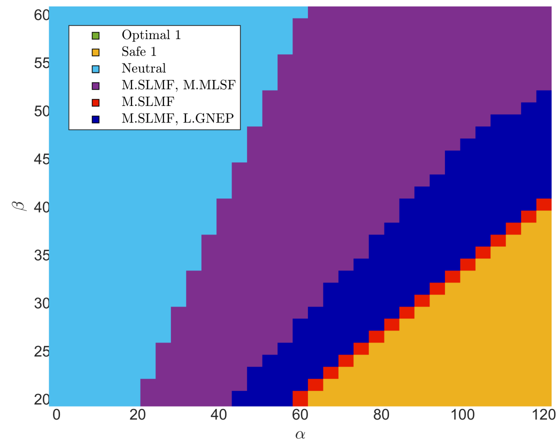

FIGURE 4. Illustrated result of Theorem 3.1 for input $\mathscr{I}_1$ with replacing $w_3 = 0.2$.

Let us first compare cases depicted in Figures 1-4 respectively by changing coefficients α and β. Figure 1 corresponds to a situation where all possibility of Theorem 3.1 are represented or, in other words, all colours appear. The Figures 2, 3, and 4 show different states at which w_3 varies. It is easy to see that, as the value of w_3 increases, the overall graph moves down. The partitions green ■, yellow ■ and cyan ■ become narrower and give place to indigo ■, blue ■, and red ■. This implies that if one player (not only player $n+1$) suffers more disadvantage because the w_i value is too large, the opportunity to achieve safe strategies is gradually reduced.

As soon as the price α decreases (resp. cost coefficient increases), the frequency of appearing of the cyan colour ■ is higher, meaning that the strategy will be neutral when all kinds of payoffs are the same. If α is too high (or c_{n+1} is too low), it infers that the value of $x^\star_{n+1}$ will be extremely high such that it cannot be a payoff of player $n+1$ since the payoff mainly depends on $\bar{X}_{n+1}$, $\bar{X}^w_{n+1}(n+1,0)$ and χ^F. As a result, all the other colours will replace a part of cyan ■.

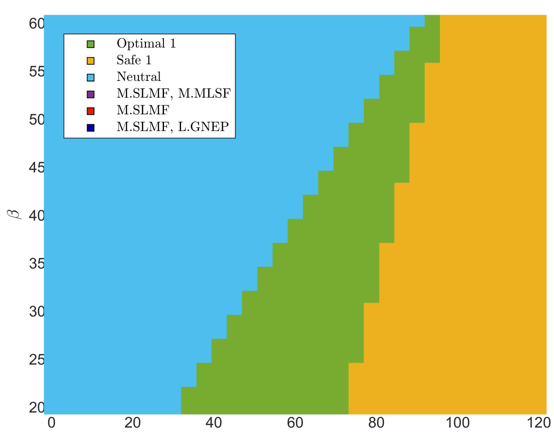

FIGURE 5. Illustrated result of Theorem 3.1 for input $\mathscr{I}_2$.

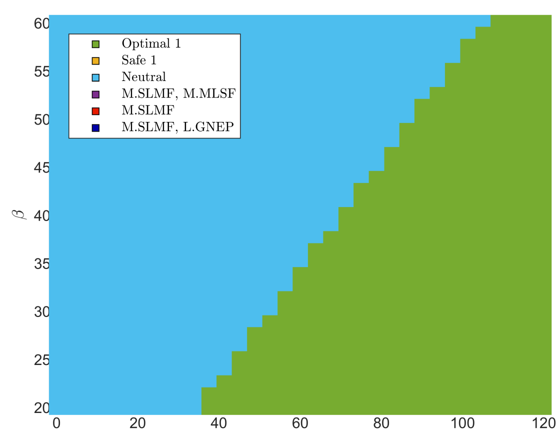

FIGURE 6. Illustrated result of Theorem 3.1 for input $\mathscr{I}_2$ with replacing $c_3 = 7$.

With the specific strictly concave quadratic objective function that we are considering in these simulations, it is clear that the solution $\bar{x}_{n+1}$ of player $(n+1)$'s problem belongs to the interval

$]0,x^{\star}_{n+1}]$. But since $\overline{X}^{w}_{n+1}(n+1,0) \leq \overline{X}_{n+1}$, condition $\min\{x^{\star}_{n+1},\overline{X}_{n+1}\} \leq \overline{X}^{w}_{n+1}(n+1,0)$ will not happen unless either $x^{\star}_{n+1} \leq \overline{X}^{w}_{n+1}(n+1,0) \leq \overline{X}_{n+1}$ or $\overline{X}^{w}_{n+1}(n+1,0) = \overline{X}_{n+1} \leq x^{\star}_{n+1}$ (with $w_{n+1}=0$). The case $w_{n+1}=0$, is clearly a rare situation where player $n+1$ does not make any sacrifice for his production reserves comparing to other n players. Hence, the former case will be more likely to happen that means with a high value of c_{n+1} such that $x^{\star}_{n+1} \leq \overline{X}_{n+1}$, then a part of inequality conditions in Theorem 3.1 (i) is formed for optimal strategy, and the green colour ■ has more chance to appear. Figures 5 and 6 reflect this observation, since in the first one c_3 equals to 4 but rises to 7 in the second. It results that the green area of Figure 6 is visually larger than the one of the other, and the yellow part (safe strategy) is empowered to become the green part (optimal strategy).

Remark 4.1. A rather special case to note is the red colour ■ matching with item (iv) in Theorem 3.1. Indeed, one can wonder why these points are so sparsely spread on the figures. Actually the case corresponds to the condition $\overline{X}^{w}_{n+1}(n+1,0) = \chi^{F} < \min\{x^{\star}_{n+1},\overline{X}_{n+1}\}$ and thus involves an equality which is, from a simulation point of view, difficult to exactly reach. Reducing the mesh size of simulation would possibly generate more such red points.

Simulation 4.2. The results in Figures 7 and 8 correspond to input data $\mathscr{I}_3$ (expanding value of β) and $\mathscr{I}_4$ (reducing values of $\overline{X}$).

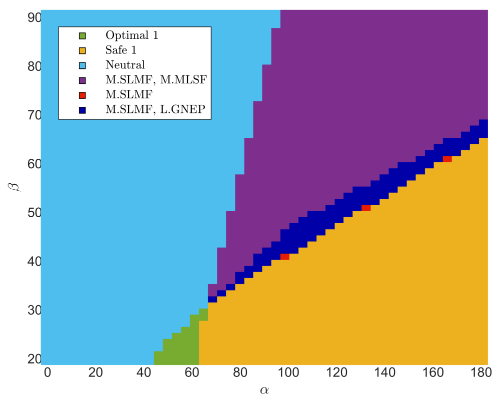

FIGURE 7. Illustrated result of Theorem 3.1 for input $\mathscr{I}_3$.

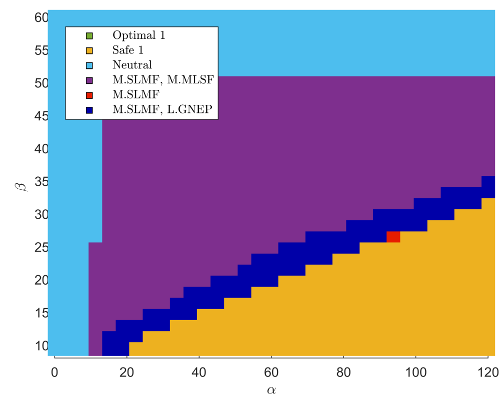

FIGURE 8. Illustrated result of Theorem 3.1 for input $\mathscr{I}_4$.

Comparing Figures 1 and 7, the general shape is more or less the same, the main difference concerns the case in which for a fixed α_t by increasing value of β over a certain threshold β_t the strategy will become neutral (see Figure 7).

For instance, at $\alpha_t = 80$, for all $\beta \geq \beta_t \approx 65$, the strategies of player $n+1$ are surely neutral. This threshold also exists in Figure 8 but with $\beta_t \approx 50$ since the consumption bounds are not the same. Indeed, this threshold β_t actually corresponds to $\beta_t = \sum_{i=1}^{n+1} \overline{X}_i$. This comes from the fact that when $\sum_{i=1}^{n+1} \overline{X}_i < \beta$ then $\overline{X}^{w}(n+1,0) = \overline{X}_{n+1}$and thus $\bar{x}^{L}_{n+1} = \bar{x}^{F}_{n+1} = \bar{x}^{G}_{n+1}$ leading to a neutral situation where any strategy of player $n+1$ will give him the same payoff.

Now take a deeper look at Figure 8, which will reveal the effect of changing vector $\overline{X}$ in the input $\mathscr{I}_4$, namely, the shrinking consumption bounds from the input $\mathscr{I}_1$. Let us analyse the "flat part behaviour" in this figure. Assume that $\beta < \beta_t \approx 50$ is fixed and α is increasing. Since $x^{\star}_{n+1} = \alpha/(2c_{n+1})$, then for α greater than a certain threshold (for example, $\alpha > \alpha_t \approx 15$ in

Figure 8) $\min\{x^{\star}_{n+1}, \bar{X}_{n+1}\} = \bar{X}_{n+1}$. Thus, the value of α has no longer influence and the case (iii) of Theorem 3.1 always occurs.

The next two simulations are built to illustrate Corollaries 3.1 and 3.2, which are extensions of Theorem 3.1 in the case where player $n+1$ has information on the period chosen by the group of n players.

Simulation 4.3 (for Corollary 3.1)**.** The results in Figures 9 and 10 describe cases $\mathscr{I}_1$ and $\mathscr{I}_2$ when the group of n players plays in period 1.

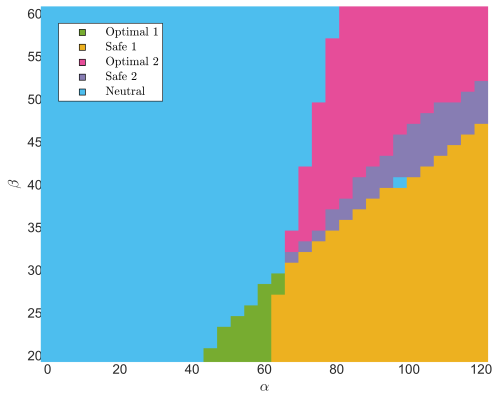

FIGURE 9. Illustrated result of Corollary 3.1 for input $\mathscr{I}_1$.

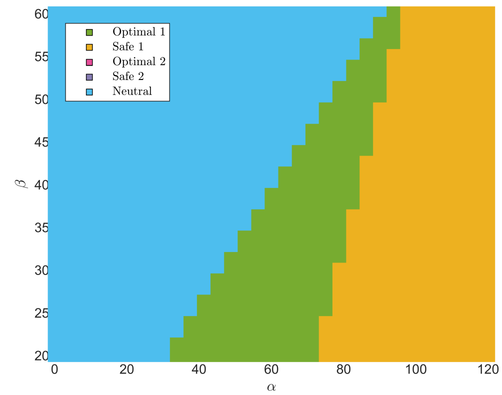

FIGURE 10. Illustrated result of Corollary 3.1 for input $\mathscr{I}_2$.

Obviously, for player $n+1$, with this additional information, stronger conclusions of the decision-making (than for example "most beneficial") can be reached. Specifically, player $n+1$ will possibly achieve Optimal 2, Safe 2. This leads to a remarkable fact. If the group of n players claims to play in period 1, then depending on the input data, strategic decisions of player $n+1$ will be very diversified containing various choices such as Optimal 1, Safe 1, Optimal 2, Safe 2 and Neutral.

When comparing Figures 9 and 10 with Figures 1 and 5, one can observe that colours ■, ■ and ■ are substituted for ■, ■ and ■ respectively.

Simulation 4.4 (for Corollary 3.2)**.** The results in Figures 11 and 12 describe cases $\mathscr{I}_1$ and $\mathscr{I}_2$ when the group of n players plays in period 2.

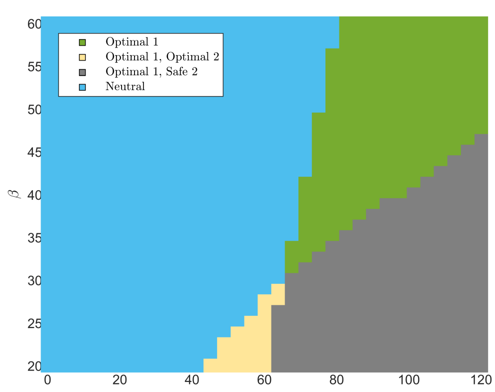

FIGURE 11. Illustrated result of Corollary 3.2 for input $\mathscr{I}_1$.

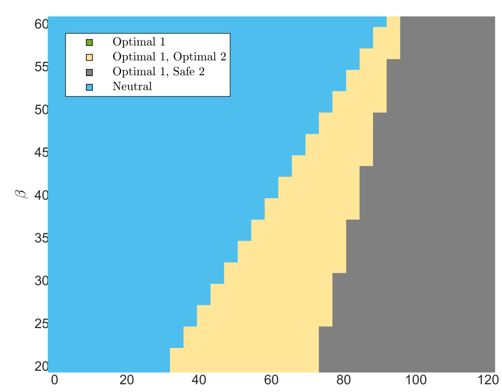

FIGURE 12. Illustrated result of Corollary 3.2 for input $\mathscr{I}_2$.

A noticeable comment in Figures 11 and 12 is that, if not neutral, the strategies for player $n+1$ are at least safe. Here a new category of decision-making appears in these two figures, a strategy which can be considered as "more effective" than any previously mentioned one: yellow case ■ (Optimal 1 and Optimal 2). Alike neutral strategy, the player can play in any period without concerning. While the neutral strategy provides the same payoff for each case (leader, follower or player of a Nash game), in this case, player $n+1$ can avoid the lowest payoff $P^F_{n+1} = \min_{\kappa \in \{L,G,F\}} P^\kappa_{n+1}$ in period 2. In this situation, choices of player $n+1$ seem very positive. The strategies are most likely Optimal and Safe (■, ■, ■) or Neutral ■ in the worst case.

These last figures clearly confirm an evident fact: more information player $n+1$ has, better and less risky will be his decision-making.

5. Conclusion

As highlighted in Part 1, the concepts of weighted Nash equilibrium and of "safe/optimal/neutral/beneficial" decisions have been introduced. Taking advantage of the theoretical tools and results proved in Part 1, in this second paper, we investigated the strategic decisions of player $(n+1)$ when entering in the game where n players are already in. In particular, by assuming that the objective function is diagonally strictly concave, we can fully characterize the most favourable strategies for player $n+1$ to enter into the game. Numerical simulations were also conducted bringing to the fore the sensitivity of the "favourable strategy" to the value of the price α and the maximal exchange volume β of the market. Even though, the treatment was used for a specific model, the results obtained can be used as a test for real case problems. One possible extension of this framework would be to change the initial hypothesis that whenever some additional players join the market, the group of n players has a common strategy/decision on the period they want to play. The situation would be then much more complicated since one can face a multi-leader-multi-follower game with a lot of possible combinations of leaders/followers groups.

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