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STRATEGIC DECISION IN A TWO-PERIOD GAME USING A MULTI-LEADER-FOLLOWER APPROACH. PART 1 - GENERAL SETTING AND WEIGHTED NASH EQUILIBRIUM

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Abstract. In the situation where a new player wants to join a group of players, which is interacting in a non-cooperative way through a generalized Nash game, this new player can face three different situations: playing together with the other players in a generalized Nash game, playing first and waiting for the response of the opponent group, or letting the group play first and act then as a follower. This two-period game can thus lead to a generalized Nash game, a single-leader-multi-follower game, or a multi-leader-single-follower game. Our aim in this couple of papers is to elaborate a decision-making strategy to help this new player when choosing the most beneficial game, beside the fact that he does not know what game the group of other players would like to select. This work, composed of a couple of papers, extends to $n+1$ players the previous research of B. von Stengel (Games and Economic Behaviour - 2010) done for a two-player symmetric duopoly game. In this first part, we present the main concepts, introduce in particular the new notion of *weighted Nash equilibrium*, and provide an adapted analysis in the case of a specific model. In the companion second part, the decision-making policy is developed and numerical simulations are conducted.

Keywords. Multi-Leader-Single-Follower game; Nash equilibrium problem; Quasi-concavity; Single-Leader-Multi-Follower game; Two-period game.

1. INTRODUCTION

Historically, the earliest *duopoly model* was introduced by the French economist Cournot [1]. It states that if, in a two-player game, each player's strategy is completely independent on the other's strategy, they will obtain a lower profit than considering the opponent's choice through periods. Therefore, players should consider their interdependence, which is defined as the best response to the opponent for improving their profits. Besides, the concept of the *leadership game* was introduced by von Stackelberg [2, 3]. With the same payoff functions, the game introduced is a sequential pattern in which the first mover is called the leader with a strategy based on the best response of the player playing after (follower).

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Many recent papers show that in order to solve *sequential games* they have to consider an endogenous timing problem. This is important because it determines the role of players being leaders or followers based on the time they join the game or make their decisions. For instance, in [4, 5], the authors compared explicitly the follower payoff to the payoff the player would obtain as a leader in sequential play or as Nash player in simultaneous play. In von Stengel [6], the leader and follower payoffs in a duopoly game, whose payoff arises in sequential play, are compared with the Nash payoff in simultaneous play. If the game is symmetric, it will admit a unique symmetrical equilibrium and each player's payoff is monotonic in the opponent's choice along with their own best reply function.

In this work, the authors develop von Stengel's idea for the scheme of multi-leader-follower games in order to analyse non-cooperative/hierarchical games between $n+1$ players. More precisely, we intend to evaluate the gap of payoff of a player between two possible models. On one hand, a non-cooperative model in which this player is one of the players of a Nash game (one-level game). On the other hand, a bi-level game in which this player plays the role of a common follower or common leader.

By defining this model as a two-period game, we have that, similarly to [6], two situations may occur: sequential play and simultaneous play. Apart from the order of play, we distinguish two types of players: one is a group of n players, and the other is the $(n+1)^{\text{th}}$ independent player. Here, there appears three specific cases that are:

- Generalized Nash equilibrium problem (GNEP) for $n+1$ players, together giving the strategy in same phase/period;
- Multi-leader-single-follower game (MLSF): the group of n players choose period 1 to play and the $(n+1)^{\text{th}}$ player chooses period 2;
- Single-leader-multi-follower (SLMF) game when the $(n+1)^{\text{th}}$ player chooses to play in period 1 and becomes the leader while the rival group plays in the other period.

Based on this, possible outcomes for the payoff of the $(n+1)^{\text{th}}$ player have been studied. The aim is to propose a decision-making policy in order to advice the new player in taking a strategy to play in period 1 or 2 by which it generates the above three situations (SLMF, GNEP, and MLSF). Of course, this strategic decision of player $n+1$ must be chosen without knowing which period the group of n players will prefer to play. The problem examined here is not only controlled by the endogenous timing factor (meaning that the chronological order of the strategy also affects the outcome of the player) but also the predicted choice between one-level Nash and bi-level game. Different from von Stengel's orientation, which only focuses on the behaviour of the follower (due to the symmetry), we try to clarify the possibilities of the $(n+1)^{\text{th}}$ player depending on the situations. Player $n+1$, indeed, can not influence the strategy of the group of other n players, but by choosing to play in first or second period he is able to change the nature of the game that becomes SLMF, MLSF, or GNEP for improving further his own payoff.

In order to present the obtained results in a systematic way, we divide the paper into two parts. In Part 1, we define the general settings, we provide a classification of decisions for player $n+1$, and moreover we introduce the new concept of weighted generalized Nash equilibrium, a notion which allows us to obtain the uniqueness of generalized Nash equilibrium under mild hypotheses. Part 2 is devoted to the study of a particular setting, where the utility function is a concave quadratic function, and the constraint set is defined by inequalities. In this context, a complete decision-making policy is developed.

Part 1 is organized as follows. In Section 2, the problem setting and the decisions concepts are introduced. In Section 3, the notion of the weighted generalized Nash equilibrium problem is defined, and the relationship between equilibria in the GNEP and the weighted GNEP is studied. Finally, we analyse the first properties of the weighted multi-leader-follower exchange models.

2. Problem Setting

The general setting of this section is as follows: while n players are interacting in a non-cooperative way, thus playing a generalized Nash game, an $(n+1)^{\text{th}}$ player wants to enter the game. Nevertheless, in this asymmetric situation, this new player faces several possibilities to inter-operate with the group of n players. The game can be analysed in strategic form as a leadership game where players commit to a strategy without knowing what the other players will choose. Then, we have a sequential game with a leader-follower approach.

2.1. **Multi-leader-follower models in two-period game context.** Our aim is here to consider the asymmetric situation, where a player, called here player $n+1$, plans to start a non-cooperative interaction with a group of n players already playing a generalized Nash game between them.

Thus there are three different possible interactions: of course the more "natural" is that player $n+1$ is inserted into the generalized Nash game with the other n players. This new $(n+1)$-player generalized Nash game would generate Nash payoffs for all $n+1$ players including player $(n+1)$'s. But the player $n+1$ can also consider to have other hierarchical interactions with the group of n players. Indeed, he can have a position of *sole leader* in SLMF or a position of *single follower* in a MLSF. In the first case, it means that he will "play first" and that the group of n players will play a GNEP, which is therefore parametrized by the decision of player $n+1$ (in the objective functions and/or in the constraint sets). This conducts to a result that player $n+1$ will gain a *leader payoff* while others' will obtain the *follower payoffs*. In the second case, the group of n players will play a GNEP game between them but in which the optimization problem of a member among n players will be actually connected to a bi-level problem. In particular, each member will maximize his payoff at the upper level constrained by a lower-level optimization problem of player $n+1$. Player $n+1$ thus receives a *follower payoff* comparing to *leader payoffs* of others, and his optimization problems is parametrized by n players' decision.

Now the decision to follow one of these three possible interaction models will be made by a "two-step process". The group of n players and player $n+1$ will simultaneously declare if they want to play as leader (period 1) or as follower (period 2). There are thus four possible combinations that lead to the three mentioned models.

The n players \ **New player $n+1$**	Single Leader (plays in period 1)	Single Follower (plays in period 2)
Multiple Leaders (play in period 1)	GNEP_{n+1}	MLSF_{n+1}
Multiple Followers (play in period 2)	SLMF_{n+1}	GNEP_{n+1}

TABLE 1. Description for the situations of $(n+1)$-player games.

Clearly, in the table, it is indifferent to have both choosing "leader" or both choosing "follower" and these two situations will lead to the same generalized Nash game.

Let us now describe the game more in detail. So let us consider a market in which q commodities can be bought, and let β be a positive real number vector of $\mathbb{R}^q$ representing a maximum initial endowment or a maximum exchange volume of the commodities on the market. Let us assume that, for any $i = 1, \ldots, n+1$, $C_i \subset \mathbb{R}^q$ denotes the constraint strategy set of player i while the real valued function θ_i, defined on $\mathbb{R}^q \times \mathbb{R}^{nq}$ stands for his payoff function. Thus, the vector $x_i \in C_i$ is the strategies vector for player i, while x_{-i} is the vector of the strategies of all players but excluding i. The notation $C(\beta)$ will be used here to describe the common constraint set parametrized by the initial endowment β and shared by all the players. Three different models can be introduced as follows.

a) **Single-leader-multi-follower game $\mathbf{SLMF}_{n+1}(\beta)$**:
A single-leader-multi-follower game between $n+1$ players (after the arrival of player $n+1$) is defined as

$$\big(\bar{P}(\beta)\big) \qquad \max_{x_{n+1}} \max_{x_1,\ldots,x_n} \; \theta_{n+1}(x_{n+1}, x_{-(n+1)})$$
$$\text{s.t.} \begin{cases} x_{n+1} \in C_{n+1}, \\ (x_1, \ldots, x_n) \in Eq(\beta - x_{n+1}), \end{cases}$$

where $Eq(\beta - x_{n+1})$ is the set of generalized Nash equilibria of the n-player $\text{GNEP}_n(\beta - x_{n+1})$ defined by

$$\forall i = 1, \ldots, n, \; \big(\bar{P}_i(\beta - x_{n+1})\big) \qquad \max_{x_i} \; \theta_i(x_i, x_{-i})$$
$$\text{s.t.} \begin{cases} x_i \in C_i, \\ (x_1, \ldots, x_{n+1}) \in C(\beta). \end{cases}$$

Note that this problem will be well-posed only if for each possible value of x_{n+1} the equilibrium problem $\text{GNEP}_n(\beta - x_{n+1})$ admits at least an equilibrium. It will be the case if, for example, the subsets C_i, $i = 1, \ldots, n$ are non-empty, convex compact, the functions θ_i are quasi-concave with regard to x_i and continuous on $\mathbb{R}^{(n+1)q}$ and, for any $x_{n+1} \in C_{n+1}$, the subset $C(\beta)$ is non-empty convex and compact (see, e.g., [7]). In the case of several possible equilibria, a choice must be done by the leader (player $n+1$): optimistic, pessimistic or selection approach and the "$\max\limits_{x_{n+1}}$" formulation adapted accordingly (see, e.g., [8, 9]).

b) **Generalized Nash equilibrium problem GNEP$_{n+1}(\beta)$**:
A generalized Nash game between $n+1$ players (after the arrival of player $n+1$) is defined as

$$\forall i=1,\dots,n+1,\ \left(\widetilde{P}_i(\beta)\right) \qquad \max_{x_i}\ \theta_i(x_i,x_{-i}) \quad \text{s.t.} \begin{cases} x_i \in C_i, \\ (x_1,\dots,x_{n+1}) \in C(\beta). \end{cases}$$

c) **Multi-leader-single-follower game MLSF$_{n+1}(\beta)$**:
A multi-leader-single-follower game between $n+1$ players (after the arrival of player $n+1$) is defined by

$$\forall i=1,\dots,n,\ \left(\widehat{P}_i(\beta)\right) \qquad \max_{x_i}\ \theta_i(x_i,x_{-i}) \quad \text{s.t.} \begin{cases} x_i \in C_i, \\ x_{n+1} \text{ is the unique solution of the} \\ \text{optimization problem } \left(\widehat{P}(\beta,x_1,\dots,x_n)\right), \end{cases}$$

where

$$\left(\widehat{P}(\beta,x_1,\dots,x_n)\right) \qquad \max_{x_{n+1}}\ \theta_{n+1}(x_{n+1},x_{-(n+1)}) \quad \text{s.t.} \begin{cases} x_{n+1} \in C_{n+1}, \\ (x_1,\dots,x_{n+1}) \in C(\beta). \end{cases}$$

Regarding the well-posedness, problem MLSF$_{n+1}(\beta)$ is well-defined if, for any $(x_1,\dots,x_n) \in \prod_{i=1}^{n} C_i$, the lower level problem $\left(\widehat{P}(\beta,x_1,\dots,x_n)\right)$ admits a unique solution. It will be the case if, for example, function θ_{n+1} is strictly quasi-concave upper semi-continuous with regard to variable x_{n+1} and the subsets C_{n+1} and $C(\beta)$ are non-empty convex and compact; or θ_{n+1} can be even strongly concave to drop the compactness of constraint set. This uniqueness assumption avoids to deal with the intrinsic ambiguity of multi-leader-follower games coming from the fact that the different leaders can consider different optimum of the lower level problem.

The interested reader can refer to [10, 11, 12, 13] for more information about Nash games and multi-leader-follower games.

All along the work we will make the following assumptions.

Assumption 2.1 (*Well-posedness assumption*)**.** For the considered maximum exchange volume β, each of the three problems SLMF$_{n+1}(\beta)$, GNEP$_{n+1}(\beta)$, and MLSF$_{n+1}(\beta)$ are assumed to be well-posed, which are

a) for each possible value of x_{n+1} the equilibrium problem GNEP$_n(\beta - x_{n+1})$ admits at least an equilibrium;
b) GNEP$_{n+1}(\beta)$ admits at least a generalized Nash equilibrium;
c) for any $(x_1,\dots,x_n) \in \prod_{i=1}^{n} C_i$, the lower level problem $\left(\widehat{P}(\beta,x_1,\dots,x_n)\right)$ admits a unique solution.

Sufficient conditions for the well-posedness of SLMF$_{n+1}(\beta)$ and MLSF$_{n+1}(\beta)$ have been given above. Note that the uniqueness assumption *c*) is important for the well-posedness of model MLSF$_{n+1}(\beta)$ since it allows to avoid a classical ambiguity of these models. Indeed, without this uniqueness hypothesis, one can face the unacceptable situation where the different leaders consider different solutions of the lower problem. It is well-known that when considering a SLMF a selection process can be implemented to obtain artificially this uniqueness of the follower response (see, e.g., [10]). Nevertheless, this is not possible in the case of MLSF since each leader can decide about his own selection process thus leading to a possible equilibrium in terms of the leader variables but corresponding to different conjectures of the follower's response (see, e.g., [14, 15]).

Assumption 2.2 (*Uniqueness assumption*)**.** For the considered maximum exchange volume β, each of the three problems $\mathrm{SLMF}_{n+1}(\beta)$, $\mathrm{GNEP}_{n+1}(\beta)$, and $\mathrm{MLSF}_{n+1}(\beta)$ admits at most a solution $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_{n+1})$.

Knowing that it is quite difficult to ensure the uniqueness of equilibrium of a GNEP, the *uniqueness assumption* appears to be quite restrictive. Nevertheless, as it will be seen in the forthcoming Section 3, for the case of "bounded strategies", this uniqueness assumption will be fulfilled under a mild assumption thanks to the concept of the weighted generalized Nash equilibrium.

Thus, the question which we want to address is the following. Taking into account some information that player $n+1$ has collected from the former GNEP_n game that the group of n players was having, and with player $n+1$ not knowing what will be the choice of this group, what is the *more advantageous* choice for player $n+1$? Our aim in the forthcoming section is to define these notions and to provide some sufficient conditions under which such *better* strategy is possible.

2.2. **Decision concepts.** In endogenous timing problems, the order of playing determines the difference in the player's benefit. An endogenous game is a game in which a leader and a follower arise spontaneously as a consequence of each player attempting to maximize their payoffs. Considering the ability to choose a specific moment to devise a strategy is called a decision. Both leader and follower may prefer to adopt sequential roles rather than engage in simultaneous competition in case that the sequential competition may bring them a higher payoff. There is no conflict over who moves first. In any case, player $n+1$ has the right to make a move in period 1 or 2 pro-actively and autonomously, just like the group of n players. Such these games, as mentioned, are said to exhibit endogenous timing behaviour because the roles of first and second movers are formed naturally from attempting to optimize their benefit, rather than being assigned exogenously by an existing rule.

Before entering into further details, under Assumption 2.1 and Assumption 2.2, let us introduce some notations concerning the payoffs of player $n+1$:

- P^L_{n+1} : optimal payoff of player $n+1$ in role of a leader in $\mathrm{SLMF}_{n+1}(\beta)$ that is P^L_{n+1} is optimal value of problem $\big(\bar{P}(\beta)\big)$;
- P^G_{n+1} : optimal payoff of player $n+1$ in role of a player in $\mathrm{GNEP}_{n+1}(\beta)$ that is $P^G_{n+1} = \theta_{n+1}(\bar{x}_{n+1}, \bar{x}_{-(n+1)})$ where $(\bar{x}_{n+1}, \bar{x}_{-(n+1)})$ is the unique equilibrium of $\mathrm{GNEP}_{n+1}(\beta)$;
- P^F_{n+1} : optimal payoff of player $n+1$ in role of a follower in $\mathrm{MLSF}_{n+1}(\beta)$ that is $P^F_{n+1} = \theta_{n+1} = (\bar{x}_{n+1}, \bar{x}_{-(n+1)})$ where $\bar{x}_{-(n+1)}$ is the unique equilibrium of $\mathrm{MLSF}_{n+1}(\beta)$ and $\bar{x}_{n+1}$ is the associated solution of $\big(\widehat{P}(\beta, \bar{x}_{-(n+1)})\big)$;
- P^m_{n+1} : lowest payoff of player i between the three payoffs P^L_{n+1}, P^G_{n+1} and P^F_{n+1}, therefore $P^m_{n+1} = \min_{\kappa \in \{L,G,F\}} P^\kappa_{n+1}$;

P^M_{n+1} : highest payoff of player i between the three payoffs P^L_{n+1}, P^G_{n+1} and P^F_i, and thus $P^M_{n+1} = \max_{\kappa \in \{L,G,F\}} P^\kappa_{n+1}$.

Notice that, P^L_{n+1}, P^G_{n+1}, and P^F_{n+1} represent 3 different payoffs coming from distinct games. They are not related to each other and depend on the period that player $n+1$ takes. Based on these payoffs, let us now describe the meaning of *favourable strategies*.

Definition 2.1. Let us assume that player $n+1$ took a decision (either to play period 1 or to play period 2) which generates a payment P_{n+1}. This decision is said to be

$i)$ a *safe strategy* for player $n+1$ if $P^m_{n+1} < P_{n+1}$;

$ii)$ an *optimal strategy* for player $n+1$ if $P^m_{m+1} < P_{n+1} = P^M_{n+1}$;

$iii)$ a *neutral strategy* for player $n+1$ if $P_{n+1} = P^m_{n+1} \leq P^M_{n+1}$.

Let us illustrate the three above definitions. A safe strategy is a strategy such that when applied, allows the player to avoid getting the lowest payoff and keeps the chance to gain the maximum without having to know about the others' decision. The term "lowest" here has to be understood in the sense of *lowest among three optimal payoffs in the three types of game*. The player's payoff can reach the maximum or not. Nevertheless, even though the best situation which is maximum can not occur, player $n+1$ is at least able to avoid the worst situation and to "feel safe". The fear that one player would get the lowest payoff reflects risk-aversion mentality of players. The lowest payoff is not necessarily a bad result, but the player could be "unsatisfied" if it is the smallest of the possible outcomes the player could reach. Therefore, the so-called "safe", which is based on a risk scale, comes from optimizing what a player can achieve.

Suppose now that player $n+1$ possesses a strategy which guarantees the maximum payoff, that is $P_{n+1} = P^M_{n+1}$. Two cases can occur: if $P_{n+1} > P^m_{n+1}$, then the decision will be an *optimal strategy*; otherwise $P_{n+1} = P^m_{n+1}$, which implies that $P^m_{n+1} = P^M_{n+1}$ and then we have $P^L_{n+1} = P^G_{n+1} = P^F_{n+1}$. The strategy is a *neutral one*.

Remark 2.1. An optimal strategy is a safe strategy, but the reverse is not true. Although a neutral strategy is not the least, it is neither an optimal nor a safe strategy.

Let us assume, as a first step, that the three different payoffs P^L_{n+1}, P^F_{n+1}, and P^G_{n+1} are known. Then proposition below describes some sufficient conditions for decision of player $n+1$ to be safe or optimal strategy.

Proposition 2.1. *Assume Assumption 2.1 and Assumption 2.2. Then*

$i)$ *If $P^L_{n+1} > P^F_{n+1}$ and $P^m_{n+1} \neq P^G_{n+1}$, then the safe strategy for the player is to play in period* 1. *In addition, if $P^L_{n+1} = P^G_{n+1}$, then this safe strategy becomes optimal strategy.*

$ii)$ *If $P^F_{n+1} > P^L_{n+1}$ and $P^m_{n+1} \neq P^G_{n+1}$, then the safe strategy for the player is to play in period* 2. *In addition, if $P^F_{n+1} = P^G_{n+1}$, then this safe strategy becomes optimal strategy.*

Proof. Let us preliminarily list the possible strategies and payoff of player $n+1$. Player $n+1$ has to decide if entering the game in period 1 or period 2, without knowing the strategy of the other n players. Let us, then, distinguish the different cases:

$a)$ player $n+1$ decides to enter the game in period 1. Then, the following two situations can occur:

a_1) the other n players enter the game in period 1, then a $\text{GNEP}_{n+1}(\beta)$ will be performed and the payoff of player $n+1$ will be P^G_{n+1};

a_2) the other n players enter the game in period 2, then a $\text{SLMF}_{n+1}(\beta)$ will be performed and the payoff of player $n+1$ will be P^L_{n+1}.

b) player $n+1$ decides to enter the game in period 2. Then, the following two situations can occur:

b_1) the other n players enter the game in period 1, then a $\text{MLSF}_{n+1}(\beta)$ will be performed and the payoff of player $n+1$ will be P^F_{n+1};

b_2) the other n players enter the game in period 2, then a $\text{GNEP}_{n+1}(\beta)$ will be performed and the payoff of player $n+1$ will be P^G_{n+1}.

Now in case (i), since $\min_{k\in\{L,G,F\}} P^k_{n+1} \neq P^G_{n+1}$ and $P^L_{n+1} > P^F_{n+1}$, then

$$\min_{k\in\{L,G,F\}} P^k_{n+1} = P^F_{n+1}$$

and the only safe strategy for player $n+1$ is to play period 1. This strategy clearly becomes optimal if additionally $P^L_{n+1} = P^G_{n+1}$.

Symmetrically, in case (ii), since $\min_{k\in\{L,G,F\}} P^k_{n+1} \neq P^G_{n+1}$ and $P^F_{n+1} > P^L_{n+1}$, then

$$\min_{k\in\{L,G,F\}} P^k_{n+1} = P^L_{n+1}.$$

Playing period 2 is thus the only safe strategy for player $n+1$, which becomes optimal if $P^F_{n+1} = P^G_{n+1}$. □

The claim of Proposition 2.1 can be illustrated by the following strategic tables.

Player $n+1$ / Group of n players	Period 1		Period 2
Period 1	P^G_{n+1}	$>$	P^F_{n+1}
Period 2	P^L_{n+1}	$\geq$	P^G_{n+1}

TABLE 2. Playing in period 1 provides safe/optimal strategy to player $n+1$.

Player $n+1$ / Group of n players	Period 1		Period 2
Period 1	P^G_{n+1}	$\leq$	P^F_{n+1}
Period 2	P^L_{n+1}	$<$	P^G_{n+1}

TABLE 3. Playing in period 2 provides safe/optimal strategy to player $n+1$.

Player $n+1$ / Group of n players	Period 1		Period 2
Period 1	P^G_{n+1}	$\geq$	P^F_{n+1}
Period 2	P^L_{n+1}	$\leq$	P^G_{n+1}
Additionally	$P^L_{n+1} > P^F_{n+1}$		

TABLE 4. Another case gaining safe/optimal strategy in period 1.

Player $n+1$ / Group of n players	Period 1		Period 2
Period 1	P^G_{n+1}	$\geq$	P^F_{n+1}
Period 2	P^L_{n+1}	$\leq$	P^G_{n+1}
Additionally	$P^L_{n+1} < P^F_{n+1}$		

TABLE 5. Another case gaining safe/optimal strategy in period 2.

Let us now introduce two last senses/understandings of *favourable strategies* for player $n+1$.

Definition 2.2. Let us assume that player $n+1$ took a decision (either to play period 1 or to play period 2) which generates a payment P_{n+1}. This decision is said to be

$i)$ *most beneficial* for player $n+1$ if $P_{n+1} = P^M_{n+1}$;
$ii)$ *lowest beneficial* for player $n+1$ if $P_{n+1} = P^m_{n+1}$.

Remark 2.2. If a decision of player $n+1$ is at the same time the most and the lowest beneficial for player $n+1$, then any decision of this player is neutral.

As mentioned in Remark 2.1, if a player has a strategy which is optimal, safe or neutral, he can decide in which period (1 or 2) to play to optimize his payoff. Nevertheless, there are some cases in which we cannot provide a good-enough advice for the player since those cases can not guarantee to avoid the lowest payoff. Our aim is then to give as much information as possible to the player. If there is no knowledge about the best strategy to adopt, he can at least know which type of game he should take part in. If player $n+1$ knows which period the other n players will enter in the game (in period 1 or 2), he can adopt a suitable strategy that maximizes his profit or at least avoids the worst payoff by choosing the appropriate game to play. The following proposition states the condition under which playing a SLMF, MLSF or GNEP game is the most or lowest beneficial strategies for player $n+1$. Following that, as a direct consequence of Definition 2.1, Definition 2.2, and Remark 2.2, one easily obtains the following conclusions.

Proposition 2.2. *Assume Assumption 2.1 and Assumption 2.2. Then,*

$i)$ *If $P^G_{n+1} < P^F_{n+1} \leq P^L_{n+1}$ or $P^G_{n+1} = P^F_{n+1} < P^L_{n+1}$, then there exists at least a most beneficial game for player $n+1$ that is $\mathrm{SLMF}_{n+1}(\beta)$ and at least a lowest beneficial game $\mathrm{GNEP}_{n+1}(\beta)$.*
$ii)$ *If $P^G_{n+1} < P^L_{n+1} \leq P^F_{n+1}$ or $P^G_{n+1} = P^L_{n+1} < P^F_{n+1}$, then there exists at least a most beneficial game for player $n+1$ that is $\mathrm{MLSF}_{n+1}(\beta)$ and at least a lowest beneficial game $\mathrm{GNEP}_{n+1}(\beta)$.*
$iii)$ *If $P^G_{n+1} = P^F_{n+1} = P^L_{n+1}$, then any decision of player $n+1$ is neutral.*

3. A SELECTION PROCESS FOR BOUNDED STRATEGY MODELS

One of the difficulties of the above presented models is that they required, for the well-posedness of models $\mathrm{SLMF}_{n+1}(\beta)$, $\mathrm{GNEP}_{n+1}(\beta)$, and $\mathrm{MLSF}_{n+1}(\beta)$, that the generalized Nash equilibrium problem considered in these models admits a unique equilibrium (see Assumption 2.2). This hypothesis is known to be quite difficult to guarantee. We thus propose here an adaptation of the formulation for which this uniqueness hypothesis will be automatically satisfied. It consists of a selection process on the different possible generalized Nash equilibria. This selection of generalized Nash equilibrium will be based on specific forms of the constraints sets C_i and $C(\beta)$ which correspond to the case of bounded strategy sets. Namely, for the rest of the content, we will assume as follows.

- For any $i = 1, \ldots, n+1$, the constraint strategy set of player i is given by

$$C_i := \prod_{l=1}^{q} [0, \overline{X}_{i,l}]$$

where the consumption upper bound $\bar{X}_{i,l}$ of player i for commodity l is such that $0 < \bar{X}_{i,l} < \beta_l$. Note that the right inequality expresses here that no player can act in a monopolistic way for commodity l.

- The common constraint set $C(\beta)$ is given by
$$C(\beta) := \left\{(x_1,\dots,x_{n+1}) \in \mathbb{R}^{(n+1)q} : \sum_{i=1}^{n+1} x_{i,l} \le \beta_l,\ \forall l = 1,\dots,q\right\},$$
where $x_{i,l}$ stands for the quantity of commodity l consumed by player i, and the set $C(\beta)$, with $\beta = (\beta_l)_l$, defines the set of feasible consumption quantities for all commodities q and all players i taking into account that, for each commodity l the total consumption can not exceed the maximum available amount β_l of the commodity.

3.1. **Weighted Nash equilibrium.** Now the selection process between the different possible generalized Nash equilibria will be based on a "weights vector" $(w_1,\dots,w_{n+1})$ of the players and the associated new concept of *weighted Nash equilibrium*.

Definition 3.1 (Weighted constraint)**.** Consider that $n+1$ players are interacting on a market with a maximum exchange volume $\beta \in \mathbb{R}^q$ such that, for any $l = 1,\dots,q$, $\sum_{i=1}^{n+1} \bar{X}_{i,l} \ne \beta_l$. Let $\mathscr{W} = \{w_1,\dots,w_{n+1}\}$ be a family of weights of the players satisfying
$$\text{for any } l \begin{cases} \text{for any } i,\ w_{i,l} \in \left[0, \dfrac{\bar{X}_{i,l}}{\left|\sum_{i=1}^{n+1} \bar{X}_{i,l} - \beta_l\right|}\right[, \\ \text{and } \sum_{i=1}^{n} w_{i,l} = 1. \end{cases} \tag{3.1}$$
Then, for any *pre-booking vector* $\delta \in [0,\beta_l]^q$, the *weighted consumption bounds* of player $i = 1,\dots,n+1$ for commodity l is defined as follows:

- for a generalized Nash game between players $\{1,\dots,n\}$, for any $l = 1,\dots,q$,
$$\begin{aligned} \bar{X}^w_{i,l}(n,\delta_l) &= \bar{X}_{i,l} - w_{i,l}\max\left\{0, \left[\sum_{j=1}^{n} \bar{X}_{j,l} - (\beta_l - \delta_l)\right]\right\} \\ &= \begin{cases} \bar{X}_{i,l} & \text{if } \sum_{j=1}^{n} \bar{X}_{j,l} \le \beta_l - \delta_l, \\ \bar{X}_{i,l} - w_{i,l}\left[\sum_{j=1}^{n} \bar{X}_{j,l} - (\beta_l - \delta_l)\right] & \text{otherwise;} \end{cases} \end{aligned}$$
- for a generalized Nash game between players $\{1,\dots,n+1\}$, for any $l = 1,\dots,q$,
$$\begin{aligned} \bar{X}^w_{i,l}(n+1,\delta_l) &= \bar{X}_{i,l} - w_{i,l}\max\left\{0, \left[\sum_{j=1}^{n+1} \bar{X}_{j,l} - (\beta_l - \delta_l)\right]\right\} \\ &= \begin{cases} \bar{X}_{i,l} & \text{if } \sum_{j=1}^{n+1} \bar{X}_{j,l} \le \beta_l - \delta_l, \\ \bar{X}_{i,l} - w_{i,l}\left[\sum_{j=1}^{n+1} \bar{X}_{j,l} - (\beta_l - \delta_l)\right] & \text{otherwise.} \end{cases} \end{aligned}$$

An economical interpretation of the weights $w_i = (w_{i,l})_l$ is the following. Suppose that the players in the market of commodity l have different market shares. In this case, their power in negotiating the consumption quantities can reflect their market share, then $(w_{i,l})$ can be interpreted as an evaluation in their power in negotiating while the pre-booking vector $\delta = (\delta_l)_l$ stands for a part of the maximum exchange volume $\beta = (\beta_l)_l$, which has already been bought, for example, by player $n+1$ in the $\text{SLMF}_{n+1}(\beta)$.

It can appear to be quite surprising that the weights $w_{i,l}$ are assumed to satisfy the equality $\sum_{i=1}^{n} w_{i,l} = 1$ and not equality $\sum_{i=1}^{n+1} w_{i,l} = 1$. This specific choice of the weights is motivated by the fact that the weights are chosen before knowing what kind of game will be faced by the group of n players and player $n+1$ (SLMF, MLSF, or GNEP) and the fact that condition $\sum_{i=1}^{n} w_{i,l} = 1$ is needed to ensure that multi-leader-follower games (SLMF and MLSF) will be well-posed. This will be explicitly proved in item (i) of the forthcoming Proposition 3.1.

Example 3.1. Let us consider a game with 4 players and a market in which a goods is sold. Suppose that $\{x_1, x_2, x_3, x_4\}$ represents the quantity of the goods bought by each player, respectively. The maximum requests of the goods for each player are $\bar{X}_1 = 3$, $\bar{X}_2 = 5$, $\bar{X}_3 = 6$ and $\bar{X}_4 = 4$, respectively. If $\beta = 20$, then the four players can buy as much as they want, since $\sum_{i=1}^{4} \bar{X}_i \le \beta$. Suppose now the total availability of the goods reduces to $\hat{\beta} = 15$ and suppose further the last player 4 has a chance to buy first. If player 4 decides to buy a quantity $x_4 = 4$, then the three other players need to play in a Nash game such that $x_1 + x_2 + x_3 \le \hat{\beta} - x_4 = 11$. Since the weighted coefficients of three players are such that $\sum_{i=1}^{3} w_i = 1$, all new weighted endowments $\bar{X}_i^w$ of the players will adapt to be fit the new volume $\hat{\beta} - x_4$.

Replacing, in a generalized Nash game, the consumption bounds $\bar{X}_{i,l}$ by the weighted consumption bounds, leads to the concept of *weighted Nash equilibrium*.

Definition 3.2 (Weighted generalized Nash equilibrium problem)**.** Consider that $n+1$ players are interacting on a market with a maximum exchange volume $\beta \in \mathbb{R}^q$ such that, for any $l = 1, \ldots, q$ and $\sum_{i=1}^{n+1} \bar{X}_{i,l} \neq \beta_l$. Let $\mathscr{W} = \{w_1, \ldots, w_{n+1}\}$ be a given family of weights of the players satisfying condition (3.1).

Then, for $p = n$ or $p = n+1$, and for any pre-booking $\delta \in [0, \beta_l]^q$, the *weighted generalized Nash equilibrium problem* $\mathrm{GNEP}_p^w(\beta - \delta)$ consists of:

$$\begin{aligned}
&\text{Finding } \bar{x} = (\bar{x}_1, \ldots, \bar{x}_p) \in \mathbb{R}^{qp} \text{ such that }, \forall i = 1, \ldots, p,\ \bar{x}_i \text{ is a solution}\\
&\text{of the problem of player } i\\
&\quad \big(P_i^w(\bar{x}_{-i})\big) \qquad \max_{x_i}\ \theta_i(x_i, \bar{x}_{-i})\\
&\qquad\qquad\qquad\qquad \text{s.t. } \ x_{i,l} \in \big[0, \bar{X}_{i,l}^w(p, \beta_l - \delta_l)\big], \quad \forall l = 1, \ldots, q.
\end{aligned}$$

The equilibria $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_p)$ are called weighted generalized Nash equilibria and their set will be denoted by $\mathrm{GNE}_p^w(\beta - \delta)$.

It follows immediately from definitions 3.1 and 3.2 that if, for any $l = 1, \ldots, q$, $\sum_{j=1}^{p} \bar{X}_{j,l} \le \beta_l - \delta_l$, then for any $i = 1, \ldots, p$ (with $p = n$ or $n+1$), one has $\bar{X}_{i,l}^w(p, \delta) = \bar{X}_{i,l}$ and thus $\mathrm{GNE}_p^w(\beta - \delta) = \mathrm{GNE}_p(\beta - \delta)$. This situation corresponds to the case where the maximum exchange volume is higher than the maximum cumulative consumption of the players.

A natural question arising is the link between the set of generalized Nash equilibria of $\mathrm{GNE}_p(\beta - \delta)$ and the set of weighted Nash equilibria $\mathrm{GNE}_p^w(\beta)$. As shown in the forthcoming proposition, any weighted generalized Nash equilibrium is actually a generalized Nash equilibrium of $\mathrm{GNEP}_p(\beta - \delta)$ thus bringing to the fore that replacing the consumption bounds by the weighted consumption bounds leads to a selection process on the Nash equilibria.

Proposition 3.1. *Consider that $n+1$ players are interacting on a market with a maximum exchange volume $\beta \in \mathbb{R}^q$ such that, for any $l = 1, \ldots, l$, $\sum_{i=1}^{n+1} \bar{X}_{i,l} \neq \beta_l$. Let $\mathscr{W} = \{w_1, \ldots, w_{n+1}\}$ be a family of weights of the players satisfying condition* (3.1).

Then, for $p=n$ or $p=n+1$ and any pre-booking $\delta \in [0,\beta_l]^q$, one has:

i) Let $l \in \{1,\dots,q\}$. If, for any $i=1,\dots,p$, $x_{i,l} \in \left[0,\overline{X}^w_{i,l}(p,\beta_l-\delta_l)\right]$, then $\sum_{i=1}^{p} x_{i,l} \le \beta_l - \delta_l$;

ii) Any weighted generalized Nash equilibrium is actually a generalized Nash equilibrium of $\mathrm{GNEP}_p(\beta-\delta)$, that is,

$$\mathrm{GNE}^w_p(\beta-\delta) \subseteq \mathrm{GNE}_p(\beta-\delta).$$

Proof. Let us first observe that, as an immediate consequence of the definition of weighted consumption bounds, one has, for any $i=1,\dots,p$ and any $l=1,\dots,q$, $\overline{X}^w_{i,l}(p,\delta) \le \overline{X}_{i,l}$.

To prove i), let us first consider the case $p=n$. If $\sum_{j=1}^{n} \overline{X}_{j,l} \le \beta_l - \delta_l$, then, for any i, $\overline{X}^w_{i,l}(p,\beta_l-\delta_l) = \overline{X}_{i,l}$ and the desired inequality is trivially fulfilled. So let us assume that $\sum_{j=1}^{n} \overline{X}_{j,l} > \beta_l - \delta_l$ and, for any $i=1,\dots,n$, $x_{i,l} \in \left[0,\overline{X}^w_{i,l}(n,\beta_l-\delta_l)\right]$. Then one can deduce that

$$\begin{array}{rcl} \sum_{i=1}^{n} x_{i,l} & \le & \sum_{i=1}^{n} \overline{X}^w_{i,l}(n,\beta_l-\delta_l) \\ & = & \sum_{i=1}^{n} \overline{X}_{i,l} - \sum_{i=1}^{n} w_{i,l}\left(\sum_{k=1}^{n}\overline{X}_{k,l} - (\beta_l-\delta_l)\right) \\ & = & \sum_{i=1}^{n} \overline{X}_{i,l} - \sum_{k=1}^{n} \overline{X}_{k,l} + (\beta_l-\delta_l) = \beta_l - \delta_l. \end{array}$$

Now in the case $p=n+1$, since the case $\sum_{j=1}^{n+1} \overline{X}_{j,l} \le \beta_l - \delta_l$ is as immediate as above, let us assume that $\sum_{j=1}^{n+1} \overline{X}_{j,l} > \beta_l - \delta_l$ and, for any $i=1,\dots,n+1$, $x_{i,l} \in \left[0,\overline{X}^w_{i,l}(n+1,\beta_l-\delta_l)\right]$. Then, similarly,

$$\begin{array}{rcl} \sum_{i=1}^{n+1} x_{i,l} & \le & \sum_{i=1}^{n+1} \overline{X}^w_{i,l}(n+1,\beta_l-\delta_l) \\ & = & \sum_{i=1}^{n+1} \overline{X}_{i,l} - \sum_{i=1}^{n+1} w_{i,l}\left(\sum_{k=1}^{n+1}\overline{X}_{k,l} - (\beta_l-\delta_l)\right) \\ & = & \sum_{i=1}^{n+1} \overline{X}_{i,l} - (1+w_{n+1})\left(\sum_{k=1}^{n+1}\overline{X}_{k,l} - (\beta_l-\delta_l)\right) \\ & = & \sum_{i=1}^{n+1} \overline{X}_{i,l} - \sum_{k=1}^{n+1} \overline{X}_{k,l} + (\beta_l-\delta_l) - w_{n+1}\left(\sum_{k=1}^{n+1}\overline{X}_{k,l} - (\beta_l-\delta_l)\right) \\ & \le & \beta_l - \delta_l. \end{array}$$

Now ii) is a direct consequence of (i). Indeed, according to (i), $\mathrm{GNEP}_p(\beta-\delta)$ can be simplified in

Finding $\bar{x} = (\bar{x}_1,\dots,\bar{x}_p) \in \mathbb{R}^{qp}$ such that, for any $i=1,\dots,p$, $\bar{x}_i$ is a solution of the problem of player i

$$\begin{array}{rl} & \max_{x_i} \ \theta_i(x_i,\bar{x}_{-i}) \\ \text{s.t.} & x_{i,l} \in \left[0,\overline{X}_{i,l}\right], \quad l = 1\dots,q. \end{array}$$

and thus any weighted generalized Nash equilibrium of $\mathrm{GNEP}^w_p(\beta-\delta)$ is actually a generalized Nash equilibrium of $\mathrm{GNEP}_p(\beta-\delta)$. □

Let us first observe that, as a consequence of item (i), the fundamental inequality "$\sum_{i=1}^{p} x_{i,l} \le \beta_l - \delta_l$" stating that the total consumption of a commodity l cannot exceed the maximum exchange volume $\beta_l - \delta_l$ of this commodity can be dropped from the GNEP of the three models

$\text{SLMF}_{n+1}(\beta)$, $\text{GNEP}_{n+1}(\beta)$, and $\text{MLSF}_{n+1}(\beta)$ as soon as one considers "weighted formulation".

Thus, according to Proposition 3.1 (*ii*), the weighted Nash equilibria can be interpreted as a selection process of the generalized Nash equilibria of $\text{GNEP}_n(\beta)$. In addition to the fact that the formulation of $\text{GNEP}_n^w(\beta)$ is simpler than the one of $\text{GNEP}_n(\beta)$, one can wonder what is the real improvement considering this selection process. There are two main reasons:

- First as it will be shown in the forthcoming Proposition 3.2, it can be proved that, given a family $\mathscr{W} = \{w_1,\dots,w_{n+1}\}$ of weights of the players satisfying conditions (3.1) and under mild assumptions, there exists a unique weighted generalized Nash equilibrium. This uniqueness property will be of course of main importance when considering, in Section 3.2 and afterwards, weighted version of problems $\text{SLMF}_{n+1}(\beta)$, $\text{GNEP}_{n+1}(\beta)$ and $\text{MLSF}_{n+1}(\beta)$. Indeed the "GNEP part" of Assumption 2.2-*b*) will be automatically satisfied while no "optimistic" or "pessimistic" formulations will be needed in the weighted version of $\text{SLMF}_{n+1}(\beta)$ and $\text{MLSF}_{n+1}(\beta)$.
- The second main reason to consider weighted Nash equilibrium is that it will drastically simplify the structure of the three models $\text{SLMF}_{n+1}(\beta)$, $\text{GNEP}_{n+1}(\beta)$ and $\text{MLSF}_{n+1}(\beta)$ (see Remark 3.1 and Proposition 3.3).

Proposition 3.2. *Consider that $n+1$ players are interacting on a market with a maximum exchange volume $\beta \in \mathbb{R}^q$ such that, for any $l = 1,\dots,q$, $\sum_{i=1}^{n+1} \bar{X}_{i,l} \neq \beta_l$. Let $\mathscr{W} = \{w_1,\dots,w_{n+1}\}$ be a given family of weights of the players satisfying conditions* (3.1).

Let $p = n$ or $p = n+1$ and $\delta \in [0,\beta_l]^q$ be a pre-booking vector. Assume that, for any $i = 1,\dots,p$, the function $\theta_i(\cdot,x_{-i})$ is continuous in both variables x_i and x_{-i} and for any $x_{-i} \in \prod_{k=1, k\neq i}^{p} \prod_{l=1}^{q} [0,\bar{X}_{k,l}]$, the function $\theta_i(\cdot,x_{-i})$ is diagonally strictly concave.

Then,

i) $\text{GNEP}_p^w(\beta-\delta)$ *admits a unique weighted Nash equilibrium* $\bar{x}^w = (\bar{x}_1^w,\dots,\bar{x}_p^w)$*;*

ii) *If $q = 1$ and $\operatorname*{argmax}_{x_i \in \mathbb{R}_+^q} \theta_i(x_i,x_{-i})$ is a singleton (denoted by $\{x_i^\star\}$) independently of the value of x_{-i}, then*

$$\forall i = 1,\dots,p, \quad \bar{x}_i^w = \begin{cases} \bar{X}_i^w(p,\beta-\delta) & \text{if } \bar{X}_i^w(p,\beta-\delta) < x_i^\star, \\ x_i^\star & \text{otherwise.} \end{cases}$$

Note that in case *ii*) (that is with $q = 1$), the notation $\bar{X}_i^w(p,\beta-\delta)$ is a shortcut for $\bar{X}_{i,1}^w(p,\beta-\delta)$.

Remark 3.1. The proof of Proposition 3.2 is a direct consequence of the following important observation: the use of the selection process through weighted Nash equilibrium allows to transform the bounded strategy generalized Nash game $\text{GNEP}_p(\beta)$ into the (classical) Nash game $\text{GNEP}_p^w(\beta)$. Indeed one can easily observe that in Definition 3.2 the constraint set of problem $\left(P_i^w(\bar{x}_{-i})\right)$ does not depend on the values of the other player strategies. It is thus an important advantage of the proposed selection process.

Proof. Taking into account Remark 3.1, one simply has to prove the existence and uniqueness of the classical Nash game $\text{GNEP}_p^w(\beta)$. For each $i = 1,\dots,p$, the constraints set $\prod_{l=1}^{q} [0,\bar{X}_{i,l}^w]$ of the corresponding optimization problem $P_i^w(x_{-i})$ is non-empty, convex and compact in $\mathbb{R}_+^*$. On the other hand, the function $(x_i,x_{-i}) \longmapsto \theta_i(x_i,x_{-i})$ is continuous in both x_i and x_{-i} and diagonally strictly concave in x_i. Thus the existence of a weighted Nash equilibrium can be derived from [16].

Now let $q=1$ and $\bar{x}$ be an equilibrium of $\mathrm{GNEP}_p^w(\beta)$. Then, for every $i=1,\dots,p$,

$$\bar{x}_i \in \underset{[0,\bar{X}_i^w(p,\beta-\delta)]}{\operatorname{argmax}} \ \theta_i(x_i,\bar{x}_{-i}).$$

If $x_i^\star \in \left[0,\bar{X}_i^w(p,\beta-\delta)\right]$, then one clearly has $\bar{x}_i = x_i^\star$. Otherwise $x_i^\star > \bar{X}_i^w(p,\beta_l-\delta_l)$, then we have $\underset{[0,\bar{X}_i^w]}{\operatorname{argmax}}\,\theta_i(x_i,\bar{x}_{-i}) = \bar{X}_i^w(p,\beta-\delta)$ since the function $\theta_i(\cdot,x_{-i})$ is strictly increasing on $\left[0,\bar{X}_i^w(p,\beta-\delta)\right]$. □

Example 3.2. In a game where there are 3 players sharing a single-commodity market, let us consider a generalized Nash equilibrium problem between 2 players with exchange volume $\beta=25$, then $\mathrm{GNEP}_2(25)$ is defined as follows.

$$\begin{array}{ll} \max\limits_{x_1} \ 60x_1 - 2x_1^2 & \\ \text{s.t.} \ \begin{cases} x_1 \in [0,12], \\ x_1+\bar{x}_2 \le 25, \end{cases} \end{array} \quad \text{and} \quad \begin{array}{ll} \max\limits_{x_2} \ 60x_2 - 1.5x_2^2 & \\ \text{s.t.} \ \begin{cases} x_2 \in [0,18], \\ x_2+\bar{x}_1 \le 25. \end{cases} \end{array}$$

We are searching for a vector $\bar{x}=(\bar{x}_1,\bar{x}_2)\in\mathbb{R}^2$ such that $\bar{x}_1$ and $\bar{x}_2$ are solutions of the $\mathrm{GNEP}_2(25)$. The equilibria for this problem are

$$\mathrm{GNE}_2(25) = \left\{(x_1,x_2)\in\mathbb{R}_+^2 \ : \ x_1+x_2=25,\ x_1\le 12,\ x_2\le 18\right\},$$

which clearly contains several (actually an infinite number of) couples. However, for each weighted coefficient $w=(w_1,w_2)=(w_1,1-w_1)\in[0,2/5]\times[0,3/5]$, there is a unique weighted equilibrium

$$\begin{aligned} \mathrm{GNE}_2^w(25) &= \left\{ (x_1,x_2)\in\mathbb{R}_+^2 \ : \ \begin{array}{l} x_1+x_2=25, \\ x_1 \le 12 - w_1\max\{0,12+18-25\}, \\ x_2 \le 18 - w_2\max\{0,12+18-25\} \end{array} \right\} \\ &= \left\{(12-5w_1, 18-5w_2)\right\} \\ &= \left\{(12-5w_1, 13+5w_1)\right\}. \end{aligned}$$

Then, it can be easily observed that a weighted Nash equilibrium is a selection of the classical Nash equilibrium corresponding to the coefficient w, that yields the uniqueness of $\mathrm{GNEP}_2^w(25)$ coming from original $\mathrm{GNEP}_2(25)$.

Let us now consider the same setting except for a change to the value of $\tilde{\beta}=32$. The new $\mathrm{GNEP}_2(32)$ has a unique equilibrium and both classical and weighted GNEPs admit the same equilibrium, $\mathrm{GNE}_2(32)=\mathrm{GNE}_2^w(32)=(12,18)$. To conclude, under some appropriate assumptions on the value of exchange volume and constraint sets, GNEP can achieve the equilibrium uniqueness, while for GNEP^w it is surely guaranteed.

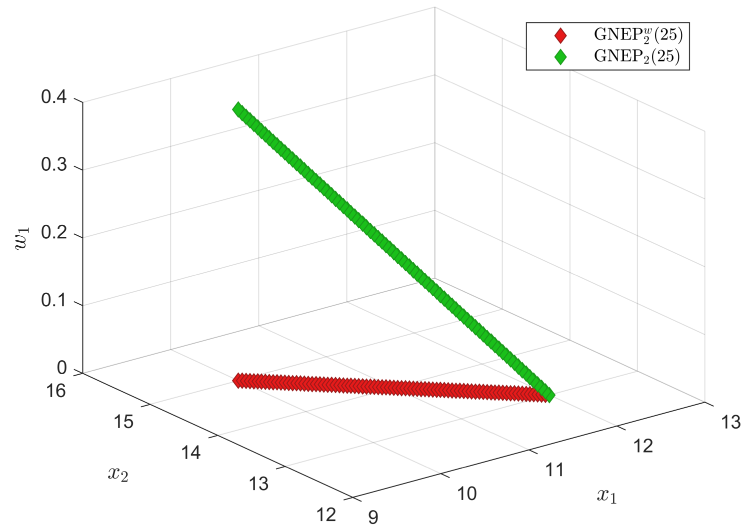

FIGURE 1. An illustration for classical and weighted Nash equilibria.

In Figure 1, a comparison between the unique equilibrium of the $\text{GNEP}_2^w(25)$ and the equilibria in the $\text{GNEP}_2(25)$ is depicted.

3.2. **Weighted multi-leader-follower exchange models.** Let us now come back to the three initial models $\text{SLMF}_{n+1}(\beta)$, $\text{GNEP}_{n+1}(\beta)$, and $\text{MLSF}_{n+1}(\beta)$. By taking into account Proposition 3.1 (i) and Proposition 3.2, we can now replace them with $\text{SLMF}^w_{n+1}(\beta)$, $\text{GNEP}^w_{n+1}(\beta)$ and $\text{MLSF}^w_{n+1}(\beta)$, by considering weighted Nash equilibrium instead of generalized Nash equilibrium. The three new formulations are defined as follows.

a) **The weighted single-leader-multi-follower game $\text{SLMF}^w_{n+1}(\beta)$**:
An $(n+1)$-player single-leader-multi-follower game after the arrival of player $n+1$ is defined as

$$\left(\bar{P}^w(\beta)\right) \qquad \max_{x_{n+1}} \; \theta_{n+1}(x_{n+1}, x_{-(n+1)})$$
$$\text{s.t.} \quad \begin{cases} 0 \le x_{n+1,l} \le \bar{X}_{n+1,l}, \quad l = 1,\ldots,q, \\ (x_1^w,\ldots,x_n^w) = Eq^w(\beta - x_{n+1}), \end{cases}$$

where $Eq^w(\beta - x_{n+1})$ is the unique weighted Nash equilibrium of $\text{GNEP}^w_n(\beta - x_{n+1})$ defined by

$$\forall i = 1,\ldots,n, \; \left(\bar{P}^w_i(\beta - x_{n+1})\right) \qquad \max_{x_i} \; \theta_i(x_i, x_{-i})$$
$$\text{s.t.} \quad x_{i,l} \in \left[0, \bar{X}^w_{i,l}(n, x_{n+1,l})\right], \quad l = 1,\ldots,q.$$

The uniqueness assumption (Assumption 2.2) for $\text{SLMF}^w_{n+1}(\beta)$ can be satisfied by combining a diagonally strict quasi-convexity hypothesis of function $\theta_{n+1}(\cdot, x_{-(n+1)})$ with Proposition 3.2. The unique solution of $\text{SLMF}^w_{n+1}(\beta)$ will be denoted by $(\bar{x}^L_{n+1}, \bar{x}^F_1, \ldots, \bar{x}^F_n)$ while the payoff of player $n+1$ will thus be $P^L_{n+1}(\beta)$.

b) **The weighted generalized Nash equilibrium problem GNEP$^w_{n+1}(\beta)$**:
An $(n+1)$-player generalized Nash game after the arrival of player $n+1$ is defined as

$$\forall i=1,\ldots,n+1,\ \left(\widetilde{P}_i^w(\beta)\right) \qquad \begin{array}{ll} & \max_{x_i}\ \theta_i(x_i,x_{-i}) \\ \text{s.t.} & x_{i,l}\in\left[0,\overline{X}^w_{i,l}(n+1,0)\right], \quad l=1,\ldots,q. \end{array}$$

The uniqueness assumption (Assumption 2.2) for GNEP$^w_{n+1}(\beta)$ can be satisfied simply through Proposition 3.2. The unique solution of GNEP$^w_{n+1}(\beta)$ will be denoted by $(\bar{x}_1^G,\ldots,\bar{x}_n^G,\bar{x}_{n+1}^G)$ while the payoff of player $n+1$ will thus be $P^G_{n+1}(\beta)$.

c) **The weighted multi-leader-single-follower game MLSF$^w_{n+1}(\beta)$**:
An $(n+1)$-player multi-leader-single-follower game after the arrival of player $n+1$ is defined as

$$\forall i=1,\ldots,n,\ \left(\widehat{P}_i^w(\beta)\right) \qquad \begin{array}{ll} & \max_{x_i}\ \theta_i(x_i,x_{-i}) \\ \text{s.t.} & \begin{cases} 0\le x_{i,l}\le \overline{X}^w_{i,l}(n,0), \quad l=1,\ldots,q, \\ x_{n+1} \text{ solves } \left(\widehat{P}(\beta-\sum_{j=1}^n x_j)\right), \end{cases} \end{array}$$

where

$$\left(\widehat{P}(\beta-\sum_{j=1}^n x_j)\right) \qquad \begin{array}{ll} & \max_{x_{n+1}}\ \theta_{n+1}(x_{n+1},x_{-(n+1)}) \\ \text{s.t.} & \begin{cases} 0\le x_{n+1,l}\le \overline{X}_{n+1,l}, \quad l=1,\ldots,q, \\ \sum_{i=1}^{n+1} x_{i,l}\le \beta_l, \quad l=1,\ldots,q. \end{cases} \end{array}$$

The uniqueness of *best response* x_{n+1} of player $n+1$ can be obtained thanks to a diagonally strict quasi-concavity hypothesis of function $\theta_{n+1}(\cdot,x_{-(n+1)})$ while the uniqueness of the equilibrium of the upper level generalized Nash game can be inferred from Proposition 3.2. The unique solution of MLSF$^w_{n+1}(\beta)$ will be denoted by $(\bar{x}_1^L,\ldots,\bar{x}_n^L,\bar{x}_{n+1}^F)$ while the payoff of player $n+1$ will thus be $P^F_{n+1}(\beta)$.

Note that, besides the use of weighted consumption bounds, at the upper level, the equation $\sum_{i=1}^{n+1} x_i\le\beta$ is maintained in the lower level because Proposition 3.1 (i) cannot be used here.

Let us now end this section by providing explicit formulae for the solutions of the three models that player $n+1$ can face, that are SLMF$^w_{n+1}$, GNEP$^w_{n+1}$ and MLSF$^w_{n+1}$.

Proposition 3.3. *Consider that $n+1$ players are interacting on a market with one commodity ($q=1$) and a maximum exchange volume $\beta\in\mathbb{R}$ such that $\sum_{i=1}^{n+1}\overline{X}_i\neq\beta$. Let $\mathscr{W}=\{w_1,\ldots,w_{n+1}\}$ be a family of weights of the players satisfying conditions (3.1). Assume that, for any $i=1,\ldots,n+1$, the function $\theta_i(\cdot,x_{-i})$ is continuous in both variables x_i and x_{-i}. Assume moreover that for any $x_{-i}\in\prod_{k=1, k\neq i}^{n+1}[0,\overline{X}_k]$, the function $\theta_i(\cdot,x_{-i})$ is diagonally strictly concave and $\operatorname{argmax}_{x_i\in\mathbb{R}^q_+}\theta_i(x_i,x_{-i})$ is a singleton (denoted by $\{x_i^\star\}$).*

Then, the following assertions hold:

i) *The game* $\mathrm{SLMF}^w_{n+1}(\beta)$ *admits a unique solution* $\bar{x} = (\bar{x}^L_{n+1}, \bar{x}^F_1, \ldots, \bar{x}^F_n)$*, where the leader-type solution of player* $n+1$ *is given by*

$$\bar{x}^L_{n+1} = \min\{x^\star_{n+1}, \overline{X}_{n+1}\},$$

coupled with the follower-type solutions in the lower-level $\mathrm{GNEP}^w_n(\beta - x^L_{n+1})$*,*

$$\bar{x}^F_i = \min\{x^\star_i, \overline{X}^w_{n+1}(n, \bar{x}^L_{n+1})\}, \quad \forall i = 1, \ldots, n.$$

ii) *The* $\mathrm{GNEP}^w_{n+1}(\beta)$ *admits a unique solution* $\tilde{x} = (\bar{x}^G_1, \ldots, \bar{x}^G_n, \bar{x}^G_{n+1})$*, where the solution of each player i is given by*

$$\bar{x}^G_i = \min\{x^\star_i, \overline{X}^w_i(n+1, 0)\}, \quad \forall i = 1, \ldots, n+1.$$

iii) *The game* $\mathrm{MLSF}^w_{n+1}(\beta)$ *admits a unique solution* $\hat{x} = (\bar{x}^L_1, \ldots, \bar{x}^L_n, \bar{x}^F_{n+1})$*, where the leader-type solutions of the upper-level* $\mathrm{GNEP}^w_n(\beta)$*,*

$$\bar{x}^L_i = \min\{x^\star_i, \overline{X}^w_i(n, 0)\}, \quad \forall i = 1, \ldots, n,$$

coupled the follower-type solution of player $n+1$ *is given by*

$$\bar{x}^F_{n+1} = \min\{x^\star_{n+1}, \overline{X}_{n+1}, \beta - \sum_{i=1}^{n} \bar{x}^L_i\}.$$

Proof. *i)* (SLMF^w_{n+1}): Let us first observe that for solving the upper-level optimization problem, the equilibrium in the lower-level $\mathrm{GNEP}^w_n(\beta - x_{n+1})$ has to be inferred. Since the objective function of each player i in the Nash game at the lower level is continuous and diagonally strictly concave, according to Proposition 3.2 (*ii*), a unique weighted Nash equilibrium can be obtained for the $\mathrm{GNEP}^w_n(\beta - x_{n+1})$, that is,

$$\begin{aligned} x^F_l &= \{x_1, \ldots, x_n\} \\ &= \Big\{ \min\{x^\star_1, \overline{X}^w_{n+1}(n, x_{n+1})\}, \ldots, \min\{x^\star_n, \overline{X}^w_{n+1}(n, x_{n+1})\} \Big\}. \end{aligned}$$

Thus, the single leader problem $(\bar{P}(\beta))$ turns out to express as

$$\big(\bar{P}(\beta)\big) \qquad \max_{x_{n+1}} \ \theta_{n+1}(x_{n+1}, x_{-(n+1)})$$
$$\text{s.t.} \ \begin{cases} 0 \le x_{n+1} \le \overline{X}_{n+1}, \\ (x_1, \ldots, x_n) = x^F_l. \end{cases}$$

Thanks to concavity of function θ_{n+1}, this problem admits a unique solution

$$\bar{x}^L_{n+1} = \min\{x^\star_{n+1}, \overline{X}_{n+1}\}.$$

and

$$\bar{x}^F_i = \min\{x^\star_i, \overline{X}^w_{n+1}(n, \bar{x}^L_{n+1})\}, \quad \forall i = 1, \ldots, n.$$

ii) (GNEP^w_{n+1}): The results follows directly from Proposition 3.2 (*ii*) for $p = (n+1)$ and the $(n+1)$-player game $\mathrm{GNEP}^w_{n+1}(\beta)$.

iii) (MLSF^w_{n+1}): Let us consider the lower-level optimization problem. Player $n+1$ optimizes the objective function taking as already settled the optimal decision of the upper-level problem. Looking at the constraint set, we have $0 \le x_{n+1} \le \min\{\overline{X}_{n+1}, \beta - \sum_{i=1}^{n} x^L_i\}$. Then, since the objective function is diagonally strictly concave and the constraint set is boxed, we can conclude that the parametrized optimal solution of player $n+1$ is

$$x_{n+1}^F = \min\left\{x_{n+1}^\star, \min\left\{\bar{X}_{n+1}, \beta - \sum_{i=1}^n x_i^L\right\}\right\}$$
$$= \min\left\{x_{n+1}^\star, \bar{X}_{n+1}, \beta - \sum_{i=1}^n x_i^L\right\}.$$

And thus the leader's problems $\left(\widehat{P}_i^w(\beta)\right)_{i=1,\dots,n}$ or namely $\mathrm{GNEP}_n^w(\beta)$ turns out to be expressed as

$$\forall i = 1,\dots,n,\ \left(\widehat{P}_i(\beta)\right) \qquad \max_{x_i}\ \theta_i(x_i, x_i),$$
$$\text{s.t.} \quad \begin{cases} 0 \le x_i \le \bar{X}_i^w(n,0), \\ x_{n+1} = x_{n+1}^F. \end{cases}$$

Then by applying Proposition 3.2 (ii) with $p = n$, we obtain a unique weighted Nash equilibrium

$$\bar{x}_l^L = \{\bar{x}_1^L, \dots, \bar{x}_n^L\}$$
$$= \left\{\min\left\{x_1^\star, \bar{X}_1^w(n,0)\right\}, \dots, \min\left\{x_n^\star, \bar{X}_n^w(n,0)\right\}\right\}.$$

And thus,

$$\bar{x}_{n+1}^F = \min\left\{x_{n+1}^\star, \bar{X}_{n+1}, \beta - \sum_{i=1}^n \bar{x}_i^L\right\}.$$

Finally, the $\mathrm{MLSF}_{n+1}^w(\beta)$ admits a unique equilibrium $\hat{x}_l = (\bar{x}_1^L, \dots, \bar{x}_n^L, \bar{x}_{n+1}^F)$ and the proof is complete. □

4. Conclusion

A game in which n players are interacting in a generalized Nash game with an additional player $n+1$ waiting for entry has been analysed. This work generalizes the approach by von Stengel [6] in the case of a duopoly game. The new player can face several possibilities to interoperate with the group of n players. In particular, the gap of the new player's payoff between two possible models is evaluated: on the one hand a non-cooperative model in which this player is one of the players of a Nash game (one-level game) and on the other hand a bi-level game in which this player plays the role of a common follower or common leader. Then, three different kinds of games, SLMF_{n+1}, GNEP_{n+1}, and MLSF_{n+1} are taken into account in order to estimate the $(n+1)$-player equilibria. The new concepts of optimal, safe and neutral strategy have been introduced for player $n+1$, in the case where the exact period to move can be determined. When an optimal choice of the period to play is not available, the concepts of most/lowest beneficial strategies can be used to support the decision.

A special attention has been given on defining the set of practicable equilibria in case of GNEP, and the new notion of weighted Nash equilibrium has been introduced. The weighted Nash equilibria can be interpreted as a selection process of the generalized Nash equilibria. Then it has been proved that, given a family of weights of the players satisfying some conditions (see (3.1)) and under mild assumptions, there exists a unique weighted generalized Nash equilibrium. This uniqueness property has been then used when considering a weighted version of problems SLMF_{n+1}^w, GNEP_{n+1}^w and MLSF_{n+1}^w, to avoid "optimistic" or "pessimistic" formulations. Accordingly, a comprehensive framework for equilibria of three kinds of games is established. This former setting, finally, is the basis for the latter step, namely Part 2 of this couple of papers, that is devoted to the study of a particular case where the utility function is a quadratic concave function and the constraint set is defined by inequalities. In that context, a complete decision-making policy is developed.

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