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A NEW CLASS OF VECTOR OPTIMIZATION PROBLEMS WITH LINEAR FRACTIONAL OBJECTIVE CRITERIA

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Abstract. This paper studies a new class of vector optimization problems where the objective criteria are linear fractional functions, the ordering cone can be any nonempty closed convex pointed and solid cone, and the constraint set can be any nonempty closed convex set. Necessary optimality conditions, as well as sufficient optimality conditions, are obtained. In addition, two theorems on the connectedness of the weakly efficient solution set and the efficient solution set are established. The results are analyzed by concrete examples.

Keywords. Connectedness; Monotone variational inequality; Linear fractional function; Optimality condition; Vector optimization.

1. Introduction

Due to their significant applications in management science, remarkable properties, and theoretical importance, *linear fractional vector optimization problems* (LFVOPs) have been studied intensively in the last forty years. The book by Steuer [1], the papers of Choo and Atkins [2, 3], Choo [4], and Malivert [5] are among the first research works on LFVOPs. For subsequent studies on this class of nonconvex vector optimization problems, we refer to Benoist [6], Malivert and Popovici [7], Yen and Phuong [8], Hoa *et al.* [9, 10, 11], Yen and Yao [12], and Yen [13]. Recently, new results on the connectedness of the solution sets and properly efficient solutions of LFVOPs were obtained by Huong et al. [14, 15, 16, 17, 18] and Tuyen [19]. Detailed surveys on the available research results on LFVOPs can be found in [13] and [16]. In 2018, Yen and Yang [20] suggested an extension of linear fractional vector optimization problems to a normed space setting. All LFVOPs in the above-mentioned books and papers are vector optimization problems with *linear fractional objective criteria*, *standard ordering cones* (that is, the nonnegative orthants in Euclidean spaces), and *polyhedral convex constraint sets* (or *generalized polyhedral convex constraint sets*).

In this paper, we are interested in studying a new class of vector optimization problems in finite dimensions where the objective criteria are linear fractional functions, the ordering cone can be any nonempty closed convex pointed and solid cone, and the constraint set can be any

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nonempty closed convex set. To the best of our knowledge, those problems are considered for the first time here. Our main results on the new type of LFVOPs are the following:

- Three theorems on necessary conditions and/or sufficient conditions for a feasible point to be a weakly efficient solution or an efficient solution;

- Two theorems on the connectedness of the weakly efficient solution set and a certain part the efficient solution set.

The separation theorem [21, Theorem 11.3], an increment formula for linear fractional functions (see Lemma 2.1 below), the concept of monotone variational inequality [22, p. 83], the Minty Lemma, the connectedness preservation under certain upper semicontinuous set-valued maps [23, 24], and the treatment of LFVOPs via monotone affine vector variational inequalities of Yen and Phuong [8] will be our main tools. The obtained results are analyzed by some examples. To highlight the difficulties of obtaining complete results on the new class of LFVOPs, several nontrivial open questions will be formulated.

The paper organization is as follows. After giving some preliminaries in the next section, we establish optimality conditions in Section 3, connectedness of the efficient solution sets in Section 4. Section 5, the last section, is devoted to conclusions and topics for further investigations.

2. Preliminaries

As usual, the scalar product of two vectors x, y in an Euclidean space is denoted by $\langle x, y\rangle$. For a set $A \subset \mathbb{R}^m$, the symbol $\bar{A}$ stands for the closure of A. By $\mathrm{ri}\,A$ we denote the *relative interior* of A that is, the interior of A in the induced (or relative) topology of its affine hull $\mathrm{aff}\,A$; see, e.g., [21, p. 44]. The open ball (resp., closed ball) centered at $y \in \mathbb{R}^m$ with radius $\varepsilon > 0$ is denoted by $B(y, \varepsilon)$ (resp., $\bar{B}(y, \varepsilon)$). The closed unit ball in $\mathbb{R}^m$ is abbreviated to $\bar{B}_{\mathbb{R}^m}$. A subset $K \subset \mathbb{R}^m$ is said to be a *cone* if $tv \in K$ for all $v \in K$ and $t \geq 0$; see, e.g. [25, p. 8] and [26, p. 1].

Let $f_i : \mathbb{R}^n \to \mathbb{R} \quad (i = 1, 2, \cdots, m)$ be *linear fractional functions*, that is,

$$f_i(x) = \frac{a_i^T x + \alpha_i}{b_i^T x + \beta_i}$$

for some $a_i \in \mathbb{R}^n, b_i \in \mathbb{R}^n, \alpha_i \in \mathbb{R}$, and $\beta_i \in \mathbb{R}$. Let $\Delta \subset \mathbb{R}^n$ be a nonempty, closed, and convex set, and $K \subset \mathbb{R}^m$ be a nonempty, closed, and convex cone. It is assumed that K is *pointed* (i.e., $K \cap (-K) = \{0\}$) and *solid* (i.e., the *interior* of K, denoted by $\mathrm{int}\,K$, is nonempty). In addition, we assume that $b_i^T x + \beta_i > 0$ for all $i \in I$ and $x \in \Delta$, where $I := \{1, \cdots, m\}$ and ^{T} denotes the matrix transposition. Let

$$f(x) = (f_1(x), \ldots, f_m(x))^T \quad (\forall x \in \Delta).$$

For any $y^2, y^1 \in \mathbb{R}^n$, if $y^2 - y^1 \in K$, one writes $y^1 \leq_K y^2$ and also $y^2 \geq_K y^1$. If $y^2 - y^1 \in \mathrm{int}\,K$, one writes $y^1 <_K y^2$ and also $y^2 >_K y^1$. Consider the *vector optimization problem with linear fractional objective criteria*

$$\min_K \{f(x) \,:\, x \in \Delta\}. \tag{2.1}$$

Specializing the general concepts of efficiency and weak efficiency in [27, pp. 33–34] and [26, p. 57] to problem (2.1), we obtain the following.

Definition 2.1. A point $x \in \Delta$ is said to be an *efficient solution* of (2.1) if there exists no $y \in \Delta$ such that $f(y) \leq_K f(x)$ and $f(y) \neq f(x)$. If $x \in \Delta$ and there exists no $y \in \Delta$ with $f(y) <_K f(x)$, then one says that x is a *weakly efficient solution* of (2.1).

The *efficient solution set* and the *weakly efficient solution set* of (2.1) are denoted, respectively, by E and E^w.

Linear fractional functions have remarkable properties; see, e.g., [5] and [28, pp. 144–145]. The next lemma will be very useful for our study of (2.1).

Lemma 2.1. *The equality*

$$f_i(y) - f_i(x) = \frac{b_i^T x + \beta_i}{b_i^T y + \beta_i} \times \frac{[(b_i^T x + \beta_i) a_i^T - (a_i^T x + \alpha_i) b_i^T](y - x)}{(b_i^T x + \beta_i)^2} \tag{2.2}$$

holds for any $x, y \in \Delta$ *and* $i \in I$.

Proof. Let $x, y \in \Delta$ and $i \in I$ be given arbitrarily. We have

$$\begin{aligned} f_i(y) - f_i(x) &= \frac{a_i^T y + \alpha_i}{b_i^T y + \beta_i} - \frac{a_i^T x + \alpha_i}{b_i^T x + \beta_i} \\ &= \frac{b_i^T x + \beta_i}{b_i^T y + \beta_i} \times \frac{(a_i^T y + \alpha_i)(b_i^T x + \beta_i) - (a_i^T x + \alpha_i)(b_i^T y + \beta_i)}{(b_i^T x + \beta_i)^2} \\ &= \frac{b_i^T x + \beta_i}{b_i^T y + \beta_i} \times \frac{a_i^T (y - x)(b_i^T x + \beta_i) - b_i^T (y - x)(a_i^T x + \alpha_i)}{(b_i^T x + \beta_i)^2}, \end{aligned}$$

where the third equality follows from the identity

$$\begin{aligned} &(a_i^T y + \alpha_i)(b_i^T x + \beta_i) - (a_i^T x + \alpha_i)(b_i^T y + \beta_i) \\ &= \left[a_i^T (y - x)(b_i^T x + \beta_i) + (a_i^T x + \alpha_i)(b_i^T x + \beta_i)\right] \\ &\quad - \left[b_i^T (y - x)(a_i^T x + \alpha_i) + (b_i^T x + \beta_i)(a_i^T x + \alpha_i)\right] \end{aligned}$$

and an obvious cancellation. So, equality (2.2) is valid. □

Several useful facts about convex cones in Euclidean spaces can be presented in a unified form in the forthcoming lemma, whose proof is given for convenience of the reader. Note that *each assertion of the lemma is given under a minimal set of assumptions*.

Lemma 2.2. *Let* $K \subset \mathbb{R}^m$ *be a nonempty convex cone, and let* $K^* := \{v^* \in \mathbb{R}^m : \langle v^*, v \rangle \geq 0 \ \forall v \in K\}$ *be its positive dual cone. The following assertions are valid:*

(a1) *If* K *is closed, then* $\operatorname{int} K^* = \{y^* \in \mathbb{R}^m : \langle y^*, v \rangle > 0 \ \forall v \in K \setminus \{0\}\}$.

(a2) *If* K *is closed and pointed, then* $\operatorname{int} K^*$ *is nonempty.*

(a3) *If* $\operatorname{int} K$ *is nonempty, then* K^* *is pointed.*

(a4) *If* $\operatorname{int} K$ *is nonempty and* K *is closed then, for any* $v^0 \in \operatorname{int} K$*, the set*

$$\Lambda := \left\{\xi \in K^* : \langle \xi, v^0 \rangle = 1\right\} \tag{2.3}$$

is convex, compact, and for every $v^* \in K^* \setminus \{0\}$ *there exists a unique* $t > 0$ *such that* $tv^* \in \Lambda$ *(thus,* K^* *has a compact base).*

(a5) *If* K *is closed and pointed then, for any* $\xi^0 \in \operatorname{int} K^*$*, the set*

$$B := \left\{u \in K : \langle \xi^0, u \rangle = 1\right\}$$

is convex, compact, and for every $v \in K \setminus \{0\}$ *there exists a unique* $t > 0$ *such that* $tv \in B$ *(thus,* K *has a compact base).*

(a6) *If* $\xi \in K^* \setminus \{0\}$*, then* $\langle \xi, v \rangle > 0$ *for all* $v \in \operatorname{int} K$.

Proof. (a1) Suppose that the convex cone K is closed. If $\xi \in \operatorname{int} K^*$, then there exists $\varepsilon > 0$ such that $\xi + \varepsilon \bar{B}_{\mathbb{R}^m} \subset K^*$. So, for any $v \in K \setminus \{0\}$, one has $\langle \xi + \varepsilon z, v\rangle \geq 0$ for all $z \in \bar{B}_{\mathbb{R}^m}$. This implies that

$$0 \leq \inf_{z\in \bar{B}_{\mathbb{R}^m}} \langle \xi + \varepsilon z, v\rangle = \langle \xi, v\rangle + \varepsilon \inf_{z\in \bar{B}_{\mathbb{R}^m}} \langle z, v\rangle = \langle \xi, v\rangle - \varepsilon \|v\|.$$

It follows that $\langle \xi, v\rangle \geq \varepsilon\|v\| > 0$. Thus,

$$\operatorname{int} K^* \subset \{y^* \in \mathbb{R}^m : \langle y^*, v\rangle > 0 \ \forall v \in K \setminus \{0\}\}. \tag{2.4}$$

Now, let ξ be a vector from the set on the right-hand-side of (2.4). Put $K_1 = K \cap \{v : \|v\| = 1\}$. Clearly, K_1 is compact and one has $\langle \xi, v\rangle > 0$ for every $v \in K_1$. For each $v \in K_1$, there exist $\varepsilon_v > 0$ and an open neighborhood U_v of v in the induced topology of K_1 such that

$$\langle \xi', v'\rangle > 0 \quad \forall \xi' \in B(\xi, \varepsilon_v),\ \forall v' \in U_v. \tag{2.5}$$

Sine $K_1 = \bigcup_{v\in K_1} U(v)$ and K_1 is compact, there exist $v^1, v^2, \ldots, v^k \in K_1$ such that

$$K_1 = \bigcup_{i=1}^{k} U(v^i). \tag{2.6}$$

Set $\varepsilon = \min\{\varepsilon_{v^1}, \ldots, \varepsilon_{v^k}\}$. For every $\xi' \in B(\xi, \varepsilon)$ and $v \in K_1$, by (2.6) we can find an index $i \in \{1, 2, \ldots, k\}$ such that $v \in U(v^i)$. So, by (2.5) we have $\langle \xi', v\rangle > 0$. Therefore,

$$\langle \xi', v\rangle > 0 \quad \forall \xi' \in B(\xi, \varepsilon),\ \forall v \in K_1. \tag{2.7}$$

Take any $\xi' \in B(\xi, \varepsilon)$ and $w \in K$. If $w = 0$, then $\langle \xi', w\rangle = 0$. If $w \neq 0$, then $\frac{w}{\|w\|} \in K_1$. By (2.7), $\langle \xi', \frac{w}{\|w\|}\rangle > 0$. So, $\langle \xi', w\rangle > 0$. Thus, $\langle \xi', w\rangle \geq 0$ for every $w \in K$. This means that $\xi' \in K^*$. Since the last inclusion holds for every $\xi' \in B(\xi, \varepsilon)$, we have $\xi \in \operatorname{int} K^*$. Hence, the reverse of the inclusion (2.4) is valid. Our assertion has been proved.

(a2) Suppose that K is closed and pointed. As K^* is a nonempty convex subset of $\mathbb{R}^m$, $\operatorname{ri} K^* \neq \emptyset$ (see [21, Theorem 6.2]). If $\dim(\operatorname{aff} K^*) = m$, then $\operatorname{int} K^* = \operatorname{ri} K^*$. So, $\operatorname{int} K^* \neq \emptyset$. If $\dim(\operatorname{aff} K^*) < m$, then the linear subspace $(\operatorname{aff} K^*)^\perp := \{y \in \mathbb{R}^m : \langle y^*, y\rangle = 0 \ \forall y^* \in K^*\}$ is nontrivial. Taking any $y \in (\operatorname{aff} K^*)^\perp \setminus \{0\}$, we have $y \in (K^*)^*$ and $-y \in (K^*)^*$. Since K is a nonempty convex cone, by [21, Theorem 14.1] we have $(K^*)^* = K$. Thus, the fact $y \in K \cap (-K)$ contradicts the pointedness of K. Hence, we must have $\operatorname{int} K^* \neq \emptyset$.

(a3) Suppose that $\operatorname{int} K$ is nonempty. If K^* is not pointed, then there exists a nonzero vector $v^* \in K^* \cap (-K^*)$. Take any $v^0 \in \operatorname{int} K$ and let $\varepsilon > 0$ be such that $B(v^0, \varepsilon) \subset K$. Then, for every $u \in \bar{B}_{\mathbb{R}^m}$ we have $\langle v^*, v^0 + \varepsilon u\rangle \geq 0$ and $\langle -v^*, v^0 + \varepsilon u\rangle \geq 0$. This implies that $\langle v^*, v^0 + \varepsilon u\rangle = 0$ for all $u \in \bar{B}_{\mathbb{R}^m}$. Taking supremum of the left-hand-side of the last equality w.r.t. $u \in \bar{B}_{\mathbb{R}^m}$ yields $\langle v^*, v^0\rangle + \varepsilon\|v^*\| = 0$. We have arrived at a contradiction, because $\langle v^*, v^0\rangle \geq 0$ and $\varepsilon\|v^*\| > 0$.

(a4) Suppose that $\operatorname{int} K$ is nonempty and K is closed. As K is a nonempty convex cone, by [21, Theorem 14.1] one has $K = (K^*)^*$. Then, applying the assertion (a1) for K^* instead of K, we get $\operatorname{int} K = \{v \in K : \langle v^*, v\rangle > 0 \ \forall v^* \in K^* \setminus \{0\}\}$. So, given any $v^0 \in \operatorname{int} K$, we have $\langle v^*, v^0\rangle > 0$ for all $v^* \in K^* \setminus \{0\}$. Hence, for every $v^* \in K^* \setminus \{0\}$, putting $t = \dfrac{1}{\langle v^*, v^0\rangle}$, we have $t > 0$ and $\langle tv^*, v^0\rangle = 1$. So $tv^* \in \Lambda$, where Λ is given by (2.3). The fact that $t > 0$ is uniquely defined for every $v^* \in K^* \setminus \{0\}$ by the condition $tv^* \in \Lambda$ is obvious. It is also clear that the set Λ is closed and convex. If Λ is bounded, then Λ is compact. The case where Λ is unbounded is excluded.

Indeed, if Λ is unbounded, then the *recession cone* $0^+\Lambda$ of Λ (see [21, p. 61] for the definition of recession cone) is nontrivial by [21, Theorem 8.4]. Thus, there exists some $\eta \in 0^+\Lambda \setminus \{0\}$. Let $\bar{\xi} \in \Lambda$ be chosen arbitrarily. Since $\bar{\xi} + t\eta \in \Lambda$ for all $t > 0$, we have $\langle \bar{\xi} + t\eta, v^0 \rangle = 1$ for all $t > 0$. This yields

$$\langle \eta, v^0 \rangle = 0. \tag{2.8}$$

As $\bar{\xi} + t\eta \in \Lambda \subset K^*$ for all $t > 0$, we have $\frac{1}{t}\bar{\xi} + \eta \in K^*$ for every $t > 0$. Passing the left-hand-side of the last inclusion to the limit as $t \to +\infty$, we get $\eta \in K^*$ by the closedness of K^*. Then, (2.8) contradicts the property that $\langle v^*, v^0 \rangle > 0$ for all $v^* \in K^* \setminus \{0\}$.

(a5) Suppose that K is closed and pointed. By the assertion (a2), $\operatorname{int} K^* \neq \emptyset$. Since K is a nonempty closed convex cone, according to [21, Theorem 14.1], we have $K = (K^*)^*$. Then, the desired conclusion follows from applying the assertion (a4) to the nonempty closed convex K^* whose interior is nonempty.

(a6) Let $\xi \in K^* \setminus \{0\}$ and $v \in \operatorname{int} K$ be given arbitrarily. Choose $\varepsilon > 0$ as small as $\bar{B}(v, \varepsilon) \subset K$. Then we have $\langle \xi, v + \varepsilon z \rangle \geq 0$ for every $z \in \bar{B}_{\mathbb{R}^m}$. This implies that

$$0 \leq \inf_{z \in \bar{B}_{\mathbb{R}^m}} \langle \xi, v + \varepsilon z \rangle = \langle \xi, v \rangle + \varepsilon \inf_{z \in \bar{B}_{\mathbb{R}^m}} \langle \xi, z \rangle = \langle \xi, v \rangle - \varepsilon \|\xi\|.$$

So, we get $\langle \xi, v \rangle \geq \varepsilon \|\xi\| > 0$, as desired. □

3. Optimality Conditions

To obtain optimality conditions for problem (2.1), we fix an arbitrary vector $v^0 \in \operatorname{int} K$ and define the set Λ by (2.3). By the assumptions made on K and the assertion (a4) in Lemma 2.2, Λ is a convex and compact base of K^*. Observe that $\operatorname{ri} \Lambda = \Lambda \cap \operatorname{int} K^*$.

Theorem 3.1. ***(Necessary Condition for the Weak Efficiency)*** *If $x \in \Delta$. If $x \in E^w$, then there exists $\xi = (\xi_1, \ldots, \xi_m) \in \Lambda$ such that*

$$\left\langle \sum_{i=1}^{m} \left[\xi_i \frac{(b_i^T x + \beta_i) a_i - (a_i^T x + \alpha_i) b_i}{(b_i^T x + \beta_i)^2} \right], y - x \right\rangle \geq 0 \quad \forall y \in \Delta. \tag{3.1}$$

Proof. By Lemma 2.1, for any $x, y \in \Delta$, one has

$$f(y) - f(x) = \begin{bmatrix} \dfrac{b_1^T x + \beta_1}{b_1^T y + \beta_1} \times \dfrac{[(b_1^T x + \beta_1) a_1^T - (a_1^T x + \alpha_1) b_1^T](y - x)}{(b_1^T x + \beta_1)^2} \\ \vdots \\ \dfrac{b_m^T x + \beta_m}{b_m^T y + \beta_m} \times \dfrac{[(b_m^T x + \beta_m) a_m^T - (a_m^T x + \alpha_m) b_m^T](y - x)}{(b_m^T x + \beta_m)^2} \end{bmatrix}. \tag{3.2}$$

Consider the $m \times n$ matrix

$$Q_x := \begin{bmatrix} \dfrac{(b_1^T x + \beta_1) a_1^T - (a_1^T x + \alpha_1) b_1^T}{(b_1^T x + \beta_1)^2} \\ \vdots \\ \dfrac{(b_m^T x + \beta_m) a_m^T - (a_m^T x + \alpha_m) b_m^T}{(b_m^T x + \beta_m)^2} \end{bmatrix}. \tag{3.3}$$

To prove the theorem, we first show that if $x \in E^w$, then

$$Q_x(\Delta - x) \cap \big(-\operatorname{int} K\big) = \emptyset, \tag{3.4}$$

where and $Q_x(\Delta - x) = \{Q_x(y-x) \,:\, y \in \Delta\}$. Suppose, on the contrary, that $x \in E^w$ but (3.4) does not hold. Then there exists $z \in \Delta$ such that $Q_x(z-x) \in -\operatorname{int} K$. For each $t \in (0,1)$, we define $y_t = (1-t)x + tz$. By the convexity of Δ, we have $y_t \in \Delta$. Substituting y_t for y in (3.2) and dividing both sides of the obtained equality by t, one obtains

$$\frac{f(y_t)-f(x)}{t} = \begin{bmatrix} \dfrac{b_1^T x+\beta_1}{b_1^T y_t+\beta_1} \times \dfrac{[(b_1^T x+\beta_1)a_1^T - (a_1^T x+\alpha_1)b_1^T](z-x)}{(b_1^T x+\beta_1)^2} \\ \vdots \\ \dfrac{b_m^T x+\beta_m}{b_m^T y_t+\beta_m} \times \dfrac{[(b_m^T x+\beta_m)a_m^T - (a_m^T x+\alpha_m)b_m^T](z-x)}{(b_m^T x+\beta_m)^2} \end{bmatrix}. \tag{3.5}$$

Since $y_t = x + t(z-x)$ converges to x as t tends to 0, we have

$$\lim_{t\to 0} \frac{b_i^T x+\beta_i}{b_i^T y_t+\beta_i} = 1 \quad (\forall i \in I).$$

Hence, the vector on the right-hand side of (3.5) converges to $Q_x(z-x)$ as t tends to 0. So, recalling that $Q_x(z-x) \in -\operatorname{int} K$, we can infer by (3.5) that there exists $\delta \in (0,1)$ such that

$$\frac{f(y_t)-f(x)}{t} \in -\operatorname{int} K, \quad \forall t \in (0,\delta).$$

Then, as $\operatorname{int} K$ is a cone, inequality $f(y_t) <_K f(x)$ holds for all $t \in (0,\delta)$. This is impossible because $x \in E^w$. We have thus established (3.4).

Since Q_x is an $m \times n$ matrix (see (3.3)) and Δ is a convex set by our assumption, $Q_x(\Delta - x)$ is a nonempty convex subset of $\mathbb{R}^m$. (Note that $0 \in Q_x(\Delta - x)$.) So, thanks to (3.4), we can apply the separation theorem [21, Theorem 11.3] to find a vector $\eta \in \mathbb{R}^m \setminus \{0\}$ such that

$$\langle \eta, u\rangle \geq \langle \eta, w\rangle \quad (\forall u \in Q_x(\Delta - x),\ \forall w \in -\operatorname{int} K). \tag{3.6}$$

From (3.6), we can deduce that $\eta \in K^* \setminus \{0\}$. Then, as Λ is a base of K^*, there exists $t > 0$ such that $\xi := t\eta$ belongs to Λ. Invoking (3.6), we can easily show that

$$\langle \xi, u\rangle \geq 0 \quad \forall u \in Q_x(\Delta - x).$$

Therefore,

$$\langle Q_x^T \xi, y - x\rangle \geq 0 \quad \forall y \in \Delta. \tag{3.7}$$

Let $\xi = (\xi_1, \ldots, \xi_m)$. Combining (3.7) with (3.3) establishes (3.1) and completes the proof. □

Setting

$$F_\xi(x) = \sum_{i=1}^m \left[\xi_i \frac{(b_i^T x+\beta_i)a_i - (a_i^T x+\alpha_i)b_i}{(b_i^T x+\beta_i)^2}\right] \tag{3.8}$$

for all $x \in \Delta$ and $\xi \in \Lambda$, we can consider the following parametric *variational inequality* of the Stampacchia type [22, p. 13] defined by F_ξ and Δ, where $\xi \in \Lambda$ plays the role of a parameter: *Find $x \in \Delta$ such that*

$$\big\langle F_\xi(x), y - x\big\rangle \geq 0, \quad \forall y \in \Delta. \tag{3.9}$$

The solution set of (3.9) is denoted by $S(\xi)$.

Remark 3.1. Theorem 3.1 can be reformulated equivalently as follows: *If $x \in E^w$, then x is a solution of the* (3.9) *for some $\xi \in \Lambda$.* So, *the union $\bigcup_{\xi\in\Lambda} S(\xi)$ contains the set E^w.*

Let us show that the necessary condition for the weak efficiency given by Theorem 3.1 is also a sufficient one, provided that K has a special form or all the functions $f_i(x)$, $i \in I$, are affine. As usual, $\mathbb{R}^m_+$ stands for the nonnegative orthant in $\mathbb{R}^m$, and $f_i(x) = \dfrac{a_i^T x + \alpha_i}{b_i^T x + \beta_i}$, $i \in I$, is said to be an *affine function* if $b_i = 0$ and $\beta_i = 1$.

Theorem 3.2. ***(Sufficient Conditions for the Weak Efficiency)*** *Let $x \in \Delta$. If either $K = \mathbb{R}^m_+$ or $f_i(x)$ is an affine function for every $i \in I$, and there exists a vector $\xi = (\xi_1, \dots, \xi_m) \in \Lambda$ such that* (3.1) *holds, then $x \in E^w$.*

Proof. Assume that $x \in \Delta$ and there exists a vector $\xi = (\xi_1, \dots, \xi_m) \in \Lambda$ such that (3.1) holds. Since (3.1) can be rewritten equivalently as (3.7), we have

$$\langle \xi, Q_x(y-x)\rangle \geq 0 \quad \forall y \in \Delta. \tag{3.10}$$

The property $\xi \in K^* \setminus \{0\}$ yields $\langle \xi, v\rangle > 0$ for all $v \in \operatorname{int} K$ (see the assertion (a6) in Lemma 2.2). Hence, if there is some $y \in \Delta$ with $Q_x(y-x) \in -\operatorname{int} K$, then one has $\langle \xi, Q_x(y-x)\rangle < 0$, wihch contrary to (3.10). Therefore, relation (3.4) must hold.

Arguing by contradiction, suppose that $x \notin E^w$. Then, we could find some $z \in \Delta$ such that $f(z) <_K f(x)$, i.e.,

$$f(z) - f(x) \in -\operatorname{int} K. \tag{3.11}$$

By Lemma 2.1, we have the equality (3.2) for any $y \in \Delta$. Substituting z for y in (3.2), we obtain

$$f(z) - f(x) = \begin{bmatrix} \dfrac{b_1^T x + \beta_1}{b_1^T z + \beta_1} \times \dfrac{[(b_1^T x + \beta_1)a_1^T - (a_1^T x + \alpha_1)b_1^T](z-x)}{(b_1^T x + \beta_1)^2} \\ \vdots \\ \dfrac{b_m^T x + \beta_m}{b_m^T z + \beta_m} \times \dfrac{[(b_m^T x + \beta_m)a_m^T - (a_m^T x + \alpha_m)b_m^T](z-x)}{(b_m^T x + \beta_m)^2} \end{bmatrix}. \tag{3.12}$$

CASE 1: $K = \mathbb{R}^m_+$. In this case, (3.11) means that $f_i(z) - f_i(x) < 0$ for all $i \in I$. Hence, using (3.12) and the condition $b_i^T y + \beta_i > 0$ for all $i \in I$ and $y \in \Delta$, we have

$$Q_x(z-x) = \begin{bmatrix} \dfrac{[(b_1^T x + \beta_1)a_1^T - (a_1^T x + \alpha_1)b_1^T](z-x)}{(b_1^T x + \beta_1)^2} \\ \vdots \\ \dfrac{[(b_m^T x + \beta_m)a_m^T - (a_m^T x + \alpha_m)b_m^T](z-x)}{(b_m^T x + \beta_m)^2} \end{bmatrix} \in -\operatorname{int} \mathbb{R}^m_+.$$

But this contradicts (3.4). We have thus proved that $x \in E^w$.

CASE 2: *$f_i(x)$ is an affine function for every $i \in I$.* In this case, $b_i^T x + \beta_i = 1$ for any $i \in I$ and $x \in \Delta$. So, combining (3.12) with (3.3) yields

$$f(z) - f(x) = \begin{bmatrix} [(b_1^T x + \beta_1)a_1^T - (a_1^T x + \alpha_1)b_1^T](z-x) \\ \vdots \\ [(b_m^T x + \beta_m)a_m^T - (a_m^T x + \alpha_m)b_m^T](z-x) \end{bmatrix} = Q_x(z-x). \tag{3.13}$$

Therefore, by (3.11), we obtain $Q_x(z-x) \in -\operatorname{int} K$. Since this contradicts (3.4), we have proved that $x \in E^w$. □

As $E \subset E^w$, Theorem 3.1 also gives a necessary condition for the efficiency.

Theorem 3.3. ***(Sufficient Conditions for the Efficiency)*** *Let $x \in \Delta$. If either $K = \mathbb{R}^m_+$ or $f_i(x)$ is an affine function for every $i \in I$, and there exists a vector $\xi = (\xi_1, \ldots, \xi_m) \in \operatorname{ri}\Lambda$ such that (3.1) holds, then $x \in E$.*

Proof. Suppose that $x \in \Delta$ and there exists a vector $\xi = (\xi_1, \ldots, \xi_m) \in \operatorname{ri}\Lambda$ such that (3.1) holds. Repeating the first argument of the proof of Theorem 3.2, we get (3.10).

Firstly, consider the case where $K = \mathbb{R}^m_+$. In this situation, we have $\xi_i > 0$ for all $i \in I$. If $x \notin E$, then there exits $z \in \Delta$ such that $f(z) \leq_K f(x)$ and $f(z) \neq f(x)$, i.e.,

$$f(z) - f(x) \in -\mathbb{R}^m_+ \setminus \{0\}. \tag{3.14}$$

Similarly as it was done in the proof of Theorem 3.2, we can show that equality (3.12) holds. By (3.14), $f_i(z) - f_i(x) \leq 0$ for all $i \in I$ and at least one inequality must be strict. From (3.12) and the condition $b_i^T y + \beta_i > 0$ for all $i \in I$ and $y \in \Delta$, it follows that

$$Q_x(z-x) = \begin{bmatrix} \dfrac{[(b_1^T x + \beta_1)a_1^T - (a_1^T x + \alpha_1)b_1^T](z-x)}{(b_1^T x + \beta_1)^2} \\ \vdots \\ \dfrac{[(b_m^T x + \beta_m)a_m^T - (a_m^T x + \alpha_m)b_m^T](z-x)}{(b_m^T x + \beta_m)^2} \end{bmatrix} \in -\mathbb{R}^m_+ \setminus \{0\}.$$

As $\xi_i > 0$ for all $i \in I$, this implies that $\langle \xi, Q_x(z-x) \rangle < 0$. We have arrived at a contradiction, because $\langle \xi, Q_x(z-x) \rangle \geq 0$ by (3.10). So, we must have $x \in E$.

Secondly, consider the case where $f_i(x)$ is an affine function for every $i \in I$. If $x \notin E$, then we can find some $z \in \Delta$ such that

$$f(z) - f(x) \in -K \setminus \{0\}. \tag{3.15}$$

Substituting z for y in (3.2), we obtain (3.12). As $b_1^T x + \beta_1 = 1$ for any $i \in I$ and $x \in \Delta$, from (3.12) and (3.3), we obtain (3.13). Then, (3.15) implies that $Q_x(z-x) \in -K \setminus \{0\}$. Recalling that $\xi \in \operatorname{ri}\Lambda$, from this we can deduce that $\langle \xi, Q_x(z-x) \rangle < 0$, contradicting (3.10). Thus, we have shown that $x \in E$. □

Remark 3.2. The sufficient condition for the efficiency provided by Theorem 3.3 is not a sufficient one. To justify this observation, choose $n = m = 2$, $\Delta = \bar{B}(0,1) := \left\{x \in \mathbb{R}^2 : \|x\| \leq 1\right\}$, $K = \mathbb{R}^2_+$, $f_i(x) = x_i$ for all $x = (x_1, x_2) \in \mathbb{R}^2$, and $i \in I = \{1,2\}$. For $v^0 := (1,1) \in \operatorname{int} K$, one sees that $\Lambda = \left\{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2_+ : \xi_1 + \xi_2 = 1\right\}$. In accordance with (3.8), here we have

$$F_\xi(x) = \xi_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \xi_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix}$$

for all $x \in \Delta$ and $\xi = (\xi_1, \xi_2) \in \Lambda$. Hence, $S(\xi) = \{x \in \Delta : \langle \xi, y - x \rangle \geq 0 \ \ \forall y \in \Delta\}$. Setting $\bar{x} = (-1, 0)$ and $\hat{x} = (0, -1)$, we can check that $\{\bar{x}, \hat{x}\} \subset E$, but $\bar{x} \notin \bigcup_{\xi \in \operatorname{ri}\Lambda} S(\xi)$ and $\hat{x} \notin \bigcup_{\xi \in \operatorname{ri}\Lambda} S(\xi)$. Thus, *it may happen that for some efficient solution of* (2.1) *there does not exist any* $\xi \in \operatorname{ri}\Lambda$ *such that* (3.1) *is fulfilled.*

Consider the variational inequality (3.9) with $\xi \in \Delta$ being fixed. In agreement with [22, p. 83], we say that the problem is *monotone* if $\langle F_\xi(y) - F_\xi(x), y-x\rangle \geq 0$ for all $x, y \in \Delta$.

Remark 3.3. The variational inequality (3.9) arisen in the general setting of (2.1) may not be monotone. Indeed, for any $x, y \in \Delta$, one has

$$\begin{aligned}
&\langle F_\xi(y) - F_\xi(x), y-x\rangle \\
=\ & \left\langle \sum_{i=1}^m \left[\xi_i \left(\frac{(b_i^Ty+\beta_i)a_i - (a_i^Ty+\alpha_i)b_i}{(b_i^Ty+\beta_i)^2} - \frac{(b_i^Tx+\beta_i)a_i - (a_i^Tx+\alpha_i)b_i}{(b_i^Tx+\beta_i)^2}\right)\right], y-x\right\rangle \\
=\ & \sum_{i=1}^m \left[\xi_i \left(\frac{(b_i^Ty+\beta_i)a_i^T(y-x) - (a_i^Ty+\alpha_i)b_i^T(y-x)}{(b_i^Ty+\beta_i)^2}\right.\right. \\
&\qquad \left.\left. - \frac{(b_i^Tx+\beta_i)a_i^T(y-x) - (a_i^Tx+\alpha_i)b_i^T(y-x)}{(b_i^Tx+\beta_i)^2}\right)\right] \\
=\ & \sum_{i=1}^m \left[\xi_i \left(\frac{(b_i^Ty+\beta_i)\left[(a_i^Ty+\alpha_i)-(a_i^Tx+\alpha_i)\right] - (a_i^Ty+\alpha_i)\left[(b_i^Ty+\beta_i)-(b_i^Tx+\beta_i)\right]}{(b_i^Ty+\beta_i)^2}\right.\right. \\
&\qquad \left.\left. - \frac{(b_i^Tx+\beta_i)\left[(a_i^Ty+\alpha_i)-(a_i^Tx+\alpha_i)\right] - (a_i^Tx+\alpha_i)\left[(b_i^Ty+\beta_i)-(b_i^Tx+\beta_i)\right]}{(b_i^Tx+\beta_i)^2}\right)\right] \\
=\ & \sum_{i=1}^m \left\{\xi_i \left[(a_i^Ty+\alpha_i)(b_i^Tx+\beta_i) - (b_i^Ty+\beta_i)(a_i^Tx+\alpha_i)\right]\left(\frac{1}{(b_i^Ty+\beta_i)^2} - \frac{1}{(b_i^Tx+\beta_i)^2}\right)\right\}.
\end{aligned}$$

So, one may not have $\langle F_\xi(y) - F_\xi(x), y-x\rangle \geq 0$ for all $x, y \in \Delta$, unless $f_i(x)$ is an *affine function* for every $i \in I$. In the latter case, $b_i^Tx+\beta_i = 1$ for all $x \in \Delta$ and $i \in I$. So, the above calculation yields $\langle F_\xi(y) - F_\xi(x), y-x\rangle = 0$ for all $x, y \in \Delta$, which signifies the monotonicity of the variational inequality (3.9) for every $\xi \in \Lambda$.

Remark 3.4. If $K = \mathbb{R}^m_+$ and $\xi \in \Lambda$ such that (3.1) holds, then by setting $\tilde{\xi} = (\tilde{\xi}_1, \ldots, \tilde{\xi}_m)$ where $\tilde{\xi}_i := \dfrac{\xi_i}{(b_i^Tx+\beta_i)^2}$ for all $i \in I$, we have $\tilde{\xi} \in \mathbb{R}^m_+ \setminus \{0\}$. Let $\tau > 0$ be such that $\tau\tilde{\xi} \in \Lambda$. Then, condition (3.1) is equivalent to

$$\left\langle \sum_{i=1}^m (\tau\tilde{\xi})_i \left[(b_i^Tx+\beta_i)a_i - (a_i^Tx+\alpha_i)b_i\right], y-x\right\rangle \geq 0 \quad \forall y \in \Delta.$$

Therefore, *if $K = \mathbb{R}^m_+$, then* (3.1) *can be put in the following simpler form:*

$$\left\langle \sum_{i=1}^m \xi_i \left[(b_i^Tx+\beta_i)a_i - (a_i^Tx+\alpha_i)b_i\right], y-x\right\rangle \geq 0 \quad \forall y \in \Delta. \tag{3.16}$$

Clearly, condition (3.16) corresponds to a variational inequality of the form (3.9), where $F_\xi(x)$ is given by

$$F_\xi(x) = \sum_{i=1}^m \xi_i \left[(b_i^Tx+\beta_i)a_i - (a_i^Tx+\alpha_i)b_i\right]. \tag{3.17}$$

Thanks to (3.17) and the calculation given in Remark 3.3, we have $\langle F_\xi(y) - F_\xi(x), y-x\rangle = 0$ for any $x, y \in \Delta$. In particular, (3.16) is a monotone variational inequality.

4. Connectedness of the Efficient Solution Sets

Let $G : X \rightrightarrows Y$ be a set-valued map between two topological spaces. One says that G is *upper semicontinuous* (usc) at $u \in X$ if, for every open set $V \subset Y$ satisfying $G(u) \subset V$, there exists a neighborhood U of u such that $G(u') \subset V$ for all $u' \in U$.

A topological space Z is said to be *connected* if one cannot represent $Z = W_1 \cup W_2$ with W_1 and W_2 being nonempty disjoint open subsets of Z; see [29, p. 53] for an equivalent definition. One says (see, e.g., [30, Definition 27.1]) that Z is *pathwise connected* if for any $u^1, u^2 \in Z$ there exits a continuous mapping $\gamma : [0,1] \to Z$ such that $\gamma(0) = u^1$, $\gamma(1) = u^2$. It is well known (see, e.g., [30, Theorem 27.1]) that every pathwise connected space is connected. In particular, every convex subset of $\mathbb{R}^n$ is connected.

Theorem 4.1. ***(Connectedness Preservation)*** *(*See [23, 24]*) Assume that X is connected. If for every $x \in X$ the set $G(x)$ is nonempty and connected, and G is upper semicontinuous at every $u \in X$, then the image set $G(X) := \bigcup_{x \in X} G(x)$, which is equipped with the induced topology, is connected.*

Based on Theorem 4.1 and the results obtained in the preceding section, we can establish some sufficient conditions for the connectedness of the weakly efficient solution set and the efficient solution set of the vector optimization problem (2.1).

Theorem 4.2. ***(Sufficient Conditions for the Connectedness of E^w)*** *Assume that either $K = \mathbb{R}^m_+$ or $f_i(x)$ is an affine function for every $i \in I$. If Δ is compact, then E^w is a connected set.*

Proof. First, consider the case where $K = \mathbb{R}^m_+$ and the set Δ is compact. According to Theorems 3.1 and 3.2, a vector $x \in \Delta$ belongs E^w if and only if there is $\xi = (\xi_1, \ldots, \xi_m) \in \Lambda$ such that (3.1) holds. By Remark 3.4, this is equivalent to saying that $x \in \Delta$ is a solution of (3.16), which is a monotone variational inequality. Denote the solution set of (3.16) by $S_1(\xi)$. As the map $F_\xi : \Delta \to \mathbb{R}^n$, $x \mapsto F_\xi(x)$, given in (3.17) is continuous, (3.16) has a solution by [22, Theorem 1.4 in Chapter III]. Moreover, by the Minty Lemma (see [22, Lemma 1.4 in Chapter III]), $x \in S_1(\xi)$ if and only if $x \in \Delta$ and

$$\langle F_\xi(y), y - x \rangle \geq 0 \quad \forall y \in \Delta.$$

Since $x \mapsto \langle F_\xi(y), y - x \rangle$ is an affine function, it follows that $S_1(\xi)$ is a closed convex subset of Δ. Thus, the set-valued map $S_1 : \Lambda \rightrightarrows \Delta$ has nonempty, compact, and connected images.

As Δ is compact and $S_1(\xi) \subset \Delta$ for all $\xi \in \Lambda$, the map S_1 is *uniformly compact* near any point of Λ (see [31, p. 594]). Hence, by [31, Theorem 3], to prove that S_1 is usc at any $\bar{\xi} = (\bar{\xi}_1, \ldots, \bar{\xi}_m)$ belonging to Λ, we need only to show that S_1 is *closed* (see [31, p. 592]) at $\bar{\xi}$. To do so, take any sequence $\{\xi^k\} \subset \Lambda$ converging to $\bar{\xi}$, a sequence $\{x^k\} \subset \mathbb{R}^n$ with $x^k \in S_1(\xi^k)$ for all k, which converges to $\bar{x}$. Then, for every index k, we have

$$\left\langle \sum_{i=1}^m \xi_i^k \left[(b_i^T x^k + \beta_i) a_i - (a_i^T x^k + \alpha_i) b_i \right], y - x^k \right\rangle \geq 0 \quad \forall y \in \Delta. \tag{4.1}$$

For each $y \in \Delta$, passing the inequality in (4.1) to the limit as $k \to \infty$, one gets

$$\left\langle \sum_{i=1}^m \bar{\xi}_i \left[(b_i^T \bar{x} + \beta_i) a_i - (a_i^T \bar{x} + \alpha_i) b_i \right], y - \bar{x} \right\rangle \geq 0 \quad \forall y \in \Delta,$$

which shows that $\bar{x} \in S_1(\bar{\xi})$. Therefore, S_1 is closed at any $\bar{\xi} \in \Lambda$.

Now, applying Theorem 4.1 to the set-valued map $S_1 : \Lambda \rightrightarrows \Delta$, we can assert that the set $\bigcup\limits_{\xi \in \Lambda} S_1(\xi)$ is connected. As $E^w = \bigcup\limits_{\xi \in \Lambda} S_1(\xi)$, the conclusion of our theorem has been proved.

Second, consider the case where $f_i(x)$, $i \in I$, are affine functions, and the set Δ is compact. By Theorems 3.1 and 3.2, $x \in \Delta$ belongs E^w if and only if there is $\xi = (\xi_1, \ldots, \xi_m)$ from Λ such that (3.1) holds. As it has been shown in Remark 3.2, this means that $x \in \Delta$ is a solution of (3.9), a monotone variational inequality. Repeating the above arguments, we can show that the solution map $S : \Lambda \rightrightarrows \Delta$, $\xi \mapsto S(\xi)$, has nonempty, compact, connected images, and is usc at any $\bar{\xi} \in \Lambda$. Hence, thanks to Theorem 4.1, we can conclude that the set $E^w = \bigcup\limits_{\xi \in \Lambda} S(\xi)$ is connected. □

Theorem 4.3. ***(Connectedness of some Part of*** *E) The following assertions are valid:*

(a) *If $f_i(x)$ is an affine function for every $i \in I$ and Δ is compact, then $E_0 := \bigcup\limits_{\xi \in \operatorname{ri} \Lambda} S(\xi)$ is a connected subset of E.*

(b) *If $K = \mathbb{R}^m_+$ and Δ is compact, then $E_1 := \bigcup\limits_{\xi \in \operatorname{ri} \Lambda} S_1(\xi)$, where $S_1(\xi)$ denotes the solution set of* (3.16)*, is a connected subset of E.*

Proof. To prove the assertions (a) and (b), it suffices to repeat the arguments of the preceding proof, use Theorem 3.3 instead of Theorem 3.2, and apply Theorem 4.1 to some set-valued maps defined on $\operatorname{ri} \Lambda$, which is a connected set. □

Remark 4.1. If a set-valued map $G : X \rightrightarrows Y$ between two topological spaces is usc at any $u \in X$, X is compact, and $G(u)$ is nonempty and compact for every $u \in X$, then $G(X)$ is a compact set. By this fact, the assumptions of Theorem 4.2 imply that the weakly efficient solution set of (2.1) *is not only connected, but also compact.*

Example 4.1. Let n, m, K, f_i for $i \in I = \{1, 2\}$, v^0, Λ be the same as in Remark 3.2, and let $\Delta = \left\{x \in \mathbb{R}^2 : \|x\| \le 1,\ x_1 \ge -\frac{3}{4}\right\}$. Setting $\bar{x} = \left(-\dfrac{3}{4}, -\dfrac{\sqrt{7}}{4}\right)$ and $\hat{x} = (0, -1)$, we have $\{\bar{x}, \hat{x}\} \subset E$, $\bar{x} \in \bigcup\limits_{\xi \in \operatorname{ri} \Lambda} S(\xi)$, but $\hat{x} \notin \bigcup\limits_{\xi \in \operatorname{ri} \Lambda} S(\xi)$. Set $\bar{\xi} = \left(\dfrac{9 - 3\sqrt{7}}{2}, \dfrac{3\sqrt{7} - 7}{2}\right)$, $\tilde{\xi} = (1, 0)$, $\hat{\xi} = (0, 1)$, and observe that $\{\bar{\xi}, \tilde{\xi}, \hat{\xi}\} \subset \Lambda$. For every $\xi \in \{(1-t)\bar{\xi} + t\tilde{\xi} : 0 \le t < 1\} \subset \operatorname{ri} \Lambda$, one has $S_1(\xi) = S(\xi) = \{\bar{x}\}$. In addition, on the interval $\{(1-t)\hat{\xi} + t\bar{\xi} : 0 \le t < 1\}$, the map $\xi \mapsto S_1(\xi)$ coincides with the map $\xi \mapsto S(\xi)$. In fact, the maps have single values and they draw the curve

$$\left\{x = (x_1, x_2) \in \mathbb{R}^2 : \|x\| \le 1,\ -\frac{3}{4} < x_1 \le 0,\ x_2 \le 0\right\}.$$

Remark 4.2. In Example 4.1, the multiplier $\tilde{\xi} = (1, 0)$ is very special. Indeed, for $\tilde{x} := \left(-\dfrac{3}{4}, \dfrac{\sqrt{7}}{4}\right)$, we see that both sets $S_1(\tilde{\xi})$ and $S(\tilde{\xi})$ collapse to the segment

$$\{(1-t)\bar{x} + t\tilde{x} : 0 \le t \le 1\},$$

which is a part of E^w and which has $\bar{x}$ as a unique efficient point. Therefore, *for some multipliers $\xi \in \Lambda$, $S(\xi)$ may contain weakly efficient points not belonging to E, while $S(\xi) \cap E \neq \emptyset$.* The same circumstance also happens with the map $S_1(\cdot)$. This does allow us to apply Theorem 4.1 and the techniques for proving Theorem 4.2 to establish results on the connectedness of E.

Remark 4.3. If A and B are subsets of a topological space, A is connected, and $A \subset B \subset \bar{A}$, then B is connected (see, e.g., [30, Theorem 26.8]). So, in the setting of Theorem 4.3, if we can show that E_0 (resp., E_1) is dense in E, i.e., $E_0 \subset E \subset \bar{E}_0$ (resp., $E_1 \subset E \subset \bar{E}_1$), then E is connected.

5. Conclusions and Topics for Further Investigations

In this paper, we studied a new class of vector optimization problems with linear fractional objective criteria. Necessary optimality conditions, as well as sufficient optimality conditions, were obtained. In addition, two theorems on the connectedness of the weakly efficient solution set and the efficient solution set were established.

There are some open questions:

Question 1. *Are the sufficient conditions for the weak efficiency in Theorem 3.2 valid without any additional assumption, or not?*

Question 2. *Are the sufficient conditions for the efficiency in Theorem 3.3 valid without any additional assumption, or not?*

Question 3. *Are the sufficient conditions for the connectedness of E^w in Theorem 4.2 valid without any additional assumption, or not?*

Question 4. *Are the sufficient conditions for the connectedness of some part of E in Theorem 4.3 valid without any additional assumption, or not?*

It is also worthy to carry some investigations on the *Benson proper efficiency* [32] of this class of the vector optimization problems with linear fractional objective criteria similarly as the ones which were done in [4, 16, 17, 18]. Besides, we still do not know whether an analogue of Theorem 7 of [8] on the *solution stability* of LFVOPs can be obtained for (2.1), or not.

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