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SET-VALUED VARIATIONAL PRINCIPLES. WHEN MIGRATION IMPROVES QUALITY OF LIFE

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Abstract. In this paper, in the context of quasi-metric spaces, we obtain two set-valued versions of the Ekeland variational-type principle by means of lower and upper set less relations, for the case where the perturbations need not satisfy the triangle inequality. An application in terms of migration problems and quality of life is given.

Keywords. Lower and upper set less relations; Set-valued mapping; Trap; Quasi-metric space; Variational rationality.

1. Introduction

Ekeland's variational principle [1] (briefly, denoted by EVP) provides the existence of a strict minimum of a perturbed lower semicontinuous function on complete metric spaces. It is well-known that the EVP is equivalent to Caristi's fixed point theorem, the petal theorem, Takahashi's nonconvex minimization theorem, and the drop theorem. Since the EVP has significant applications in mathematics and various other areas of science, behavioral sciences, and economics and management, a number of generalizations of the EVP (in particular, the case of vector-valued and set-valued mappings) and their equivalent formulations have received much attention from researchers in the last three decades; see, e.g., [2]-[28]. Recently, Bao *et al.* [5, 6] developed variational principles and applied them to the models in behavioral sciences that are mainly related to human behavior. The latter applications are based on the variational rationality approach in behavioral sciences, initiated by Soubeyran in [29, 30]. More recently, some researchers obtained several extensions of the EVP with applications in behavioral sciences; see, e.g., [7, 9, 10, 11, 15, 16, 24, 27] and the references therein. However, there are "two types of EVP in the case of set-valued mappings": the vector criterion (see, e.g., [2, 7, 8, 9, 13, 14, 19, 23]) and the set criterion (see, e.g., [3, 4, 20, 21, 22, 28]). The first criterion is to find an approximate minimal (efficient) point of the range set of a set-valued mapping. The second criterion is to find a strict minimum of a perturbed set-valued mapping with respect to

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the set order relation in the class of all nonempty subsets of the image space. Following Fakhar *et al.* [15, 16] and Qiu *et al.* [24], and driven by behavioral science applications, we give new formulations of set-valued versions of the EVP by using lower and upper set less relations with variable preferences. An application is given in relation to migration problems and the quality of life in the framework of the variational rationality approach of stay (continue) and change (stop and start) human dynamics [31, 32, 33, 34].

2. Preliminaries

In this section, we give some notions, notation, and basic results that will be used in this paper.

A quasi-metric q on a nonempty set X is a bifunction $q : X \times X \to \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the set of non-negative real numbers, such that

(i) $q(x,y) = 0$ if and only if $x = y$,
(ii) $q(x,y) \leq q(x,z) + q(z,y)$, for all $x,y,z \in X$.

A set X equipped with a quasi-metric q is called a quasi-metric space, and it is denoted by (X,q). Let (X,q) be a quasi-metric space. A sequence $\{x_n\}$ in X is said to be left convergent to $x \in X$ if $\lim_{n\to\infty} q(x_n,x) = 0$. The sequence $\{x_n\}$ is said to be left Cauchy if, for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that $q(x_n,x_m) < \varepsilon$ for all $m \geq n \geq N_\varepsilon$. A quasi-metric space (X,q) is said to be left-complete if every left Cauchy sequence is left convergent. Let Y be a linear space, and let S be a nonempty subset of Y. The cone generated by S is the set $cone(S) = \{tx : t \geq 0, x \in S\}$. The set S is said to be free disposal with respect to a convex cone D if $S + D = S$; see [35]. For any $k_0 \in Y$, we define the k_0-vector closure of S as follows:

$$vcl_{k_0}(S) = \{y \in Y : \exists \lambda_n \geq 0, \lambda_n \to 0 \text{ such that } y + \lambda_n k_0 \in S, \forall n \in \mathbb{N}\}.$$

The set A is said to be k_0-closed if and only if $S = vcl_{k_0}(S)$. The set

$$\mathrm{vcl}(S) := \bigcup_{k\in Y} \mathrm{vcl}_k(S),$$

is called the vector closure of S. For more details, one can refer to [25, 36].

Let $\mathscr{P}(Y)$ be the class of all nonempty subsets of Y, and S be a nonempty subset of Y. For $A, B \in \mathscr{P}(Y)$, the upper set less relation $\preceq^u_S$ (resp. the lower set less relation $\preceq^l_S$) is defined as follows: $A \preceq^u_S B$ (resp. $A \preceq^l_S B$) if and only if $A \subseteq B - S$ (resp. $B \subseteq A + S$). If S is a convex cone, then the upper (lower) set less relation is transitive; see [37]. Let K be a nonempty subset of Y and $k_0 \in Y \setminus \{0\}$. The nonlinear scalarization function $\xi_{k_0} : Y \to \mathbb{R} \cup \{\pm\infty\}$ is defined as follows:

$$\xi_{k_0}(y) := \begin{cases} +\infty & \text{if } y \notin \mathbb{R}k_0 - K, \\ \inf\{t \in \mathbb{R} : y \in tk_0 - K\} & \text{otherwise.} \end{cases}$$

The function ξ_{k_0} is also called the Gerstewitz's function generated by K and k_0. For more details, see [12, 17, 18, 24, 25, 38] and the references therein. Furthermore, for every $y \in Y$, $\xi_{k_0}(y) > -\infty$ if and only if $k_0 \notin -vcl(K)$; see [25].

We say that ξ_{k_0} is C-nondecreasing if $\xi_{k_0}(y_1) \leq \xi_{k_0}(y_2)$ for all $y_1, y_2 \in Y$, $y_2 - y_1 \in C$. If $k_0 \in K \setminus (-K)$, then ξ_{k_0} is subadditive, positively homogeneous, and K-nondecreasing, i.e, $\xi_{k_0}(y_1) \leq \xi_{k_0}(y_2)$, whenever $y_1 - y_2 \in -K$.

Proposition 2.1. [18, 38] *Let K, C be two nonempty subset of a linear space Y and $k_0 \in Y \setminus \{0\}$. Then the Gerstewitz function ξ_{k_0} has the following properties:*

(i) $\xi_{k_0}(y) < +\infty$ *if and only if* $y \in \mathbb{R}k_0 - vcl_{k_0}(K)$*;*
(ii) ξ_{k_0} *is C-nondecreasing if and only if* $K + C \subseteq cone(\{k_0\}) + vcl_{k_0}(K)$*;*
(iii) $\xi_{k_0}(y + rk_0) = \xi_{k_0}(y) + r$.

The purpose of this paper is to consider the set-valued versions of the Ekeland variational-type principle by means of lower and upper set less relations. An application in terms of migration problems and quality of life is given.

In order to obtain our main results, we also need to state a pre-order principle provided by Qiu [26]. A binary relation $\preceq$ on X is called a pre-order if the transitive property is satisfied. Let $(X, \preceq)$ be a pre-order set. An extended real-valued function $\eta : (X, \preceq) \to \mathbb{R} \cup \{\pm\infty\}$ is said to be monotone with respect to $\preceq$ if and only if, for any $x_1, x_2 \in X$,

$$x_1 \preceq x_2 \Rightarrow \eta(x_1) \leq \eta(x_2).$$

For any given $x_0 \in X$, denote by $S(x_0)$ the set $\{x \in X : x \preceq x_0\}$.

Theorem 2.1. [26] *Let $(X, \preceq)$ be a pre-order set, $x_0 \in X$ such that $S(x_0) \neq \emptyset$, and $\eta : (X, \preceq) \to \mathbb{R} \cup \{\pm\infty\}$ be an extended real-valued function, which is monotone with respect to $\preceq$. Suppose that the following conditions are satisfied*

(A) $-\infty < \inf\{\eta(x) : x \in S(x_0)\} < \infty$*;*
(B) *for any $x \in S(x_0)$ with $-\infty < \eta(x) < +\infty$ and $x' \in S(x) \setminus \{x\}$, one has $\eta(x) > \eta(x')$;*
(C) *for any sequence $\{x_n\} \subset S(x_0)$ with $x_n \in S(x_{n-1})$ $\forall n$, such that $\eta(x_n) - \inf\{\eta(x) : x \in S(x_{n-1})\} \to 0, (n \to \infty)$, there exists $\bar{x} \in X$ such that $\bar{x} \in S(x_n)$ for all $n \in \mathbb{N}$.*

Then there exists $u \in X$ such that

(a) $u \in S(x_0)$*;*
(b) $S(u) \subset \{u\}$.

3. Main Results

In order to prove our main results, we need the following definition and proposition.

Definition 3.1. Let Y be a linear space, (X, q) be a quasi-metric space, and $C : X \rightrightarrows Y$ be a set-valued map with nonempty values. Let $\psi : X \times X \to]0, +\infty[$ be a real-valued bifunction, and $\Phi : X \rightrightarrows Y$ be a set-valued mapping with nonempty values.

(i) Φ is said to be C-u-left sequentially lower monotone (briefly, denoted C-u-left slm) (resp. C-l-left sequentially lower monotone) if $\{x_n\}$ is a left convergent to an element $\bar{x} \in X$ and $\Phi(x_{n+1}) \leq^u_{C(x_n)} \Phi(x_n)$ (resp. $\Phi(x_{n+1}) \leq^l_{C(x_n)} \Phi(x_n)$), for all $n \in \mathbb{N}$, we have $\Phi(\bar{x}) \leq^u_{C(x_n)} \Phi(x_n)$, (resp. $\Phi(\bar{x}) \leq^l_{C(x_n)} \Phi(x_n)$) for each $n \in \mathbb{N}$.
(ii) ψ is said to be u-Φ-decreasing (resp. l-Φ-decreasing) in the first argument iff, for every $x, x' \in X$ with $\Phi(x') \leq^u_{C(x)} \Phi(x)$ (resp. $\Phi(x') \leq^l_{C(x)} \Phi(x)$), $\psi(x', y) \geq \psi(x, y)$, for each $y \in X$.
(iii) We say that ψ is u-Φ-increasing (resp. l-Φ-increasing) in the second argument iff for every $x, x' \in X$ with $\Phi(x') \leq^u_{C(x)} \Phi(x)$ (resp. $\Phi(x') \leq^l_{C(x)} \Phi(x)$), we have $\psi(y, x') \leq \psi(y, x)$, for each $y \in X$.

Proposition 3.1. *Suppose that Φ is C-u-left slm (resp. C-l-left slm), $\psi : X \to \mathbb{R}$ is u-Φ-increasing (resp. l-Φ-increasing) in the second argument, $\{x_n\} \subset X$ is left convergent to $\bar{x} \in X$, and $\Phi(x_{n+1}) \leq^u_{C(x_n)} \Phi(x_n)$ (resp. $\Phi(x_{n+1}) \leq^l_{C(x_n)} \Phi(x_n)$), for all $n \in \mathbb{N}$. Then, $\psi(x,\bar{x}) \leq \psi(x,x_n)$, for every $n \in \mathbb{N}$ and $x \in X$.*

Proof. Suppose on the contrary that there exist $n_0 \in \mathbb{N}$ and $x \in X$ such that $\psi(x,x_{n_0}) < \psi(x,\bar{x})$. Since ψ is u-Φ-increasing (resp. l-Φ-increasing), we have $\Phi(\bar{x}) \not\leq^u_{C(x_{n_0})} \Phi(x_{n_0})$, (rep. $\Phi(\bar{x}) \not\leq^l_{C(x_{n_0})} \Phi(x_{n_0})$), which contradicts the fact that Φ is C-u-slm (resp. C-l-slm). □

Now, we are ready to give our main results.

Theorem 3.1. *Let (X,q) be a left-complete quasi-metric space, Y be real linear space, $k_0 \in Y \setminus \{0\}$, and $K \subset Y$ be a nonempty, k_0-vectorially closed and free disposal with respect to cone$(\{k_0\})$. Let $\Phi : X \rightrightarrows Y$ be a set-valued mapping with nonempty values, $C : X \rightrightarrows Y$, $C(x)$ be k_0-vectorially closed and free disposal with respect to cone$(\{k_0\})$, $C(x) + K \subseteq K$ and $0 \in C(x)$ for any $x \in X$. Let $\psi : X \times X \to]0,+\infty[$ be a real valued bifunction such that $inf_{x\in X} \psi(\cdot,x) > 0$ and the following conditions hold.*

(i) *Φ is C-l-left slm in (X,q).*
(ii) *ψ is l-Φ-decreasing in the first argument and l-Φ-increasing in the second argument.*
(iii) *If $\Phi(u) \leq^l_{C(v)} \Phi(v)$, then $C(v) + C(u) \subseteq C(v)$.*
(iv) *Suppose that $x_0 \in X$, $y_0 \in Y$, and $\varepsilon > 0$ such that $\Phi(x) \not\leq^l_K y_0 - \varepsilon k_0$, for all $x \in X$ with $\Phi(x) \leq^l_{C(x_0)} \Phi(x_0)$, and $\Phi(x_0) \leq^l_{C(x_0)} y_0 + s_0$, for some $s_0 \in \mathbb{R}k_0 - K$.*

Then, there exists $\hat{x} \in X$ such that

(a) $\Phi(\hat{x}) + \psi(x_0,\hat{x})q(x_0,\hat{x})k_0 \leq^l_{C(x_0)} \Phi(x_0)$;
(b) $\Phi(x) + \psi(\hat{x},x)q(\hat{x},x)k_0 \not\leq^l_{C(\hat{x})} \Phi(\hat{x})$, *for all $x \neq \hat{x}$.*

Proof. We will prove this result by applying Theorem 2.1. For this purpose, we define a relation $\preceq$ on X as follows: for any $x,x' \in X$,

$$x' \preceq x \iff \Phi(x') + \psi(x,x')q(x,x')k_0 \leq^l_{C(x)} \Phi(x).$$

Since $C(x)$ is free disposal with respect to $cone(\{k_0\})$ for $x' \preceq x$, we have

$$\Phi(x) \subseteq \Phi(x') + \psi(x,x')q(x,x')k_0 + C(x) \subseteq \Phi(x') + C(x).$$

Therefore, we deduce

$$x' \preceq x \Rightarrow \Phi(x') \leq^l_{C(x)} \Phi(x). \tag{3.1}$$

In the first step, we show that $\preceq$ is a pre-order. Now, suppose that $x' \preceq x, x \preceq y$. Since ψ is l-Φ-decreasing in the first argument (resp. l-Φ-increasing in the second argument) by (3.1), we have

$$\psi(y,x') \leq \psi(x,x') \text{ (resp. } \psi(y,x') \leq \psi(y,x)). \tag{3.2}$$

Now, from (3.2), condition (iii), and the fact that the values of C is free disposal with respect to $cone(\{k_0\})$, we have

$$\begin{aligned}
&\Phi(y) \subseteq \Phi(x) + \psi(y,x)q(y,x)k_0 + C(y)\\
&\subseteq \Phi(x') + \psi(y,x)q(y,x)k_0 + \psi(x,x')q(x,x')k_0 + C(x) + C(y)\\
&\subseteq \Phi(x') + (\psi(y,x) - \psi(y,x'))q(y,x)k_0 + \psi(y,x')q(y,x)k_0\\
&+ (\psi(x,x') - \psi(y,x'))q(x,x')k_0 + \psi(y,x')q(x,x')k_0 + C(y)\\
&\subseteq \Phi(x') + \psi(y,x')(q(y,x) + q(x,x'))k_0 + C(y)\\
&= \Phi(x') + \psi(y,x')(q(y,x) + q(x,x') - q(y,x'))k_0 + \psi(y,x')q(y,x')k_0 + C(y)\\
&\subseteq \Phi(x') + \psi(y,x')q(y,x')k_0 + C(y).
\end{aligned}$$

Hence, $x' \preceq y$, and $\preceq$ has the transitive property. Therefore, the relation $\preceq$ is a pre-order. Now, we set

$$S(x) := \{x' \in X : x' \preceq x\}, \quad \forall x \in X,$$

and we define $\eta : (X,\preceq) \to \mathbb{R} \cup \{+\infty\}$ as follows:

$$\eta(x) := \inf\{\xi_{k_0}(y - y_0 + \varepsilon k_0) : y \in \Phi(x)\}, \;\; \forall x \in X.$$

We will show that all the conditions of Theorem 2.1 are satisfied. In the first step, we show that η is monotone with respect to $\preceq$. If $x, x' \in X$ and $x' \preceq x$, then $\Phi(x) \subseteq \Phi(x') + \psi(x,x')q(x,x')k_0 + C(x)$. So, for every $y \in \Phi(x)$, there exists $y' \in \Phi(x')$ such that $y \geq_{C(x)} y' + \psi(x,x')q(x,x')k_0$. Since $C(x) + K \subseteq K$, then by Proposition 2.1(ii), ξ_{k_0} is $C(x)$-monotone. It follows that

$$\begin{aligned}
&\xi_{k_0}(y' - y_0 + \varepsilon k_0 + \psi(x,x')q(x,x')k_0)\\
&= \xi_{k_0}(y' - y_0 + \varepsilon k_0) + \psi(x,x')q(x,x') \leq \xi_{k_0}(y - y_0 + \varepsilon k_0).
\end{aligned}$$

So,

$$\eta(x') + \psi(x,x')q(x,x') \leq \eta(x).$$

This shows that η is monotone with respect to $\preceq$. Moreover, if $x' \preceq x$, $x \neq x'$ and $\eta(x) < \infty$, then

$$\eta(x') \leq \eta(x) - \psi(x,x')q(x,x') < \eta(x). \tag{3.3}$$

Since $x_0 \in S(x_0)$, $S(x_0) \neq \emptyset$. Now, it is enough to show that conditions (A), (B), and (C) in Theorem 2.1 hold.

Checking condition A. If $x \in S(x_0)$, then $\Phi(x) \leq^l_{C(x_0)} \Phi(x_0)$. In view the of assumption, we have $\Phi(x) \not\leq^l_K y_0 - \varepsilon k_0$. Therefore, $y - y_0 + \varepsilon k_0 \notin -K$ for all $y \in \Phi(x)$. Thus, $\xi_{k_0}(y - y_0 + \varepsilon k_0) \geq 0$, for each $y \in \Phi(x)$, and hence $\eta(x) \geq 0$. Observe that $\Phi(x_0) \leq^l_{C(x_0)} y_0 + s_0$. If $y \in \Phi(x_0)$ such that $y_0 + s_0 - y \in C(x_0)$, then $y - y_0 + \varepsilon k_0 \in \varepsilon k_0 + s_0 - C(x_0)$. So,

$$\eta(x_0) \leq \xi_{k_0}(y - y_0 + \varepsilon k_0) \leq \xi_{k_0}(\varepsilon k_0 + s_0) = \varepsilon + \xi_{k_0}(s_0) < +\infty.$$

Hence, $\eta(x) \leq \eta(x_0) < +\infty$, for each $x \in S(x_0)$. Therefore, condition (**A**) is satisfied.

Checking condition B. Consider $x \in S(x_0)$ with $-\infty < \eta(x) < +\infty$, and let $x' \in S(x) \setminus \{x\}$. By relation (3), we have $\eta(x') < \eta(x)$. So, condition **B** is satisfied.

Checking condition C. Let $\{x_n\} \subset S(x_0)$ be a sequence such that $x_n \in S(x_{n-1})$ for each $n \in \mathbb{N}$. Therefore,

$$x_n \preceq x_{n-1} \quad \forall n \in \mathbb{N}. \tag{3.4}$$

Hence,

$$\eta(x_n) + \psi(x_{n-1}, x_n) q(x_{n-1}, x_n) \leq \eta(x_{n-1}) \quad \forall n \in \mathbb{N}. \tag{3.5}$$

From (3.4), we obtain

$$\Phi(x_n) \leq^l_{C(x_{n-1})} \Phi(x_{n-1}) \leq^l_{C(x_{n-2})} \cdots \leq^l_{C(x_0)} \Phi(x_0). \tag{3.6}$$

Since ψ is l-Φ-decreasing in the first argument and l-Φ-increasing in the second argument, we obtain from relation (3.6) that

$$\psi(x_0, x_n) \leq \psi(x_0, x_i) \leq \psi(x_{i-1}, x_i),\ \forall i = 1, \cdots, n. \tag{3.7}$$

Now, by (3.5) and (3.7), we deduce

$$\begin{aligned} &\sum_{i=1}^n \eta(x_i) + \psi(x_0, x_n) \sum_{i=1}^n q(x_{i-1}, x_i) \leq \\ &\sum_{i=1}^n \eta(x_i) + \sum_{i=1}^n \psi(x_{i-1}, x_i) q(x_{i-1}, x_i) \leq \sum_{i=1}^n \eta(x_{i-1}). \end{aligned} \tag{3.8}$$

Therefore,

$$\sum_{i=1}^n q(x_{i-1}, x_i) \leq \frac{1}{\psi(x_0, x_n)} (\eta(x_0) - \eta(x_n)) \leq \frac{1}{\inf_{x \in X} \psi(x_0, x)} \eta(x_0) < +\infty,\ \forall n \in \mathbb{N}.$$

Thus,

$$\sum_{i=1}^\infty q(x_{i-1}, x_i) \leq \frac{1}{\inf_{x \in X} \psi(x_0, x)} \eta(x_0) < +\infty.$$

So, for every $m > n$,

$$q(x_n, x_m) \leq \sum_{i=n}^{m-1} q(x_i, x_{i+1}) \to 0 (m > n \to \infty).$$

This means that the sequence $\{x_n\}$ is a left-Cauchy sequence in (X, q). Since (X, q) is left-complete, there exists $\bar{x} \in X$ such that $q(x_n, \bar{x}) \to 0 (n \to \infty)$. Since $\Phi(x_n) \leq^l_{C(x_{n-1})} \Phi(x_{n-1})$, condition (i) implies

$$\Phi(\bar{x}) \leq^l_{C(x_n)} \Phi(x_n),\ \forall n \in \mathbb{N}. \tag{3.9}$$

Now, for fixed n and $m > n$, we have $x_m \in S(x_n)$. So,

$$\Phi(x_m) + \psi(x_n, x_m) q(x_n, x_m) k_0 \leq^l_{C(x_n)} \Phi(x_n). \tag{3.10}$$

Also, by conditions (i), (ii), and Propositions 3.1, we have

$$\psi(x_n, \bar{x}) \leq \psi(x_n, x_m). \tag{3.11}$$

Since $\Phi(x_m) \leq^l_{C(x_n)} \Phi(x_n)$, then

$$\psi(x_n, x_m) \leq \psi(x_n, x_n). \tag{3.12}$$

Hence, by (3.9), (3.10), (3.11), (3.12), and the triangle inequality of q, we obtain

$$\begin{aligned}
&\Phi(x_n) \subseteq \Phi(x_m) + \psi(x_n,x_m)q(x_n,x_m)k_0 + C(x_n)\\
&= \Phi(x_m) + \psi(x_n,x_m)q(x_n,\bar{x})k_0 - \psi(x_n,x_m)q(x_m,\bar{x})k_0\\
&+ \psi(x_n,x_m)(q(x_n,x_m) + q(x_m,\bar{x}) - q(x_n,\bar{x}))k_0 + C(x_n)\\
&\subseteq \Phi(x_m) + \psi(x_n,\bar{x})q(x_n,\bar{x})k_0 - \psi(x_n,x_m)q(x_m,\bar{x})k_0 + C(x_n)\\
&\subseteq \Phi(\bar{x}) + C(x_m) + \psi(x_n,\bar{x})q(x_n,\bar{x})k_0 - \psi(x_n,x_m)q(x_m,\bar{x})k_0 + C(x_n)\\
&\subseteq \Phi(\bar{x}) + \psi(x_n,\bar{x})q(x_n,\bar{x})k_0 + (\psi(x_n,x_n) - \psi(x_n,x_m))q(x_m,\bar{x})k_0 - \psi(x_n,x_n)q(x_m,\bar{x})k_0 + C(x_n)\\
&\subseteq \Phi(\bar{x}) + \psi(x_n,\bar{x})q(x_n,\bar{x})k_0 - \psi(x_n,x_n)q(x_m,\bar{x})k_0 + C(x_n).
\end{aligned}$$

If $z \in \Phi(x_n)$, then there exists $u \in \Phi(\bar{x})$ such that

$$z - u - \psi(x_n,\bar{x})q(x_n,\bar{x})k_0 + \psi(x_n,x_n)q(x_m,\bar{x})k_0 \in C(x_n).$$

Since $q(x_m,\bar{x}) \to 0, (m \to \infty)$, and $C(x_n)$ is a k_0-closed, we have $z - u - \psi(x_n,\bar{x})q(x_n,\bar{x})k_0 \in vcl_{k_0}(C(x_n)) = C(x_n)$. Then,

$$z \in u + \psi(x_n,\bar{x})q(x_n,\bar{x})k_0 + C(x_n) \subseteq \Phi(\bar{x}) + \psi(x_n,\bar{x})q(x_n,\bar{x})k_0 + C(x_n),$$

and so

$$\Phi(x_n) \subset \Phi(\bar{x}) + \psi(x_n,\bar{x})q(x_n,\bar{x})k_0 + C(x_n).$$

Therefore, $\bar{x} \preceq x_n$ for each $n \in \mathbb{N}$, and so $\bar{x} \in S(x_n)$ for any $n \in \mathbb{N}$. Thus, condition **C** is satisfied. Hence, by Theorem 2.1, conditions (a) and (b) hold. The proof is completed. □

In the following, we give another version of the EVP for set-valued mappings.

Theorem 3.2. *Let (X,q) be a left-complete quasi-metric space, Y be real linear space, $k_0 \in Y \setminus \{0\}$, and $K \subset Y$ be a nonempty, k_0-vectorially closed, and free disposal with respect to $cone(\{k_0\})$. Let $\Phi : X \rightrightarrows Y$ be a set-valued mapping with nonempty values, $C : X \rightrightarrows Y$, $C(x)$ be k_0-vectorially closed, and free disposal with respect to $cone(\{k_0\})$, $C(x) + K \subseteq K$, and $0 \in C(x)$ for any $x \in X$. Let $\psi : X \times X \to]0,+\infty[$ be a real valued bifunction such that $inf_{x\in X}\psi(\cdot,x) > 0$, and the following conditions hold.*

(i) *Φ is C-u-left slm in (X,q).*
(ii) *ψ is u-Φ-decreasing in the first argument and u-Φ-increasing in the second argument.*
(iii) *If $\Phi(u) \leq^u_{C(v)} \Phi(v)$, then $C(v) + C(u) \subseteq C(v)$.*
(iv) *Suppose that $x_0 \in X$, $y_0 \in Y$ and $\varepsilon > 0$ such that $\Phi(x) \not\leq^u_K y_0 - \varepsilon k_0$, for all $x \in X$ with $\Phi(x) \leq^u_{C(x_0)} \Phi(x_0)$, and $\Phi(x_0) \leq^u_{C(x_0)} y_0 + s_0$, for some $s_0 \in \mathbb{R}k_0 - K$.*

Then, there exists $\hat{x} \in X$ such that

(a) *$\Phi(\hat{x}) + \psi(x_0,\hat{x})q(x_0,\hat{x})k_0 \leq^u_{C(x_0)} \Phi(x_0)$;*
(b) *$\Phi(x) + \psi(\hat{x},x)q(\hat{x},x)k_0 \not\leq^u_{C(\hat{x})} \Phi(\hat{x})$, for all $x \neq \hat{x}$.*

Proof. Similar to the proof of Theorem 3.1, we define a relation $\preceq$ on X as follows: for any $x,x' \in X$,

$$x' \preceq x \iff \Phi(x') + \psi(x,x')q(x,x')k_0 \leq^u_{C(x)} \Phi(x).$$

Note that $\Phi(x')+\psi(x,x')q(x,x')k_0 \leq^u_{C(x)} \Phi(x)$ if and only if $\Phi(x)-\psi(x,x')q(x,x')k_0 \leq^l_{-C(x)} \Phi(x')$. From the same proof as that of Theorem 3.1, e relation $\preceq$ is a pre-order. Now, we set

$$S(x) := \{x' \in X : x' \preceq x\}, \quad \forall x \in X,$$

and define $\eta : (X,\preceq) \to \mathbb{R}\cup\{+\infty\}$ as follows:

$$\eta(x) := \sup\{\xi_{k_0}(y-y_0+\varepsilon k_0) : y \in \Phi(x)\}, \ \forall x \in X.$$

We will show that all conditions of Theorem 2.1 are satisfied. In the first step, we show that η is monotone with respect to $\preceq$. If $x,x' \in X$ and $x' \preceq x$, then $\Phi(x')+\psi(x,x')q(x,x')k_0 \subseteq \Phi(x)-C(x)$. So, for every $y' \in \Phi(x')$ there exists $y \in \Phi(x)$ such that $y \geq_{C(x)} y'+\psi(x,x')q(x,x')k_0$. Since $C(x)+K \subseteq K$, then ξ_{k_0} is $C(x)$-monotone. Hence, we have

$$\begin{aligned}&\xi_{k_0}(y'-y_0+\varepsilon k_0+\psi(x,x')q(x,x')k_0)\\ &=\xi_{k_0}(y'-y_0+\varepsilon k_0)+\psi(x,x')q(x,x') \leq \xi_{k_0}(y-y_0+\varepsilon k_0).\end{aligned}$$

So,

$$\eta(x')+\psi(x,x')q(x,x') \leq \eta(x).$$

This shows that η is monotone with respect to $\preceq$. Moreover, if $x' \preceq x$, $x \neq x'$, and $\eta(x) < \infty$, then

$$\eta(x') \leq \eta(x)-\psi(x,x')q(x,x') < \eta(x).$$

Since $x_0 \in S(x_0)$, $S(x_0) \neq \emptyset$. Now, it is enough to show that conditions (A), (B), and (C) in Theorem 2.1 hold.

Checking condition A. If $x \in S(x_0)$, then $\Phi(x) \leq^u_{C(x_0)} \Phi(x_0)$. From condition (iv), we have $\Phi(x) \not\leq^u_K y_0-\varepsilon k_0$. Therefore, there exists $y \in \Phi(x)$ such that $y-y_0+\varepsilon k_0 \notin -K$. Thus, $\xi_{k_0}(y-y_0+\varepsilon k_0) \geq 0$. So, $\eta(x) \geq 0$. Since $\Phi(x_0) \leq^u_{C(x_0)} y_0+s_0$, then, for every $y \in \Phi(x_0)$, $y-y_0+\varepsilon k_0 \in \varepsilon k_0+s_0-C(x_0)$. Hence,

$$\eta(x_0) \leq \xi_{k_0}(y-y_0+\varepsilon k_0) \leq \xi_{k_0}(\varepsilon k_0+s_0) = \varepsilon+\xi_{k_0}(s_0) < +\infty.$$

Then, $\eta(x) \leq \eta(x_0) < +\infty$, for all $x \in S(x_0)$. Therefore, condition (**A**) is satisfied. By the same proof as that of Theorem 3.1, conditions (**B**), and (**C**) of Theorem 2.1 are satisfied. This completes the proof. □

As a consequence, we can obtain the EVP for set-valued bimaps.

Theorem 3.3. *Suppose that X, Y, K, C, and Φ are the same as in Theorem 3.2, and all conditions of Theorem 3.2 are satisfied. Let $F : X\times X \rightrightarrows Y$ be a set-valued bimap with nonempty values such that $\Phi(y) \leq^u_{C(x)} \Phi(x)+F(x,y)$ for every $x,y \in X$. Then there exists $\hat{x} \in X$ such that*

(V1) $\Phi(\hat{x})+\psi(x_0,\hat{x})q(x_0,\hat{x})k_0 \leq^u_{C(x_0)} \Phi(x_0)$*;*

(V2) $F(\hat{x},x)+\psi(\hat{x},x)q(\hat{x},x)k_0 \not\leq^u_{C(\hat{x})} 0$*, for each $x \neq \hat{x}$.*

Proof. According to Theorem 3.2, there exists $\hat{x} \in X$ such that conditions (a) and (b) hold. Therefore, condition (V1) is satisfied. Now, suppose on the contrary that there exists $x \neq \hat{x}$ such that $F(\hat{x},x)+\psi(\hat{x},x)q(\hat{x},x)k_0 \leq^u_{C(\hat{x})} 0$. Hence,

$$\Phi(x)+\psi(\hat{x},x)q(\hat{x},x)k_0 \leq^u_{C(\hat{x})} \Phi(\hat{x})+F(\hat{x},x)+\psi(\hat{x},x)q(\hat{x},x)k_0 \leq^u_{C(\hat{x})} \Phi(\hat{x}),$$

which is a contradiction. □

Remark 3.1. If F satisfies condition $F(x,z) \subset F(x,y) + F(y,z) - C(x)$ for all $x,y,z \in X$, in fact $F(x,z) \leq^u_{C(x)} F(x,y) + F(y,z)$, then we find from $\Phi(x) = F(x_0,x)$ for all $x \in X$ tat $\Phi(y) \leq^u_{C(x)} \Phi(x) + F(x,y)$, for every $x,y \in X$.

From the above remark and Theorem 3.3, we have the following result.

Theorem 3.4. *Let (X,q) be a left-complete quasi-metric space, Y be real linear space, $k_0 \in Y \setminus \{0\}$, and $K \subset Y$ be a nonempty, k_0-vectorially closed and free disposal with respect to cone$(\{k_0\})$. Let $C : X \rightrightarrows Y$, $C(x)$ be k_0-vectorially closed and free disposal with respect to cone$(\{k_0\})$, $C(x) + K \subseteq K$, and $0 \in C(x)$ for any $x \in X$. Let $F : X \times X \rightrightarrows Y$ be a set-valued bimap with nonempty values, and let $\psi : X \times X \to]0,+\infty[$ be a real valued bifunction such that $inf_{x\in X}\,\psi(\cdot,x) > 0$ and the following conditions hold.*

- (i) *Suppose that $x_0 \in X$, $y_0 \in Y$ and $\varepsilon > 0$ such that $F(x_0,x) \nleq^u_K y_0 - \varepsilon k_0$, for all $x \in X$ with $F(x_0,x) \leq^u_{C(x_0)} F(x_0,x_0)$, and $F(x_0,x_0) \leq^u_{C(x_0)} y_0 + s_0$, for some $s_0 \in \mathbb{R}k_0 - K$.*
- (ii) *$F(x_0,\cdot)$ is C-u-left slm in (X,q).*
- (iii) *ψ is u-$F(x_0,\cdot)$-decreasing in the first argument and u-$F(x_0,\cdot)$-increasing in the second argument.*
- (iv) *If $F(x_0,u) \leq^u_{C(v)} F(x_0,v)$, then $C(v) + C(u) \subseteq C(v)$.*
- (v) *$F(x,z) \subset F(x,y) + F(y,z) - C(x)$, for all $x,y,z \in X$.*

Then, there exists $\hat{x} \in X$ such that

- (V1) *$F(x_0,\hat{x}) + \psi(x_0,\hat{x})q(x_0,\hat{x})k_0 \leq^u_{C(x_0)} F(x_0,x_0)$;*
- (V2) *$F(\hat{x},x) + \psi(\hat{x},x)q(\hat{x},x)k_0 \nleq^u_{C(\hat{x})} 0$, for each $x \neq \bar{x}$.*

4. Application: when migration increases happiness

At the level of applications, we have the question: when does migration increase happiness? One study [39] found that "immigrants around the world are generally happier after migration—reporting more life satisfaction, more positive emotions, and fewer negative emotions—based on Gallup surveys of about 36,000 immigrants from more than 150 countries".

Recently, Soubeyran [33] provided a new theory of transit in migration as a direct consequence of the recent (VR) approach of stay and change human dynamics [29, 30, 31, 32]. This approach clearly distinguishes between two cases: when a migrant faces strong or weak obstacles (resistances) to migrate. For some empirical references on transit migration, we refer to Papadopoulou-Kourkoula [40] and Castagnone [41]. At the application level, we show how this paper, as a set-valued generalization of the Ekeland variational principle [1], offers an important extension of this new theory of migration dynamics in terms of transit migration [33] when resistance to migration is strong.

4.1. **The (VR) variational rationality approach.** The (VR) approach of stay and change human dynamics provides in mathematics, in the context of variational analysis and optimizing algorithms, a general and dynamic reformulation of the theory of motivation, emotion, and behavior in psychology. The goal of the motivation theory is to know why, how, and when people do what they do in a dynamic setting. Why, how, and when they stop, continue, or resume doing something every day of their lives. Why, how and when they withdraw, re-engage, or engage in different goals and activities. The first important step of VR's approach was to show (which has never been done explicitly before) that all these traditional questions in psychology prove that human dynamics can be modeled as stop and go dynamics (stop continue and start) dynamics. This new approach is unifying because it is driven by only one concept. This is the concept of worthwhile move, which provides a new and generalized formulation of sufficient descent conditions in many different and current variational principles and optimization algorithms. Thus, the VR approach has two sides. First, it illustrates and generalizes almost all the main concepts of variational principles and optimization algorithms. Second, it provides for the first time a general and mathematical theory of motivational dynamics, emotions, and behavior in psychology.

4.2. **A simple VR model of migration.** Consider three periods: an earlier (initial) period, a current period, and a future period, and an individual who wants to fulfil a long list of needs $j \in J = \{1, 2, ..., h\}$. These needs can be physiological, physical, material, financial, cognitive, emotional, and social. Partial fulfilment of a need j provides some pleasures (feelings of satisfaction). It also brings pain (feelings of frustration) as long as an individual has to wait for the complete fulfilment of that need. To satisfy these needs, an individual must perform various actions. To be able to perform these actions and to perform them is costly. It requires the expenditure of resources and efforts. This, in turn, causes various pains, including fatigue, ego depletion, boredom, stress, and opportunity costs. For example, if you expend too many resources to meet a particular need, other needs may not be met. This gives a sense of frustration. Then, in relation to the fulfilment of each $j \in J$, we can associate both pleasures and pains, i.e., feelings of satisfaction and feelings of dissatisfaction. That is, net feelings of satisfaction.

Net Satisfaction of living in a city. Suppose that an individual in the first period lives in city x of a country $\mathbf{x}$. This means that he is able to perform a bundle of activities $\alpha \in \mathscr{A}(x)$ located in that city. In this context, a bundle of activities α located in city x helps him to partially satisfy his current needs. The net satisfaction feeling (utility, pleasure) resulting from the satisfaction of need j is $g_j(\alpha, x) \in \mathbb{R}$. These feelings of satisfaction are nets of costs of doing the bundle of activities α in city x. For example, doing an activity (a pleasant job) both satisfies a need and may increase the magnitude of other needs (needs to rest more and spend more time with one's family). Let $g_j^* = \sup\left\{g_j(\alpha, x), \alpha \in \mathscr{A}(x), x \in X\right\} < +\infty$ be the desire level relative to the net satisfaction feeling $g_j(\alpha, x)$, given that this individual can expect to live in any city x of any country $\mathbf{x}$. This is the highest satisfaction feeling he can hope to obtain relative to need satisfaction $j \in J$. Then the difference $\varphi_j(\alpha, x) = g_j^* - g_j(\alpha, x) \geq 0$ models the frustration or sense of dissatisfaction with respect to partially satisfying need j when living in city α of country $\mathbf{x}$. So,

- the vector of individual's net satisfaction levels when living in city of x of country $\mathbf{x}$ is $g(\alpha, x) = (g_j(\alpha, x), j \in J)$, and the corresponding set of net satisfaction levels is $G(x) = \{g(\alpha, x), \alpha \in \mathscr{A}(x)\}$;
- the vector of frustration levels of this individual when living in city x of country $\mathbf{x}$ is $\varphi(\alpha, x) = (\varphi_j(\alpha, x), j \in J)$, and the associated set of frustration levels is $\Phi(x) = \{\varphi(\alpha, x), \alpha \in \mathscr{A}(x)\}$. Thus $\varphi(\alpha, x) = g^* - g(\alpha, x)$.

4.2.1. *Migrating from living in a city to live in another city.* In the context of a migration theory, the VR approach [33] defines a space of positions P and an associated space $\mathfrak{M} = \mathsf{P} \times \mathsf{P}$ of possible moves. In the initial period, the position of an individual is $p = (\alpha, x) \in \mathsf{P}$. In the current period, its position may be $q = (\beta, y) \in \mathsf{P}$. If this is the case, this individual has made a move $m = p \in \mathsf{P} \curvearrowright q \in \mathsf{P}$. In this first phase of the presentation, a migrant in the current period compares two alternatives (moves): to stay or to change, i.e.,

- (i) to stay with the status quo $p = (\alpha, x) \in \mathsf{P}$, which makes the stay $p \curvearrowright q = p$;
- ii) to switch implying the change $p \curvearrowright q, q \neq p, q = (\beta, y) \in \mathsf{P}$.

In this scenario, migration occurs only if $y \neq x$. If $y = x$, this individual stays in the same city x and performs the same bundle of activities in the previous and current periods or not.

The motivation to move comes from weighing the advantages and inconveniences of moving. It compares,

(i) additional satisfaction (additional pleasures) to better satisfy needs and remaining dissatisfactions to not sufficiently satisfy them (frustrations) with,

ii) additional dissatisfaction (additional pains) to become relocatable and move (additional migration costs).

These satisfactions and dissatisfaction are measured in the same units.

4.2.2. *Advantages of moving from one city to another.* **Utilities (feelings of satisfaction)**. Suppose that an individual, in order to better satisfy his various needs, moves from city x of country $\mathbf{x}$, where he lived in the past period, to city y of country $\mathbf{y}$ in the current period. Therefore, his vector of physiological, physical, material, financial, cognitive, emotional, and social satisfactions (utilities) moves from $g(p) = g(\alpha, x) = (g_j(\alpha, x), j \in J)$ to $g(q) = g(\beta, y) = (g_j(\beta, y), j \in J)$.

Migration advantages. The advantages of moving from city x to city y for an individual then define the difference $A(q/p) = g(q) - g(p) = (A_j(q/p) = g_j(q) - g_j(p), j \in J)$ between the vector of utilities $g(q)$ of living in city y and the vector of utilities $g(p)$ of living in the city x. Then the advantages of moving are also the difference between the vector of feelings of frustration of staying in city x and the vector of feelings of frustration of living in the city y : $A(q/p) = \varphi(p) - \varphi(q) = g(q) - g(p)$, provided that $\varphi_j(\alpha_j, x) - \varphi_j(\beta_j, y) = g_j(\beta_j, y) - g_j(\alpha_j, x)$ for any need $j \in J$.

4.2.3. *Inconveniences to move.* As we will see below that moving is costly. Resistance to moving can be very strong. So in deciding whether or not to move, the advantages and inconveniences must be weighed. Inconvenience of moving $I_j\ (q/p) = C_j(p,q) - C_j(p,p) \geq 0$ represents, for each physiological, physical, material, financial, cognitive, emotional, and social need $j \in J$, the difference between two costs. On the one hand, the cost of staying in the same position p, i.e. $C_j(p,p)$. On the other side the cost to changing $C_j(p,q)$ from position p to position q. The construction of such costs of changing location is very complex and requires a variety of justifications, since they model many different and simultaneous types of costs, including physiological, physical, material, financial, cognitive, motivational, effective, and social costs, see [29, 30, 31, 32]. Costs to move are not symmetric. That is, $C_j(q,p) \neq C_j(p,q)$ if $q \neq p$.

Costs of moving $C_j(p,q)$ can be, depending on the model, ability costs (costs to be able to do something), or capability plus execution costs (costs to do that thing).

4.2.4. *A worthwhile move: balancing motivation and resistance to move.* Consider a particular need j. Then, relative to the satisfaction of that need,

1) motivation to move is $M_j(q/p) = U_j\left[A_j(q/p)\right]$, where $U_j\left[A_j\right]$ is the utility (value) of advantages of moving;

2) resistance to the move is $R_j(q/p) = D_j\left[I_j(q/p)\right]$, $j \in J$, where $D_j\left[I_j\right]$ is the disability of inconvenience of moving. Resistance to a move is strong if $D_j\left[I_j\right] = I_j^{\gamma}, 0 < \gamma \leq 1$. It is weak if $\gamma > 1$.

-3) a worthwhile balance $B_j(q/p) = M_j(q/p) - \xi_j R_j(q/p)$ is a weighted difference between motivation and resistance to move. The importance given to resistance to move is the weight $\xi_j > 0$.

- 4) then a move is worthwhile with respect to the satisfaction of need j if $B_j(q/p) \geq 0$. This condition implies that motivation to move is high enough relative to the satisfaction of need j, to overcome resistance to moving. Thus, moving is worthwhile if $B_j(q/p) \geq 0$ for all $j \in J$.

- 5) The vectorial advantages, inconvenience, motivation, and resistance to moving are then $A(q/p) = (A_j(q/p), j \in J), I(q/p) = (I_j(q/p), j \in J)$,

$M(q/p) = (M_j(q/p), j \in J)$ and $R = (R_j(q/p), j \in J)$. Then, a vectorial worthwhile balance follows: $B = M - \xi * R$ if $\xi * R = (\xi_j R_j, j \in J)$.

-6) The mathematical part of this paper implicitly assumes a linear variational structure, where $U_j\left[A_j\right] = A_j$ and $D_j\left[I_j\right] = I_j$ for all $A_j, I_j \in R_+$. In this linear structure, $M = A, R = I$, and $B = M - \xi * R$.

4.2.5. *Worthwhile stop and go individual dynamics.* Starting from the status quo position p_0, these dynamics can be

a) over one period (the current period),

- a desired move going from position p_0 to position q if $A_j(q/p_0) \geq 0$ for all $j \in J$;
- a worthwhile move going from position p_0 to position q if $B_j(q/p_0) \geq 0$ for all $j \in J$;
- a desired stay $p^* = p_0 \in \mathrm{P}$ if, for all $q \neq p^*$, it exists $j \in J$ such that $\mathbf{A}_j(q/p^*) < 0$;
- a stationary trap $p_* = p_0 \in$ if for all $q \neq p^*$, it exists $j \in J$ such that $B_j(q/p^*) < 0$.

b) over two periods,

- a variational trap $p_* \neq p_0 \in \mathrm{P}$ if, starting from position $p_0 \in \mathrm{P}$,

i) it is worthwhile to change from p_0 to p_* in the current period; that is, if $B_j(p_*/p_0) \geq 0$ for all $j \in J$;

ii) p_* is a stationary trap in the future period, i.e., it is not worthwhile to move from p_* to q, for all $q \neq p^*$.

4.3. **Interpretations of the results of this paper.**

4.3.1. *Rewriting Theorem 3.1.* Let us give an interpretation of Theorem 3.1, which is the second main result of this paper. This theorem yields two inequalities (a) and (b).

Let us first consider inequality (a) $A = \Phi(\bar{x}) + I(x_0, \bar{x}) \leq^l_{C(x_0)=S} \Phi(x_0) = B$, where $\Phi(x) = \{\varphi(\alpha, x), \alpha \in \mathscr{A}(x)\}$ and $I(x_0, \bar{x}) = \psi(x_0, \bar{x}) q(x_0, \bar{x}) k_0$. Recall that $A \leq^l_S B$ if and only if $B \subseteq A + S$, i.e., for all $b \in B$, there exist $a \in A$ and $s \in S$ such that $b = a + s$. Then inequality (a) is equivalent to: for all $b = \varphi(\alpha, x_0) \in B$, $\alpha \in \mathscr{A}(x_0)$, there exist $a = \varphi(\beta, \bar{x}) + I(x_0, \bar{x}) \in A, \beta \in \mathscr{A}(\bar{x})$, and $s \in S = C(x_0)$ such that $\varphi(\alpha, x_0) = \varphi(\beta, \bar{x}) + I(x_0, \bar{x}) + s$. That is, $g^* - g(\alpha, x_0) = g^* - g(\beta, \bar{x}) + I(x_0, \bar{x}) + s$, i.e., $g(\alpha, x_0) = g(\beta, \bar{x}) - I(x_0, \bar{x}) - s$.

In summary, we have shown that for all α that belong to x_0, there exist β belong to $\bar{x}$ and $s \in S$ such that $g(\beta, \bar{x}) - I(x_0, \bar{x}) = g(\alpha, x_0) + s$, i.e.,

$g(\beta, \bar{x}) - g(\alpha, x_0) = \varphi(\alpha, x_0) - \varphi(\beta, \bar{x}) = I(x_0, \bar{x}) + s, (*)$

where $I(x_0, \bar{x}) = \psi(x_0, \bar{x}) q(x_0, \bar{x}) k_0$.

It is time to return to the VR concepts of vectorial advantages and inconveniences of moving.

Advantages to move. Let $p = (\alpha, x_0)$ and $q = (\beta, \bar{x})$. Then, the difference who appeared in the left hand side of equality (*) represents vectorial advantages to move $A(q/p) = g(\beta, \bar{x}) - g(\alpha, x_0) = g(q) - g(p)$.
Inconveniences to move. Let us show now that the first term $I(x_0, \bar{x}) = \psi(x_0, \bar{x})q(x_0, \bar{x})k_0$ who appeared in the right hand side of (*) represents a specific formulation of vectorial inconveniences to move. In this paper, costs to change $C(p,q) = \mathbb{C}(x_0, \bar{x})$ represent migration costs from city x_0 to city $\bar{x}$ (as capability costs). Costs to stay $C(p,p) = \mathbb{C}(x_0, x_0) = 0$ are equal to zero (no migration costs). Then, inconveniences to move are $I(q/p) = C(p,q) - C(p,p) = \mathbb{C}(x_0, \bar{x})$. The formulation $I(x_0, \bar{x}) = \mathbb{C}(x_0, \bar{x}) = \psi(x_0, \bar{x})q(x_0, \bar{x})k_0$ has two ingredients:

i) it supposes the existence of a **sharing rule** to solve a usual cost allocation problem. Let $q(x_0, \bar{x}) \in R_+$ be the psychological distance between city x_0 and city $\bar{x}$. This concept comes from Lewin [42]. It considers the physiological, physical, material, financial, cognitive, emotional and social aspects of a locomotion from a position to an other position. The VR approach models it as a generalized distance. For example, a quasi-distance, a pseudo-quasi-distance, a w distance, a partial quasi-distance, depending of the definition of a position in a locomotion space. In this paper, it is a quasi-distance. Let $k_0 = (k_{0,j}, j \in J) \in (R_+)^{cardJ}$, $\Sigma_{j\in J} k_{0,j} = 1, k_{0,j} > 0$ for all $j \in J$ be a vector of strictly positive shares of a global migration costs from city x_0 and city $\bar{x}$, $C(x_0, \bar{x}) = \psi(x_0, \bar{x})q(x_0, \bar{x})k_0$. Then, relative to the satisfaction of each need j, we can associate the migration cost $\mathbb{C}_j(x_0, \bar{x}) = q(x_0, \bar{x})k_{0,j}, j \in J$.

ii) its models an **ego depletion hypothesis** and a **goal gradient hypothesis**. Ego depletion refers to the idea that self-control or willpower draws upon a limited pool of mental resources that can be used up (Baumeister et al. [43]). The goal-gradient hypothesis refers to the classic finding from behaviorism that animals expend more effort when approaching a reward. ... The goal-gradient hypothesis [44] states that the tendency to approach a goal increases with proximity to the goal. The introduction of the term $\psi(x_0, \bar{x})$ models these two aspects. To save space see Fakhar et al. [16].
Changing preferences. They are modeled by the definition of moving cones. To save space, see Bao et al. [10] and Bao et al. [11] for a clear modeling of this aspect.

4.3.2. *Interpretation of Theorem 3.1: when migration increases happiness.* One of the most important results of the VR approach is to show when a stop-and-go dynamic ends up in a variational trap, both, when resistance to migration is strong and weak. Very unexpectedly, this result has to do with two important variational principles of variational analysis. The VR approach showed [29, 30, 31, 32] that

i) on one side, the celebrated Ekeland variational principle provides an example of stop and go dynamics that ends in a trap when resistance to move is strong and is modeled using a (pseudo)quasi-distance, w-distance, or partial quasi-distance.

ii) on the other side, the famous proximal algorithm [45] provides examples of stop-and-go dynamics, ending in a variational trap when resistance to move is weak, using the square of a (pseudo) quasi-distance, w-distance, or partial quasi-distance.

Theorem 3.1 shows that there are variational traps that are worth reaching but not worth leaving if resistance to moving is strong. This theorem provides an example of stop-and-go dynamics when uncertainty plays a large role. It solves a robust trap problem.

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