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VECTORIZATION IN NONCONVEX SET OPTIMIZATION

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Abstract. A new vectorization approach is presented for nonconvex set optimization problems with the set less order relation. This technique uses certain approximation problems with suitable parametric norms. The convexity of the sets is not required but one needs a strict lower and upper bound for all occuring sets. Karush-Kuhn-Tucker conditions are derived as necessary optimality conditions for set optimization problems in finite dimensional Euclidean spaces with the natural order cone. A multiplier-free necessary optimality condition is given as well.

Keywords. Approximation; Set optimization; Vectorization; Optimality conditions.

1. Introduction

For set optimization problems, i.e. for the optimization problems with a set-valued objective, vectorization is a known technique, which replaces the considered set optimization problem by an appropriate vector optimization problem. In general, these vector optimization problems are simpler to investigate.

Vectorization was introduced by Küçük et al. [1, 2] in 2012. The authors used a total order cone in finite dimensions and in separable Hilbert spaces. An extension of a scalarizing functional introduced by Gerstewitz (Tammer) [3] was used by Karaman et al. [4] to present a vectorization approach for nonconvex set optimization problems. The vectorization approach by Jahn [5] works with support functionals for convex sets, and was used for the formulation of Karush-Kuhn-Tucker conditions in finite dimensions [6] and for an application of vectorization in reflexive Banach spaces [7].

In this paper, the convexity of sets is not assumed as in [5] but we have the assumption that there exist a strict lower and upper bound of all sets, which is actually a weak condition. Because of the lack of convexity, the resulting vector optimization problems work with the approximation problems with parametric norms instead of the problems with a linear objective. Therefore, this theory is completely different in detail.

There are various order relations used in set optimization (see, e.g., [8]). In the present paper, we restrict ourselves to the well-known set less order relation introduced by Young [9] in 1931 and named by Chiriaev and Walster [10]. The presented results also subsume the weaker l-type less and u-type less order relations (see Kuroiwa [11] for first results).

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The nonconvex vectorization principle of this paper is applied in order to obtain necesary optimality conditions for the original set optimization problem in finite dimensional Euclidean spaces with the natural order cone. In a first step, we obtain Karush-Kuhn-Tucker conditions where some of the Lagrange multipliers are finite Radon measures. Simpler necessary optimality conditions are obtained in a multiplier-free form. A very simple example demonstrates the complexity of the resulting optimality conditions in set optimization.

This paper is organized as follows: Notations and background results are summarized in Section 2. The new vectorization approach with approximation problems is presented in Section 3. The last section, Section 4, contains necessary optimality conditions in a finite dimensional setting like Karush-Kuhn-Tucker conditions and multiplier-free conditions together with an illustrative very simple nonconvex example.

2. Preliminaries

Throughout this paper, we consider a real linear space Y and a convex cone $C \subset Y$. Recall that the cone C is said to be *pointed* iff $(-C) \cap C = \{0_Y\}$. A nonempty subset of Y is called *algebraically closed* and *algebraically bounded* iff its intersection with every straight line in Y, which is considered as subset of the real line $\mathbb{R}$, gives a closed and bounded set, respectively (for analytical details compare [12, Def. 1.8, (d) and (e)]). The *algebraic interior* (or *core*) of a nonempty subset $A \subset Y$ is defined as

$$\operatorname{core} A := \{\bar{y} \in A \mid \forall\, y \in Y\ \exists\, \bar{\lambda} > 0 :\ \bar{y} + \lambda y \in A\ \forall\, \lambda \in [0, \bar{\lambda}]\}.$$

A nonempty subset of Y is said to be *absolutely convex* iff it is convex and balanced (i.e. for a nonempty subset $A \subset Y$: $\alpha A \subset A$ for all $\alpha \in [-1,1]$).

For two nonempty sets $A, B \subset Y$, we use the abbreviation

$$A \pm B := \{a \pm b \in Y \mid a \in A \text{ and } b \in B\}$$

for the sum and the difference, respectively.

Next, we come to the standard assumption giving the frame of the following theory.

Assumption 2.1. Let Y be a real linear space, and let $C \subset Y$ be a pointed, algebraically closed, convex cone with nonempty algebraic interior.

It is well-known that, under Assumption 2.1, the real linear space Y can be normed via certain order intervals (see [12, Lemma 1.45]). For this approach, consider for an arbitrary $p \in \operatorname{core} C$ the order interval $[-p, p] := (\{-p\} + C) \cap (\{p\} - C)$. This order interval is absolutely convex, algebraically closed and algebraically bounded, and $0_Y \in \operatorname{core}[-p, p]$ (see [12, Lemma 1.22]). Therefore, the Minkowski functional $\|\cdot\|_p : Y \to \mathbb{R}$ with

$$\|y\|_p := \inf_{\lambda > 0} \left\{\lambda \;\middle|\; \tfrac{1}{\lambda} y \in [-p, p]\right\} \text{ for all } y \in Y \tag{2.1}$$

is a norm (see the proof of Lemma 1.45 in [12]). It is evident from the definition of this norm that the order interval $[-p, p]$ equals the unit ball $\mathscr{B}_p(0_Y, 1)$ (see Figure 1).

Although Y is originally assumed to be a real linear space, under the assumptions concerning the order cone C, it turns out that Y is even a real normed space. This means that Assumption 2.1 is actually a strong assumption.

The norm $\|\cdot\|_p$ is a parametric norm depending on a parameter $p \in \operatorname{core} C$. For standard scenarios, these parametric norms are well-known.

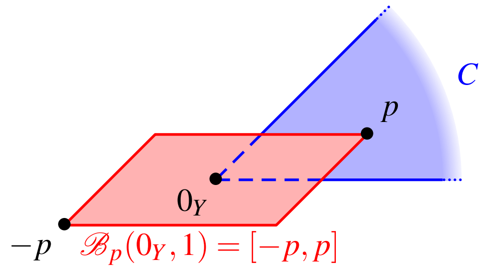

FIGURE 1. Illustration of the unit ball $\mathscr{B}_p(0_Y,1)$ for an arbitrary $p \in \operatorname{core} C$.

Proposition 2.1 (Lemmas 5.32, 5.33 and 5.34 in [12])**.** *Let Assumption 2.1 be satisfied.*

(a) For $Y := \mathbb{R}^m$ (with $m \in \mathbb{N}$), $C := \mathbb{R}^m_+$ and an arbitrary $p \in \operatorname{core} \mathbb{R}^m_+ = \{p \in \mathbb{R}^m \mid p_i > 0 \; \forall\, i \in \{1,\dots,m\}\}$, one obtains

$$\|y\|_p = \max_{i\in\{1,\dots,m\}} \frac{1}{p_i}|y_i| \text{ for all } y \in \mathbb{R}^m.$$

(b) For $Y := \mathscr{C}[0,1]$ (the real linear space of real-valued continuous functions on $[0,1]$), $C := \mathscr{C}_+[0,1] := \{y \in \mathscr{C}[0,1] \mid y(t) \geq 0 \; \forall\, t \in [0,1]\}$, and an arbitrary $p \in \operatorname{core} \mathscr{C}_+[0,1]$, one obtains

$$\|y\|_p = \max_{t\in[0,1]} \frac{1}{p(t)}|y(t)| \text{ for all } y \in \mathscr{C}[0,1].$$

(c) For $Y := \mathscr{S}^n$ (the real Hilbert space of symmetric (n,n) matrices with $n \in \mathbb{N}$), $C := \mathscr{S}^n_+ := \{A \in \mathscr{S}^n \mid A \text{ positive semidefinite}\}$, and an arbitrary $P \in \operatorname{core} \mathscr{S}^n_+$, one obtains

$$\|A\|_P = \sup_{x\neq 0_{\mathbb{R}^n}} \frac{1}{x^T P x}|x^T A x| \text{ for all } A \in \mathscr{S}^n.$$

This proposition shows that the parameter $p \in \operatorname{core} C$ defines a weight in these special norms. To be more concrete, in part (a) of Proposition 2.1, the reciprocals of the components of p equal the weights of the weighted maximum norm.

3. VECTORIZATION APPROACH

We commence this section with the comparison of sets in the real linear space Y using a specific order relation (for other types of order relations see [8]). We investigate the well-known set less order relation introduced by Young [9] in 1931 and named by Chiriaev and Walster [10].

Definition 3.1. Let Y be a real linear space, and let $C \subset Y$ be a convex cone. For nonemty subsets $A, B \subset Y$, the *set less* order relation $\preccurlyeq_s$ is defined by

$$A \preccurlyeq_s B \quad :\Longleftrightarrow \quad B \subset A + C \text{ and } A \subset B - C.$$

For the characterization of the inclusions defining the set less order relation, we work with certain approximation problems. Although we do not assume convexity of the considered sets, we require the existence of a strict lower and upper bound of these sets, which is a weak assumption in practice.

Assumption 3.1. Let Y be a real linear space, and let $C \subset Y$ be a pointed, algebraically closed, convex cone with nonempty algebraic interior. For an arbitrary $p \in \operatorname{core} C$, let $\|\cdot\|_p$ be given by (2.1). For arbitrarily chosen $\hat{y}_l, \hat{y}_u \in Y$, let $\mathscr{F}$ denote the family of nonempty subsets A of Y with $A \subset (\{\hat{y}_l\} + \operatorname{core} C) \cap (\{\hat{y}_u\} - \operatorname{core} C)$.

First, we prove a simple result concerning approximation problems with the considered parametric norms.

Lemma 3.1. *Let Assumption 3.1 be satisfied. For an arbitrarily chosen $A \in \mathscr{F}$, we obtain*

$$\inf_{a \in A} \|a - \hat{y}_l\|_p = \inf_{y \in A+C} \|a - \hat{y}_l\|_p \tag{3.1}$$

and

$$\inf_{a \in A} \|a - \hat{y}_u\|_p = \inf_{y \in A-C} \|a - \hat{y}_u\|_p. \tag{3.2}$$

Proof.

(a) For the proof of equality (3.1), notice that $\inf_{a \in A} \|a - \hat{y}_l\|_p \geq \inf_{y \in A+C} \|y - \hat{y}_l\|_p$ is obvious. For the proof of the converse inequality, we assume that $\inf_{a \in A} \|a - \hat{y}_l\|_p > \inf_{y \in A+C} \|y - \hat{y}_l\|_p$. Then there are $\bar{a} \in A$ and $\bar{c} \in C$ so that, for $\bar{y} := \bar{a} + \bar{c}$,

$$\inf_{a \in A} \|a - \hat{y}_l\|_p > \|\bar{y} - \hat{y}_l\|_p =: \lambda. \tag{3.3}$$

By the definition of the norm $\|\cdot\|_p$, we have $\bar{y} \in \{\hat{y}_l + \lambda p\} - C$ and $\bar{a} \in \{\hat{y}_l + \lambda p\} - C - \{\bar{c}\} \subset \{\hat{y}_l + \lambda p\} - C$. This means that $\|\bar{a} - \hat{y}_l\|_p \leq \lambda = \|\bar{y} - \hat{y}_l\|_p$, a contradiction to inequality (3.3).

(b) The proof of equality (3.2) follows the lines of part (a). The inequality $\inf_{a \in A} \|a - \hat{y}_u\|_p \geq \inf_{y \in A-C} \|y - \hat{y}_u\|_p$ is evident. If we assume $\inf_{a \in A} \|a - \hat{y}_u\|_p > \inf_{y \in A-C} \|y - \hat{y}_u\|_p$, then there are $\bar{a} \in A$ and $\bar{c} \in C$ such that, for $\bar{y} := \bar{a} - \bar{c}$,

$$\inf_{a \in A} \|a - \hat{y}_u\|_p > \|\bar{y} - \hat{y}_u\|_p =: \lambda. \tag{3.4}$$

It follows from the definition of the parametric norm that $\bar{y} \in \{\hat{y}_u - \lambda p\} + C$ implying $\bar{a} \in \{\hat{y}_u - \lambda p\} + C + \{\bar{c}\} \subset \{\hat{y}_u - \lambda p\} + C$. We then have $\|\bar{a} - \hat{y}_u\|_p \leq \|\bar{y} - \hat{y}_u\|_p$, which contradicts inequality (3.4).

□

Next, we characterize the set inclusions used in the definition of the set less order relation.

Proposition 3.1. *Let Assumption 3.1 be satisfied, and let the sets $A, B \in \mathscr{F}$ be arbitrarily chosen. Then*

$$B \subset A + C \implies \forall\, p \in \operatorname{core} C : \inf_{a \in A} \|a - \hat{y}_l\|_p \leq \inf_{b \in B} \|b - \hat{y}_l\|_p.$$

If $\min_{y \in A+C} \|y - \hat{y}_l\|_p$ is solvable for all $p \in \operatorname{core} C$, then the converse implication is true as well.

Proof. We prove the assertions by contraposition.

(a) Assume that there exists some $\bar{p} \in \operatorname{core} C$ with $\inf_{a \in A} \|a - \hat{y}_l\|_{\bar{p}} > \inf_{b \in B} \|b - \hat{y}_l\|_{\bar{p}}$. From Lemma 3.1, there exists some $\bar{b} \in B$ such that

$$\|\bar{b} - \hat{y}_l\|_{\bar{p}} < \inf_{a \in A} \|a - \hat{y}_l\|_{\bar{p}} = \inf_{y \in A+C} \|y - \hat{y}_l\|_{\bar{p}}.$$

This implies $\bar{b} \notin A + C$, and we conclude $B \not\subset A + C$.

(b) Now, we assume that $B \not\subset A+C$. Then there is some $\bar{b} \in B$ such that $\bar{b} \notin A+C$. Set $\bar{p} := \bar{b} - \hat{y}_l \in \operatorname{core} C$. Next, we assume that $\min_{y\in A+C}\|y-\hat{y}_l\|_{\bar{p}} \leq 1$. Then there exist elements $a \in A$ and $c \in C$ with $\|a+c-\hat{y}_l\|_{\bar{p}} \leq 1$. By the definition of the norm $\|\cdot\|_{\bar{p}}$, this implies

$$a+c-\hat{y}_l \in \{\bar{p}\} - C = \{\bar{b}-\hat{y}_l\} - C,$$

and we obtain

$$\bar{b} \in \{a+c\} + C \subset A+C+C = A+C,$$

a contradiction to condition $\bar{b} \notin A+C$. Hence, we conclude

$$\min_{y\in A+C}\|y-\hat{y}_l\|_{\bar{p}} > 1 = \|\bar{p}\|_{\bar{p}} = \|\bar{b}-\hat{y}_l\|_{\bar{p}}.$$

It then follows from Lemma 3.1 that

$$\inf_{b\in B}\|b-\hat{y}_l\|_{\bar{p}} \leq \|\bar{b}-\hat{y}_l\|_{\bar{p}} < \min_{y\in A+C}\|y-\hat{y}_l\|_{\bar{p}} = \inf_{y\in A+C}\|y-\hat{y}_l\|_{\bar{p}} = \inf_{a\in A}\|a-\hat{y}_l\|_{\bar{p}}.$$

Consequently, we have $\inf_{a\in A}\|a-\hat{y}_l\|_{\bar{p}} > \inf_{b\in B}\|b-\hat{y}_l\|_{\bar{p}}$.

□

In a similar way, one can prove the following characterization of the second inclusion given in the definition of the set less order relation.

Proposition 3.2. *Let Assumption 3.1 be satisfied, and let the sets $A, B \in \mathscr{F}$ be arbitrarily chosen. Then*

$$A \subset B - C \implies \forall\, p \in \operatorname{core} C: \inf_{a\in A}\|a-\hat{y}_u\|_p \geq \inf_{b\in B}\|b-\hat{y}_u\|_p.$$

If $\min_{y\in B-C}\|y-\hat{y}_u\|_p$ is solvable for all $p \in \operatorname{core} C$, then the converse implication is true as well.

Proof. This proof follows the lines of the proof of Proposition 3.1.

(a) Assume that there is some $\bar{p} \in \operatorname{core} C$ with $\inf_{a\in A}\|a-\hat{y}_u\|_{\bar{p}} < \inf_{b\in B}\|b-\hat{y}_u\|_{\bar{p}}$. For some $\bar{a} \in A$, we then obtain from Lemma 3.1 that

$$\|\bar{a}-\hat{y}_u\|_{\bar{p}} < \inf_{b\in B}\|b-\hat{y}_u\|_{\bar{p}} = \inf_{y\in B-C}\|y-\hat{y}_u\|_{\bar{p}}.$$

Hence, we conclude $\bar{a} \notin B-C$, which implies $A \not\subset B-C$.

(b) Assume that $A \not\subset B-C$. For some $\bar{a} \in A$ with $\bar{a} \notin B-C$, we define $\bar{p} := -(\bar{a}-\hat{y}_u) \in \operatorname{core} C$. If we assume that $\min_{y\in B-C}\|y-\hat{y}_u\|_{\bar{p}} \leq 1$, then $\|b-c-\hat{y}_u\|_{\bar{p}} \leq 1$ for some $b \in B$ and $c \in C$, and we obtain $b-c-\hat{y}_u \in \{-\bar{p}\}+C = \{\bar{a}-\hat{y}_u\}+C$, which implies $\bar{a} \in \{b\} - C$, a contradiction to our assumption. Consequently, we obtain

$$\inf_{a\in A}\|a-\hat{y}_u\|_{\bar{p}} \leq \|\bar{a}-\hat{y}_u\|_{\bar{p}} = 1 < \min_{y\in B-C}\|y-\hat{y}_u\|_{\bar{p}} = \inf_{b\in B}\|b-\hat{y}_u\|_{\bar{p}}.$$

This completes the proof.

□

The inequality $\inf_{a\in A}\|a-\hat{y}_l\|_p \leq \inf_{b\in B}\|b-\hat{y}_l\|_p$ in Proposition 3.1 means that the distance between $\hat{y}_l$ and the set A is less than the distance between this point and the set B. This fact is illustrated in Figure 2. It is obvious in this figure that the inclusion $B \subset A+C$ is fulfilled. In this special case, it can be simply checked in Figure 2 that $\inf_{a\in A}\|a-\hat{y}_u\|_p \geq \inf_{b\in B}\|b-\hat{y}_u\|_p$, and $A \subset B-C$. Assumption 3.1 does not require that the considered sets are convex but they should

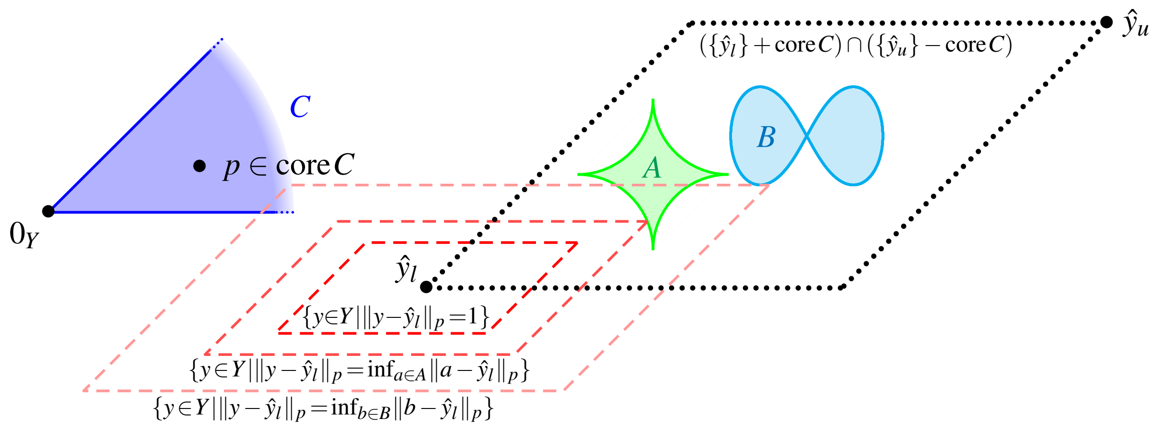

FIGURE 2. Illustration of Proposition 3.1.

have a strict lower and upper bound $\hat{y}_l$ and $\hat{y}_u$, respectively, which are also illustrated in Figure 2.

Corollary 3.1. *Let Assumption 3.1 be satisfied. For arbitrary sets $A,B\in\mathscr{F}$ with the property that the optimization problems $\min_{y\in A+C}\|y-\hat{y}_l\|_p$ and $\max_{y\in B-C}\|y-\hat{y}_u\|_p$ are solvable for all $p\in\text{core}\,C$, we have*

$$A\preccurlyeq_s B \iff \forall\, p\in\text{core}\,C:\ \inf_{a\in A}\|a-\hat{y}_l\|_p\le\inf_{b\in B}\|b-\hat{y}_l\|_p \text{ and }$$
$$\inf_{a\in A}\|a-\hat{y}_u\|_p\ge\inf_{b\in B}\|b-\hat{y}_u\|_p.$$

Proof. The assertion is a direct consequence of the definition of the set less order relation and Propositions 3.1 and 3.2. □

Notice under Assumption 3.1 that the optimization problems $\min_{y\in A+C}\|y-\hat{y}_l\|_p$ and $\min_{y\in B-C}$ $\|y-\hat{y}_u\|_p$ are solvable for all $p\in\text{core}\,C$ if real linear space Y is finite dimensional and sets $A+C$ and $B-C$ are closed.

Following the lines in [5, Def. 2.3], we now define a vector function, which uses the inf terms given in Corollary 3.1.

Definition 3.2. Let Assumption 3.1 be satisfied. Let $\mathscr{R}^2(\text{core}\,C)$ denote the space of functions on $\text{core}\,C$ with values in $\overline{\mathbb{R}}^2$. Then we define the map $v:\mathscr{F}\to\mathscr{R}^2(\text{core}\,C)$ pointwise by

$$v(A)(p):=\begin{pmatrix}\inf\limits_{a\in A}\|a-\hat{y}_l\|_p\\ -\inf\limits_{a\in A}\|a-\hat{y}_u\|_p\end{pmatrix}\ \text{for all } A\in\mathscr{F} \text{ and all } p\in\text{core}\,C.$$

With this definition, the following Proposition directly follows from Corollary 3.1.

Proposition 3.3. *Let Assumption 3.1 be satisfied. For arbitrary sets $A,B\in\mathscr{F}$, we have*

$$A\preccurlyeq_s B \iff v(A)\preccurlyeq v(B),$$

where $\preccurlyeq$ denotes the componentwise and pointwise ordering of vector functions.

This proposition is an extension of [5, Prop. 2.1] to the nonconvex case.

In order to investigate set optimization problems, we have to adapt our assumptions.

Assumption 3.2. Let Y be a real linear space, and let $C \subset Y$ be a pointed, algebraically closed, convex cone with nonempty algebraic interior. In addition, let S be a nonempty subset of a real linear space X, and let $F : S \rightrightarrows Y$ be a set-valued map with $F(x) \neq \emptyset$ for all $x \in S$. Assume that there exist $\hat{y}_l, \hat{y}_u \in Y$ with $F(x) \subset (\{\hat{y}_l\} + \operatorname{core} C) \cap (\{\hat{y}_u\} - \operatorname{core} C)$ for all $x \in S$. For an arbitrary $p \in \operatorname{core} C$, let $\|\cdot\|_p$ be given by (2.1).

Under this assumption, we investigate the set optimization problem

$$\min_{x \in S} F(x). \tag{3.5}$$

Recall that $\bar{x} \in S$ is called a *minimal* solution of set optimization problem (3.5) iff

$$F(x) \preccurlyeq_s F(\bar{x}),\ x \in S \implies F(\bar{x}) \preccurlyeq_s F(x).$$

Theorem 3.1. *Let Assumption 3.2 be satisfied and, in addition, let the optimization problems* $\min_{y \in F(x)+C} \|y - \hat{y}_l\|_p$ *and* $\max_{y \in F(x)-C} \|y - \hat{y}_u\|_p$ *be solvable for all* $x \in S$ *and all* $p \in \operatorname{core} C$. $\bar{x} \in S$ *is a minimal solution of set optimization problem (3.5) if and only if* $\bar{x} \in S$ *is a minimal solution of the vector optimization problem*

$$\min_{x \in S} v\big(F(x)\big). \tag{3.6}$$

Proof. With Proposition 3.3, we obtain

$$\begin{aligned}
& \bar{x} \in S \text{ is a minimal solution of problem (3.5)} \\
\iff\ & \big(F(x) \preccurlyeq_s F(\bar{x}),\ x \in S \implies F(\bar{x}) \preccurlyeq_s F(x)\big) \\
\iff\ & \Big(v\big(F(x)\big) \preccurlyeq v\big(F(\bar{x})\big),\ x \in S \implies v\big(F(\bar{x})\big) \preccurlyeq v\big(F(x)\big)\Big) \\
\iff\ & \Big(x \in S \text{ and } \forall\, p \in \operatorname{core} C: \inf_{y \in F(x)} \|y - \hat{y}_l\|_p \leq \inf_{y \in F(\bar{x})} \|y - \hat{y}_l\|_p, \\
& \qquad - \inf_{y \in F(x)} \|y - \hat{y}_u\|_p \leq - \inf_{y \in F(\bar{x})} \|y - \hat{y}_u\|_p \\
& \quad \text{implies } \forall\, p \in \operatorname{core} C: \inf_{y \in F(\bar{x})} \|y - \hat{y}_l\|_p \leq \inf_{y \in F(x)} \|y - \hat{y}_l\|_p, \\
& \qquad - \inf_{y \in F(\bar{x})} \|y - \hat{y}_u\|_p \leq - \inf_{y \in F(x)} \|y - \hat{y}_u\|_p\Big) \\
\iff\ & \Big(x \in S \text{ and } \forall\, p \in \operatorname{core} C: \inf_{y \in F(x)} \|y - \hat{y}_l\|_p \leq \inf_{y \in F(\bar{x})} \|y - \hat{y}_l\|_p, \\
& \qquad - \inf_{y \in F(x)} \|y - \hat{y}_u\|_p \leq - \inf_{y \in F(\bar{x})} \|y - \hat{y}_u\|_p \\
& \quad \text{implies } \forall\, p \in \operatorname{core} C: \inf_{y \in F(x)} \|y - \hat{y}_l\|_p = \inf_{y \in F(\bar{x})} \|y - \hat{y}_l\|_p, \\
& \qquad - \inf_{y \in F(x)} \|y - \hat{y}_u\|_p = - \inf_{y \in F(\bar{x})} \|y - \hat{y}_u\|_p\Big) \\
\iff\ & \bar{x} \in S \text{ is a minimal solution of problem (3.6)}
\end{aligned}$$

□

The replacement of a set optimization problem by a vector optimization problem, as done in Theorem 3.1, is called *vectorization* (see [5, p. 790]). This approach has the essential advantage that one can extend many results of vector optimization to problems of set optimization.

Although the objective functionals in the inf terms of the components of the map v are nonsmooth in general, in special cases they are Fréchet differentiable at certain points.

Proposition 3.4. *Let Y be a real linear space, let $C \subset Y$ be a pointed, algebraically closed, convex cone with nonempty algebraic interior, and let $p \in \operatorname{core} C$ and $\hat{y} \in Y$ be arbitrarily chosen. Let the norm $\|\cdot\|_p$ be given by (2.1). Then the functional $\|\cdot - \hat{y}\|_p$ is Fréchet differentiable at some $\bar{y} \in Y \backslash \{\hat{y}\}$ if there is a continuous linear functional $\ell \in Y^*$ with $\|\ell\|_{Y^*} = 1$, $\ell(\bar{y} - \hat{y}) = \|\bar{y} - \hat{y}\|_p$ and the property*

$$\forall\, \varepsilon > 0\, \exists\, \delta > 0 : \ \|\bar{y} - \hat{y} + h\|_p - \ell(\bar{y} - \hat{y} + h) \leq \varepsilon \|h\|_p \text{ for all } h \in \mathscr{B}_p(0_Y, \delta).$$

Proof. Because of $\|\ell\|_{Y^*} = 1$ and $\ell(\bar{y} - \hat{y}) = \|\bar{y} - \hat{y}\|_p$, the functional ℓ is a subgradient of the norm $\|\cdot - \hat{y}\|_p$ at $\bar{y}$ (compare [13, Example 3.24, (b)]). For every $h \in Y$, we have

$$\begin{aligned} \Big| \underbrace{\|\bar{y} - \hat{y} + h\|_p - \|\bar{y} - \hat{y}\|_p - \ell(h)}_{\geq 0 \text{ because } \ell \text{ is a subgradient}} \Big| &= \|\bar{y} - \hat{y} + h\|_p - \ell(\bar{y} - \hat{y}) - \ell(h) \\ &= \|\bar{y} - \hat{y} + h\|_p - \ell(\bar{y} - \hat{y} + h). \end{aligned}$$

Consequently, for every $\varepsilon > 0$, there is some $\delta > 0$ with

$$\begin{aligned} \frac{\big| \|\bar{y} - \hat{y} + h\|_p - \|\bar{y} - \hat{y}\|_p - \ell(h) \big|}{\|h\|_p} &= \frac{\|\bar{y} - \hat{y} + h\|_p - \ell(\bar{y} - \hat{y} + h)}{\|h\|_p} \\ &\leq \varepsilon \text{ for all } h \in \mathscr{B}_p(0_Y, \delta). \end{aligned}$$

Hence, $\|\cdot - \hat{y}\|_p$ is Fréchet differentiable at $\bar{y} \in Y$, and ℓ is the Fréchet derivative. □

Remark 3.1. Notice in Proposition 3.4 that the point $\bar{y}$ cannot be located on an edge of the ball $\mathscr{B}_p\big(\hat{y}, \|\bar{y} - \hat{y}\|_p\big)$. Fréchet differentiability at the point $\bar{y} - \hat{y} = \|\bar{y} - \hat{y}\|_p\, p$ is not possible as well.

The parameter p of the parametric norm $\|\cdot\|_p$ is chosen in the algebraic interior $\operatorname{core} C$ of C. In practice, this set is too large and one prefers a smaller set of parameters. The following proposition is a decisive key for a reduction of the set of parameters.

Proposition 3.5. *Under Assumption 3.2, we have, for all $\hat{y} \in \{\hat{y}_l, \hat{y}_u\}$, $x \in S$ and $p \in \operatorname{core} C$,*

$$\min_{y \in F(x)} \|y - \hat{y}\|_{\lambda p} = \frac{1}{\lambda} \min_{y \in F(x)} \|y - \hat{y}\|_p \text{ for all } \lambda > 0.$$

Proof. Let $\hat{y} \in \{\hat{y}_l, \hat{y}_u\}$, $x \in S$, $p \in \operatorname{core} C$, and $\lambda > 0$ be arbitrarily chosen. Then, for every $y \in Y$,

$$\begin{aligned} \|y - \hat{y}\|_{\lambda p} &= \inf\big\{\alpha > 0 \;\big|\; \tfrac{1}{\alpha}(y - \hat{y}) \in [-\lambda p, \lambda p]\big\} \\ &= \inf\big\{\alpha > 0 \;\big|\; \tfrac{1}{\alpha}(y - \hat{y}) \in \big(\{-\lambda p\} + C\big) \cap \big(\{\lambda p\} - C\big)\big\} \\ &= \inf\big\{\alpha > 0 \;\big|\; \tfrac{1}{\alpha}(y - \hat{y}) \in \lambda\big(\{-p\} + \underbrace{\tfrac{1}{\lambda}C}_{=C}\big) \cap \big(\{p\} - \underbrace{\tfrac{1}{\lambda}C}_{=C}\big)\big\} \\ &= \inf\big\{\alpha > 0 \;\big|\; \tfrac{1}{\alpha}(y - \hat{y}) \in \lambda[-p, p]\big\} \\ &= \inf\big\{\alpha > 0 \;\big|\; \tfrac{1}{\alpha}\big(\tfrac{1}{\lambda}(y - \hat{y})\big) \in [-p, p]\big\} \\ &= \|\tfrac{1}{\lambda}(y - \hat{y})\|_p \\ &= \tfrac{1}{\lambda}\|y - \hat{y}\|_p. \end{aligned}$$

This implies $\min_{y \in F(x)} \|y - \hat{y}\|_{\lambda p} = \frac{1}{\lambda} \min_{y \in F(x)} \|y - \hat{y}\|_p$, which has to be shown. □

Remark 3.2. Based on Proposition 3.5, the set $\operatorname{core} C$ of parameters can be essentially reduced if the convex cone C has a base. Recall that a nonempty convex subset B_C of C is called a *base* for C iff each $y \in C \setminus \{0_Y\}$ has a unique representation of the form

$$y = \alpha b \text{ for some } \alpha > 0 \text{ and some } b \in B_C$$

(notice that $C \neq \{0_Y\}$ holds because we assume that $\operatorname{core} C \neq \emptyset$). In the case of the existence of a base B_C for C, it makes sense to reduce the set $\operatorname{core} C$ of parameters to the set $B_C \cap \operatorname{core} C$. Then one has only to evaluate $\min_{y \in F(x)} \|y - \hat{y}\|_p$ for parameters $p \in B_C \cap \operatorname{core} C$.

4. Necessary Optimality Conditions in Finite Dimensions

The vectorization approach given in Theorem 3.1 has the advantage that necessary optimality conditions for set optimization problem (3.5) can be obtained by using known necessary optimality conditions for vector optimization problem (3.6). In general, we then have the problem that the map v is nonsmooth. But in a finite dimensional setting with the natural ordering cone, we have a richer mathematical structure, which is investigated in detail.

First, we begin with the special assumption used in this section.

Assumption 4.1. Let $Y := \mathbb{R}^m$ (with $m \in \mathbb{N}$) and $C := \mathbb{R}^m_+$ be given. Let S be a nonempty subset of $X := \mathbb{R}^n$ (with $n \in \mathbb{N}$), let $\hat{S}$ be an open superset of S, and let $F : S \rightrightarrows \mathbb{R}^m$ be a set-valued map with $F(x) \neq \emptyset$ for all $x \in S$. Let functions $g_1, \ldots, g_k, h_1, \ldots, h_q : \mathbb{R}^m \times \hat{S} \to \mathbb{R}$ (with $k, q \in \mathbb{N}$) be given with

$$F(x) \;:=\; \big\{ y \in \mathbb{R}^m \;\big|\; g_i(y,x) \leq 0 \text{ for all } i \in \{1, \ldots, k\} \text{ and } h_i(y,x) = 0 \text{ for all } i \in \{1, \ldots, q\} \big\} \text{ for all } x \in S.$$

Assume that there exist $\hat{y}_l, \hat{y}_u \in \mathbb{R}^m$ such that $F(x) \subset (\{\hat{y}_l\} + \mathbb{R}^m_{++}) \cap (\{\hat{y}_u\} - \mathbb{R}^m_{++})$ for all $x \in S$ (with $\mathbb{R}^m_{++} := \{y \in \mathbb{R}^m \mid y_i > 0 \;\forall\, i \in \{1, \ldots, m\}\}$). For an arbitrary $p \in \mathbb{R}^m_{++}$, let the parametric norm $\|\cdot\|_p$ be given by (2.1). Assume that the optimization problems $\min_{y \in F(x)} \|y - \hat{y}_l\|_p$ and $\min_{y \in F(x)} \|y - \hat{y}_u\|_p$ are solvable for all $x \in S$ and all $p \in \mathbb{R}^m_{++}$.

Since we assume that optimization problems $\min_{y \in F(x)} \|y - \hat{y}_l\|_p$ and $\min_{y \in F(x)} \|y - \hat{y}_u\|_p$ are solvable for all $x \in S$ and all $p \in \mathbb{R}^m_{++}$, we find from Lemma 3.1 that $\min_{y \in F(x) + \mathbb{R}^m_+} \|y - \hat{y}_l\|_p$ and $\min_{y \in F(x) - \mathbb{R}^m_+} \|y - \hat{y}_u\|_p$ are solvable as well.

Under Assumption 4.1, the sets $F(x)$ have a very concrete form. They are given by inequality and equality constraints. For every $x \in S$, we define the functionals $\varphi_1(x) : F(x) \to \mathscr{R}^1(\operatorname{core} C)$ and $\varphi_2(x) : F(x) \to \mathscr{R}^1(\operatorname{core} C)$ pointwise by

$$\varphi_1(x)(p) = \min_{y \in F(x)} \|y - \hat{y}_l\|_p \text{ for all } p \in \mathbb{R}^m_{++}$$

and

$$\varphi_2(x)(p) = -\min_{y \in F(x)} \|y - \hat{y}_u\|_p \text{ for all } p \in \mathbb{R}^m_{++},$$

respectively. By Proposition 2.1, (a) for every $p \in \mathbb{R}^m_{++}$ the parametric norm $\|\cdot\|_p$ is given by

$$\|y\|_p = \max_{i \in \{1, \ldots, m\}} \frac{1}{p_i} |y_i| \text{ for all } y \in \mathbb{R}^m.$$

Then, we have, for all $x \in S$ and all $p \in \mathbb{R}^m_{++}$,

$$\begin{aligned}
\varphi_1(x)(p) &= \min_{y\in F(x)} \max_{i\in\{1,\ldots,m\}} \frac{1}{p_i}(y_i - \hat{y}_{l\,i}) \\
&= \min \lambda \\
&\quad \text{subject to} \\
&\quad \tfrac{1}{p_i}(y_i - \hat{y}_{l\,i}) \le \lambda \text{ for all } i \in \{1,\ldots,m\} \\
&\quad y \in F(x),\ \lambda \in \mathbb{R} \\
&= \min \lambda \qquad (4.1) \\
&\quad \text{subject to} \\
&\quad \tfrac{1}{p_i}(y_i - \hat{y}_{l\,i}) - \lambda \le 0 \text{ for all } i \in \{1,\ldots,m\} \\
&\quad g_i(y,x) \le 0 \text{ for all } i \in \{1,\ldots,k\} \\
&\quad h_i(y,x) = 0 \text{ for all } i \in \{1,\ldots,q\} \\
&\quad (y,\lambda) \in \mathbb{R}^{m+1}.
\end{aligned}$$

This is an optimization problem with linear objective and nonlinear constraints. If $(\bar{y},\bar{\lambda})$ is a minimal solution of this optimization problem, which exists by Assumption 4.1, the functions $g_1,\ldots g_k$ are differentiable with respect to y at $\bar{y}$, the functions $h_1,\ldots h_q$ are continuously differentiable with respect to y at $\bar{y}$ and a standard constraint qualification holds, then we obtain Karush-Kuhn-Tucker (KKT) conditions as necessary optimality conditions. This means that, for every $x \in S$ and all $p_1,\ldots,p_m > 0$, there are multipliers $w_1,\ldots,w_m,u_1,\ldots,u_k \ge 0$ and $v_1,\ldots,v_q \in \mathbb{R}^m$ so that

$$\sum_{i=1}^m \frac{w_i}{p_i} e_i + \sum_{i=1}^k u_i \nabla_y g_i(\bar{y},x) + \sum_{i=1}^q v_i \nabla_y h_i(\bar{y},x) = 0_{\mathbb{R}^m},$$

$$\sum_{i=1}^m w_i = 1,$$

$$w_i\left(\frac{1}{p_i}(\bar{y}_i - \hat{y}_{l\,i}) - \bar{\lambda}\right) = 0 \text{ for all } i \in \{1,\ldots,m\}$$

and

$$u_i g_i(\bar{y},x) = 0 \text{ for all } i \in \{1,\ldots,k\},$$

where $e_1,\ldots,e_m$ denote the unit vectors in $\mathbb{R}^m$.

Under appropriate assumptions, one can evaluate the gradient of φ_1 with respect to x (see, e.g., [14, 15, 16, 17]). For this purpose, we use a result based on these KKT conditions and given by Jongen *et al.* [16].

Lemma 4.1. [16, Lemma 2.1] *Let Assumption 4.1 be satisfied, and let some $\bar{x} \in S$ and some $p \in \mathbb{R}^m_{++}$ be arbitrarily given. Let the functions g_i ($i \in \{1,\ldots,k\}$) and h_i ($i \in \{1,\ldots,q\}$) be continuously differentiable on $\mathbb{R}^m \times \hat{S}$. For all x in a neighborhood of $\bar{x}$, let $(y_{min}(x)(p),\lambda_{min}(x)(p)) \in \mathbb{R}^{m+1}$ be a KKT point of the minimization problem (4.1) with the Lagrange multipliers $u_{i_{min}}(x)(p) \ge 0$ ($i \in \{1,\ldots,k\}$) and $v_{i_{min}}(x)(\ell) \in \mathbb{R}$ ($i \in \{1,\ldots,q\}$), and let $u_{i_{min}}(\cdot)(p)$ ($i \in \{1,\ldots,p\}$) be* *continuous at $\bar{x}$ and let $(y_{min}(\cdot)(p), \lambda_{min}(\cdot)(p))$ be locally Lipschitz continuous at $\bar{x}$. Then the minimal value function $\varphi_1(\cdot)(p)$ is differentiable at $\bar{x}$ and its gradient is given by*

$$\begin{aligned}\nabla_x \varphi_1(\bar{x})(p) &= \sum_{i \in I(y_{min}(\bar{x})(p))} u_{i_{min}}(\bar{x})(p) \nabla_x g_i(y_{min}(\bar{x})(p), \bar{x}) \\ &\quad + \sum_{i=1}^{q} v_{i_{min}}(\bar{x})(p) \nabla_x h_i(y_{min}(\bar{x})(p), \bar{x}) \end{aligned} \tag{4.2}$$

(where $I(y_{min}(\bar{x})(p))$ denotes the index set of active inequality constraints at $y_{min}(\bar{x})(p)$).

Notice that only the functions g_i ($i \in I(y_{min}(\bar{x})(p))$ and h_i ($i \in \{1, \ldots, q\}$) arise in the formula (4.2) for the gradient of the minimal value functional with respect to the variable x. The objective function and the first m constraint functions of problem (4.1) do not appear in equation (4.2) because they do not depend on the variable x.

Now we turn our attention to the negative infimal value $-\inf_{y \in F(x)} \|y - \hat{y}_u\|_p$ arising in the second component of $v \circ F$. Since the aforementioned optimization problem is assumed to be solvable, we have for all $x \in S$ and all $p_1, \ldots, p_m > 0$

$$-\varphi_2(x)(p) = \min_{y \in F(x)} \max_{i \in \{1,\ldots,m\}} \frac{1}{p_i}(y_i - \hat{y}_{u_i}),$$

which can be written as problem (4.1) if one replaces $\hat{y}_l$ by $\hat{y}_u$, i.e.,

$$\begin{aligned} -\varphi_2(x)(p) = \ & \min \lambda \\ & \text{subject to} \\ & \tfrac{1}{p_i}(y_i - \hat{y}_{u_i}) - \lambda \leq 0 \text{ for all } i \in \{1, \ldots, m\} \\ & g_i(y, x) \leq 0 \text{ for all } i \in \{1, \ldots, k\} \\ & h_i(y, x) = 0 \text{ for all } i \in \{1, \ldots, q\} \\ & (y, \lambda) \in \mathbb{R}^{m+1}. \end{aligned} \tag{4.3}$$

Hence, Lemma 4.1 can immediately be applied to problem (4.3) in order to obtain a formula for the gradient of the function φ_2.

Lemma 4.2. [16, Lemma 2.1] *Let Assumption 4.1 be satisfied, and let some $\bar{x} \in S$ and some $p \in \mathbb{R}^m_{++}$ be arbitrarily given. Let the functions g_i ($i \in \{1, \ldots, k\}$) and h_i ($i \in \{1, \ldots, q\}$) be continuously differentiable on $\mathbb{R}^m \times \hat{S}$. For all x in a neighborhood of $\bar{x}$, let $(\tilde{y}_{min}(x)(p), \tilde{\lambda}_{min}(x)(p)) \in \mathbb{R}^{m+1}$ be a KKT point of the minimization problem (4.3) with the Lagrange multipliers $\tilde{u}_{i_{min}}(x)(p) \geq 0$ ($i \in \{1, \ldots, k\}$) and $\tilde{v}_{i_{min}}(x)(\ell) \in \mathbb{R}$ ($i \in \{1, \ldots, q\}$), and let $\tilde{u}_{i_{min}}(\cdot)(p)$ ($i \in \{1, \ldots, p\}$) be continuous at $\bar{x}$ and let $(\tilde{y}_{min}(\cdot)(p), \tilde{\lambda}_{min}(\cdot)(p))$ be locally Lipschitz continuous at $\bar{x}$. Then the negative minimal value function $\varphi_2(\cdot)(p)$ is differentiable at $\bar{x}$ and its gradient is given by*

$$\begin{aligned}\nabla_x \varphi_2(\bar{x})(p) &= -\sum_{i \in I(\tilde{y}_{min}(\bar{x})(p))} \tilde{u}_{i_{min}}(\bar{x})(p) \nabla_x g_i(\tilde{y}_{min}(\bar{x})(p), \bar{x}) \\ &\quad - \sum_{i=1}^{q} \tilde{v}_{i_{min}}(\bar{x})(p) \nabla_x h_i(\tilde{y}_{min}(\bar{x})(p), \bar{x}) \end{aligned} \tag{4.4}$$

(where $I(\tilde{y}_{min}(\bar{x})(p))$ denotes the index set of active inequality constraints at $\tilde{y}_{min}(\bar{x})(p)$).

With the gradients of the functions φ_1 and φ_2 given in Lemmas 4.1 and 4.2, we are then able to formulate KKT conditions as necessary optimality conditions for set optimization problem (3.5) in the special finite dimensional case.

We need some abbreviations. If $\mathscr{C}(P)$ denotes the linear space of real-valued continuous functions on a nonempty compact set $P \subset \mathbb{R}^m$, the natural ordering cone in $\mathscr{C}(P)$ is given by

$$\mathscr{C}_+(P) := \{\psi \in \mathscr{C}(P) \mid \psi(p) \geq 0 \text{ for all } p \in P\}.$$

It is well-known that the dual space $\mathscr{C}^*(P)$ can be identified with the linear space of signed finite Radon measures on P, and the dual cone $\mathscr{C}_+^*(P)$ is defined by

$$\mathscr{C}_+^*(P) := \left\{\mu \in \mathscr{C}^*(P) \;\middle|\; \int_P \psi \, d\mu \geq 0 \text{ for all } \psi \in \mathscr{C}_+(P)\right\}$$

(compare [18, 19, 20]). Under Assumption 4.1, we say that $\bar{x} \in S$ is a *locally minimal* solution of set optimization problem (3.5) iff it is a minimal solution with S being replaced by $S \cap \mathscr{B}(\bar{x}, \alpha)$ for some $\alpha > 0$.

Theorem 4.1. *Let Assumption 4.1 be satisfied, and let the constraint set S be given as*

$$S := \{x \in \mathbb{R}^n \mid \tilde{\tilde{g}}_i(x) \leq 0 \; \forall \, i \in \{1, \ldots, \tilde{\tilde{k}}\} \text{ and } \tilde{\tilde{h}}_i(x) = 0 \; \forall \, i \in \{1, \ldots, \tilde{\tilde{q}}\}\},$$

where $\tilde{\tilde{g}}_i : \mathbb{R}^n \to \mathbb{R}$ ($i \in \{1, \ldots, \tilde{\tilde{k}}\}$ with $\tilde{\tilde{k}} \in \mathbb{N}$) and $\tilde{\tilde{h}}_i : \mathbb{R}^n \to \mathbb{R}$ ($i \in \{1, \ldots, \tilde{\tilde{q}}\}$ with $\tilde{\tilde{q}} \in \mathbb{N}$) are differentiable functions. Let $\bar{x} \in S$ be a locally minimal solution of set optimization problem (3.5). Let S satisfy any constraint qualification. Let $\varepsilon > 0$ with

$$P^\varepsilon := \left\{y \in \mathbb{R}^m \;\middle|\; y_i \geq \varepsilon \text{ for all } i \in \{1, \ldots, m\} \text{ and } \sum_{1=1}^{m} y_i = 1\right\} \neq \emptyset$$

be arbitrarily chosen. For an arbitrary $p \in P^\varepsilon$, let the vector $(y_{min}(\bar{x})(p), \lambda_{min}(x)(p)) \in \mathbb{R}^{m+1}$ be a KKT point of the minimization problem (4.1) with Lagrange multipliers $u_{i_{min}}(\bar{x})(p)$ ($i \in \{1, \ldots, k\}$) and $v_{i_{min}}(\bar{x})(p)$ ($i \in \{1, \ldots, q\}$), and let the vector $(\tilde{y}_{min}(\bar{x})(p), \tilde{\lambda}_{min}(x)(p)) \in \mathbb{R}^{m+1}$ be a KKT point of the minimization problem (4.3) with Lagrange multipliers $\tilde{u}_{i_{min}}(\bar{x})(p)$ ($i \in \{1, \ldots, k\}$) and $\tilde{v}_{i_{min}}(\bar{x})(p)$ ($i \in \{1, \ldots, q\}$). Let $u_{i_{min}}(\cdot)(p)$ ($i \in \{1, \ldots, k\}$), $\tilde{u}_{i_{min}}(\cdot)(p)$ ($i \in \{1, \ldots, k\}$) be continuous at $\bar{x}$, and let $y_{min}(\cdot)(p)$, $\tilde{y}_{min}(\cdot)(p)$ be locally Lipschitz continuous at $\bar{x}$. Let the vector functions $\nabla_x \varphi_1(\bar{x})$ and $\nabla_x \varphi_2(\bar{x})$ be continuous on P^ε, and let the functions $\varphi_1(x)$ and $\varphi_2(x)$ be continuous for all $x \in \mathscr{B}(\bar{x}, \delta)$ for some $\delta > 0$. Then there exist finite Radon measures $\mu_1, \mu_2 \in \mathscr{C}_+^*(P^\varepsilon)$ with $(\mu_1, \mu_2) \neq 0_{\mathscr{C}^*(P^\varepsilon) \times \mathscr{C}^*(P^\varepsilon)}$, and multipliers $\tilde{\tilde{u}}_i \geq 0$ (for $i \in \tilde{\tilde{I}}(\bar{x})$, the index set of active inequality constraints at $\bar{x}$) and $\tilde{\tilde{v}}_i \in \mathbb{R}$ (for $i \in \{1, \ldots, \tilde{q}\}$) such that

$$\int_{P^\varepsilon} \Big(\sum_{i \in I(y_{min}(\bar{x}))} u_{i_{min}}(\bar{x}) \, \nabla_x g_i(y_{min}(\bar{x}), \bar{x}) + \sum_{i=1}^{q} v_{i_{min}}(\bar{x}) \, \nabla_x h_i(y_{min}(\bar{x}), \bar{x}) \Big) \, d\mu_1$$

$$- \int_{P^\varepsilon} \Big(\sum_{i \in I(\tilde{y}_{min}(\bar{x}))} \tilde{u}_{i_{min}}(\bar{x}) \, \nabla_x g_i(\tilde{y}_{min}(\bar{x}), \bar{x}) + \sum_{i=1}^{q} \tilde{v}_{i_{min}}(\bar{x}) \, \nabla_x h_i(\tilde{y}_{min}(\bar{x}), \bar{x}) \Big) \, d\mu_2$$

$$+ \sum_{i \in \tilde{\tilde{I}}(\bar{x})} \tilde{\tilde{u}}_i \nabla \tilde{\tilde{g}}_i(\bar{x}) + \sum_{i=1}^{\tilde{\tilde{q}}} \tilde{\tilde{v}}_i \nabla \tilde{\tilde{h}}_i(\bar{x}) = 0_{\mathbb{R}^n}.$$

Proof. Let $\bar{x} \in S$ be a locally minimal solution of set optimization problem (3.5), i.e., there is some $\alpha > 0$ so that $\bar{x}$ is a minimal solution of F on $S \cap \mathscr{B}(\bar{x}, \alpha)$. Then by Theorem 3.1, $\bar{x}$ is a minimal solution of the vector optimization problem (3.6) with S replaced by $S \cap \mathscr{B}(\bar{x}, \alpha)$. In this case, we have

$$v\big(F(x)\big)(p) = \begin{pmatrix} \min\limits_{y \in F(x)} \|y - \hat{y}_l\|_p \\ -\min\limits_{y \in F(x)} \|y - \hat{y}_u\|_p \end{pmatrix} = \begin{pmatrix} \varphi_1(x)(p) \\ \varphi_2(x)(p) \end{pmatrix} \text{ for all } x \in S \cap \mathscr{B}(\bar{x}, \alpha), p \in \mathbb{R}^m_{++}.$$

By assumption the functions $\varphi_1(x)$ and $\varphi_2(x)$ are continuous for all $x \in \mathscr{B}(\bar{x}, \delta)$. Next, we set $\eta := \min\{\alpha, \delta\} > 0$, and we obtain that $\bar{x}$ is a minimal solution of the vector optimization problem (3.6) with S replaced by $S \cap \mathscr{B}(\bar{x}, \eta)$. Since the cone $\mathbb{R}^m_+$ has a base $B_{\mathbb{R}^m_+} := \{y \in \mathbb{R}^m_+ \mid \sum_{i=1}^m y_i = 1\}$, by Remark 3.2, the parameter set $\mathbb{R}^m_{++}$ can be reduced to the set $P := B_{\mathbb{R}^m_+} \cap \mathbb{R}^m_{++} = \{y \in \mathbb{R}^m_{++} \mid \sum_{i=1}^m y_i = 1\}$. Now, we choose an arbitrary $\varepsilon > 0$, for which the set $P^\varepsilon \subset P$ is nonempty. Since, for every $x \in \mathscr{B}(\bar{x}, \eta)$, $\varphi_1(x)$ and $\varphi_2(x)$ are continuous on P, they are also continuous on the compact subset P^ε as well, i.e., $\varphi_1(x), \varphi_2(x) \in \mathscr{C}(P^\varepsilon)$ for all $x \in \mathscr{B}(\bar{x}, \eta)$. So, the product space $\mathscr{C}(P^\varepsilon) \times \mathscr{C}(P^\varepsilon)$ is the image space of the objective map with the ordering cone $\mathscr{C}_+(P^\varepsilon) \times \mathscr{C}_+(P^\varepsilon)$.

Next, we apply the Lagrange multiplier rule as given in [12, Thm. 7.4]. This means that there exist finite Radon measures $\mu_1, \mu_2 \in \mathscr{C}^*_+(P^\varepsilon)$, $(\mu_1, \mu_2) \neq 0_{\mathscr{C}^*(P^\varepsilon) \times \mathscr{C}^*(P^\varepsilon)}$, and multipliers $\tilde{\tilde{u}}_i \geq 0$ ($i \in \tilde{\tilde{I}}(\bar{x})$) and $\tilde{\tilde{v}}_i \in \mathbb{R}$ ($i \in \{1, \ldots, \tilde{\tilde{q}}\}$) with the property

$$\Bigg(\int\limits_{P^\varepsilon} \nabla_x \varphi_1(\bar{x})\, d\mu_1 + \int\limits_{P^\varepsilon} \nabla_x \varphi_2(\bar{x})\, d\mu_2 + \sum_{i \in \tilde{\tilde{I}}(\bar{x})} \tilde{\tilde{u}}_i \nabla \tilde{\tilde{g}}_i(\bar{x}) + \sum_{i=1}^{\tilde{\tilde{q}}} \tilde{\tilde{v}}_i \nabla \tilde{\tilde{h}}_i(\bar{x})\Bigg)(x - \bar{x}) \geq 0$$

$$\text{for all } x \in \mathscr{B}(\bar{x}, \eta). \tag{4.5}$$

For an arbitrary $\tilde{x} \in \mathbb{R}^n$, $\tilde{x} \neq \bar{x}$, there is some $\lambda > 0$ with $\bar{x} + \lambda\tilde{x} \in \mathscr{B}(\bar{x}, \eta)$. If we set $x := \bar{x} + \lambda\tilde{x}$ and $x := \bar{x} - \lambda\tilde{x}$ in (4.5), it follows from this inequality that

$$\int\limits_{P^\varepsilon} \nabla_x \varphi_1(\bar{x})\, d\mu_1 + \int\limits_{P^\varepsilon} \nabla_x \varphi_2(\bar{x})\, d\mu_2 + \sum_{i \in \tilde{\tilde{I}}(\bar{x})} \tilde{\tilde{u}}_i \nabla \tilde{\tilde{g}}_i(\bar{x}) + \sum_{i=1}^{\tilde{\tilde{q}}} \tilde{\tilde{v}}_i \nabla \tilde{\tilde{h}}_i(\bar{x}) = 0_{\mathbb{R}^n}.$$

If we plug in the formulas (4.2) and (4.4) for the gradients $\nabla_x \varphi_1(\bar{x})$ and $\nabla_x \varphi_2(\bar{x})$, we finally obtain the assertion. □

Remark 4.1. The original parameter set $\mathbb{R}^m_{++}$ is open. In order to obtain a compact parameter set, we reduce this set to a compact subset P^ε of the considered base of $\mathbb{R}^m_+$. Here $\varepsilon > 0$ can be chosen arbitrarily small. The advantage of this approach is that the image space of the objective map equals $\mathscr{C}(P^\varepsilon) \times \mathscr{C}(P^\varepsilon)$, and every linear functional of the dual space of $\mathscr{C}(P^\varepsilon) \times \mathscr{C}(P^\varepsilon)$ can be represented by its associated Radon measure (compare [18, Kapitel VIII, § 2, 2.19]).

Theorem 4.1 takes this point into account. Instead of the open set $P = \{y \in \mathbb{R}^m_{++} \mid \sum_{i=1}^m y_i = 1\}$ one considers the compact subset $P^\varepsilon = \{y \in \mathbb{R}^m \mid y_i \geq \varepsilon \text{ for all } i \in \{1, \ldots, m\} \text{ and } \sum_{i=1}^m y_i = 1\}$ being a "very large" compact subset of the open set P for sufficiently small $\varepsilon > 0$. In this way the result of Theorem 4.1 depends on ε. The two sets P and P^ε are illustrated in Figure 3 for $m = 2$.

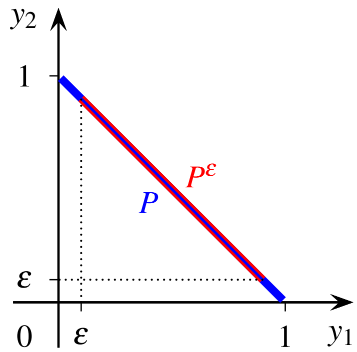

FIGURE 3. Illustration of the sets P and P^ε in Remark 4.1.

The Lagrange multipliers including the finite Radon measures are difficult to handle for concrete problems. Moreover, Theorem 4.1 formulates strong assumptions on the multipliers. Hence, it is of interest to give a multiplier-free necessary optimality condition under weaker assumptions. In this case, we do not need that the set S and the sets $F(x)$ for every $x \in S$ are explicitly defined by constraints.

Assumption 4.2. Let $Y := \mathbb{R}^m$ (with $m \in \mathbb{N}$) and $C := \mathbb{R}^m_+$ be given. Let S be a nonempty subset of $X := \mathbb{R}^n$ (with $n \in \mathbb{N}$), and let $F : S \rightrightarrows \mathbb{R}^m$ be a set-valued map with $F(x) \neq \emptyset$ for all $x \in S$. Assume that there exist $\hat{y}_l, \hat{y}_u \in \mathbb{R}^m$ such that $F(x) \subset (\{\hat{y}_l\} + \mathbb{R}^m_{++}) \cap (\{\hat{y}_u\} - \mathbb{R}^m_{++})$ for all $x \in S$. For an arbitrary $p \in \mathbb{R}^m_{++}$, let the parametric norm $\|\cdot\|_p$ be given by (2.1). Assume that the optimization problems $\min_{y \in F(x)} \|y - \hat{y}_l\|_p$ and $\min_{y \in F(x)} \|y - \hat{y}_u\|_p$ are solvable for all $x \in S$ and all $p \in \mathbb{R}^m_{++}$.

Theorem 4.2. *Let Assumption 4.2 be satisfied. Let $\bar{x} \in S$ be a minimal solution of set optimization problem (3.5). Let $\varepsilon > 0$ with $P^\varepsilon \neq \emptyset$ be arbitrarily chosen. For every $p \in P^\varepsilon$, let $\varphi_1(\cdot)(p)$ and $\varphi_2(\cdot)(p)$ be directionally differentiable at $\bar{x}$. Then, for every $x \in S$, there is some $\bar{p} \in P^\varepsilon$ with*

$$\max\left\{\varphi_1'(\bar{x}; x - \bar{x})(\bar{p}),\ \varphi_2'(\bar{x}; x - \bar{x})(\bar{p})\right\} \geq 0 \tag{4.6}$$

where $\varphi_1'(\bar{x}; x - \bar{x})(\bar{p})$ and $\varphi_2'(\bar{x}; x - \bar{x})(\bar{p})$ denote the directional derivative of $\varphi_1(\cdot)(\bar{p})$ and $\varphi_2(\cdot)(\bar{p})$, respectively, at $\bar{x}$ in the direction $x - \bar{x}$.

Proof. Let $\bar{x} \in S$ be a minimal solution of set optimization problem (3.5), and let $\varepsilon > 0$ with $P^\varepsilon \neq \emptyset$ be arbitrarily chosen. By Theorem 3.1, $\bar{x}$ is a minimal solution of vector optimization problem (3.6) as well. And by [12, Lemma 4.14], $\bar{x}$ is also a weakly minimal solution of problem (3.6), i.e., $\left[\left\{v\big(F(\bar{x})\big)\right\} - \operatorname{int}\mathscr{C}_+(P^\varepsilon)\right] \cap v\big(F(S)\big) = \emptyset$, where int denotes the topological interior of a set and $v\big(F(S)\big) := \cup_{x \in S} v\big(F(x)\big)$. By assumption, the directional derivatives $\varphi_1'(\bar{x})(\cdot)$ and $\varphi_2'(\bar{x})(\cdot)$ exist. It then follows from [12, Thm. 7.6] that

$$\begin{pmatrix} \varphi_1'(\bar{x}; x - \bar{x})(\cdot) \\ \varphi_2'(\bar{x}; x - \bar{x})(\cdot) \end{pmatrix} \notin \left[-\operatorname{int}\mathscr{C}_+(P^\varepsilon)\right] \times \left[-\operatorname{int}\mathscr{C}_+(P^\varepsilon)\right] \text{ for all } x \in S.$$

For an arbitrary $x \in S$, this means

$$\varphi_1'(\bar{x}; x - \bar{x})(\cdot) \notin -\operatorname{int}\mathscr{C}_+(P^\varepsilon)$$

or

$$\varphi_2'(\bar{x}; x - \bar{x})(\cdot) \notin -\operatorname{int}\mathscr{C}_+(P^\varepsilon).$$

Hence, there is some $\bar{p} \in P^{\varepsilon}$ with

$$\varphi_1'(\bar{x};x-\bar{x})(\bar{p}) \geq 0$$

or

$$\varphi_2'(\bar{x};x-\bar{x})(\bar{p}) \geq 0.$$

It then follows

$$\max\left\{\varphi_1'(\bar{x};x-\bar{x})(\bar{p})),\ \varphi_2'(\bar{x};x-\bar{x})(\bar{p})\right\} \geq 0,$$

which has to be shown. □

Finally, we illustrate the necessary optimality condition given in Theorem 4.2 via a very simple nonconvex example.

Example 4.1. Consider the constraint set $S := [1,3]$ and the set-valued map $F : S \rightrightarrows \mathbb{R}^2$ with

$$\begin{aligned} F(x) \quad := \quad & \left[\left(2x, \tfrac{1}{2}x^2\right), \left(\tfrac{11}{4}x, \tfrac{1}{2}x^2 + \tfrac{3}{4}x\right)\right] \\ & \setminus \left\{y \in \mathbb{R}^2 \mid 2x + \tfrac{3}{16}x < y_1 < 2x + \tfrac{9}{16}x,\ \tfrac{1}{2}x^2 + \tfrac{3}{16}x < y_2 < \tfrac{1}{2}x^2 + \tfrac{9}{16}x\right\}. \end{aligned}$$

For simplicity, only the sets $F(1)$, $F(2)$ and $F(3)$ are illustrated in Figure 4. It is evident that

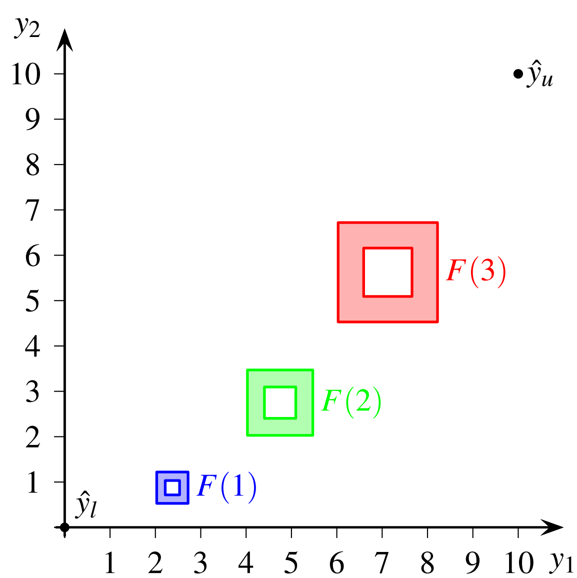

FIGURE 4. Illustration of the sets $F(1)$, $F(2)$ and $F(3)$ in Example 4.1.

$\bar{x} = 1$ is a minimal solution of the set optimization problem $\min_{x\in S} F(x)$ with $C := \mathbb{R}^2_+$. A strict lower bound and a strict upper bound of the sets $F(x)$ for $x \in S$ are chosen as $\hat{y}_l := (0,0)$ and $\hat{y}_u := (10,10)$, respectively. Because of the special form of the sets $F(x)$, φ_1 and φ_2 can be easily calculated, i.e., for all $(p_1, p_2) \in P^{\varepsilon}$ (compare Remark 4.1), we have

$$\varphi_1(x)(p_1,p_2) = \varphi_1(x)(1-p_2,p_2) = \begin{cases} \frac{x^2}{2p_2}, & \text{if } p_2 \leq \frac{x}{x+4} \\ \frac{2x}{1-p_2}, & \text{if } p_2 \geq \frac{x}{x+4} \end{cases}$$

and

$$\varphi_2(x)(p_1,p_2)=\varphi_2(x)(1-p_2,p_2)=\begin{cases}\frac{\frac{1}{2}x^2+\frac{3}{4}x-10}{p_2}, & \text{if } p_2\le\frac{\frac{1}{2}x^2+\frac{3}{4}x-10}{\frac{1}{2}x^2+\frac{7}{2}x-20}\\ \frac{\frac{11}{4}x-10}{1-p_2}, & \text{if } p_2\ge\frac{\frac{1}{2}x^2+\frac{3}{4}x-10}{\frac{1}{2}x^2+\frac{7}{2}x-20}.\end{cases}$$

At the minimal solution $\bar{x}=1$ the functions $\varphi_1(1)$ and $\varphi_2(1)$ with respect to p_2 are illustrated in Figures 5 and 6, where the parameter $\varepsilon:=0.1$ is chosen. It is evident that $\varphi_1(1)$ and $\varphi_2(1)$ are

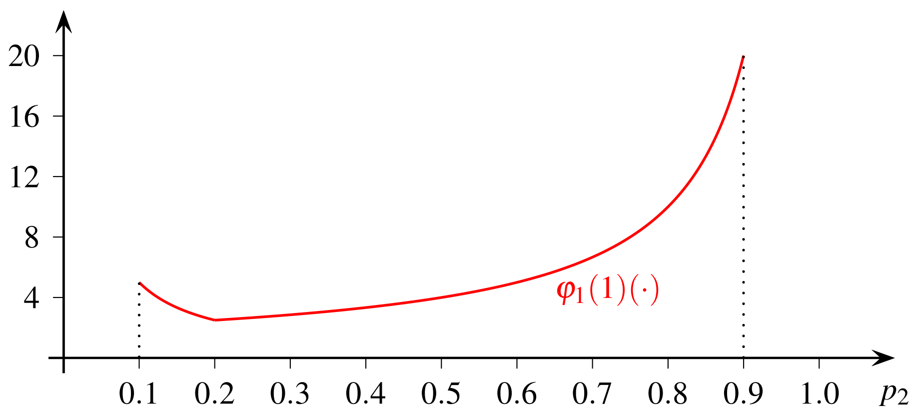

FIGURE 5. Illustration of the function $\varphi_1(1)$ in Example 4.1.

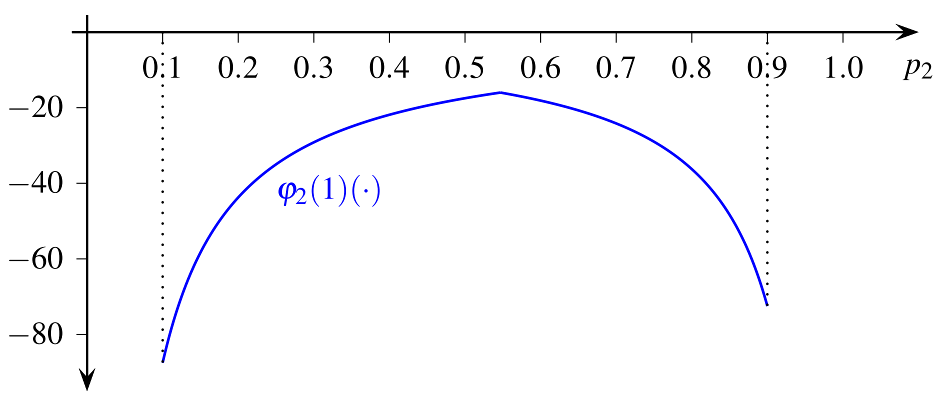

FIGURE 6. Illustration of the function $\varphi_2(1)$ in Example 4.1.

continuous on the compact interval $[0.1,0.9]$. Moreover, these functions are also directionally differentiable at 1 with the directional derivatives with respect to directions $x-1$ with arbitrary $x\in[1,3]$, i.e., for all $p_2\in[0.1,0.9]$,

$$\varphi_1'(1;x-1)(1-p_2,p_2)=\begin{cases}\frac{x-1}{p_2}, & \text{if } p_2\le\frac{1}{5}\\ \frac{2(x-1)}{1-p_2}, & \text{if } p_2>\frac{1}{5}\end{cases}$$

and

$$\varphi_2'(1;x-1)(1-p_2,p_2)=\begin{cases}\frac{7(x-1)}{4p_2}, & \text{if } p_2\le\frac{35}{64}\\ \frac{11(x-1)}{4(1-p_2)}, & \text{if } p_2>\frac{35}{64}.\end{cases}$$

These two directional derivatives with respect to the specific direction $x-1=1$ are illustrated in Figure 7. Notice that the two directional derivatives are not continuous. It is evident from

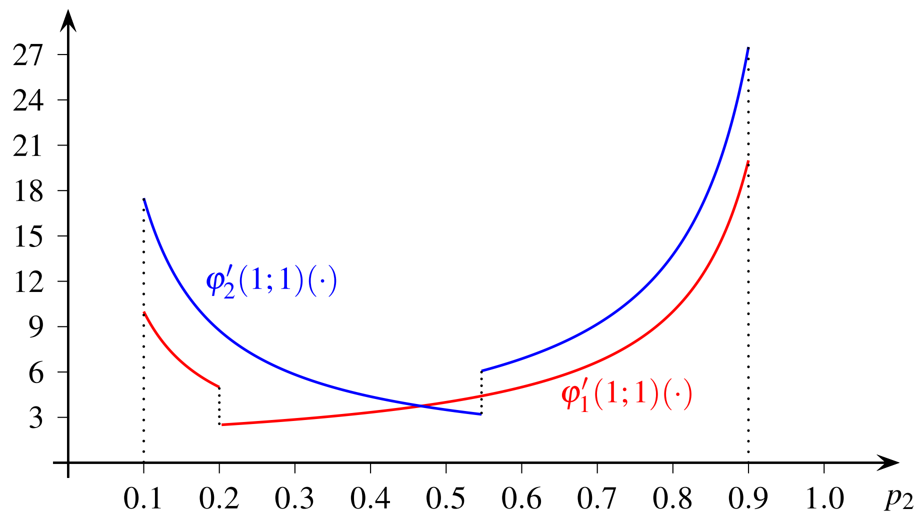

FIGURE 7. Illustration of the directional derivatives $\varphi_1'(1;1)(1-p_2,p_2)$ and $\varphi_2'(1;1)(1-p_2,p_2)$ in Example 4.1.

Figure 7 that, for every $x \in [1,3]$ and $p_2 \in [0.1,0.9]$,

$$\varphi_1'(1;x-1)(1-p_2,p_2) = (x-1)\varphi_1'(1;1)(1-p_2,p_2) \geq 0$$

and

$$\varphi_2'(1;x-1)(1-p_2,p_2) = (x-1)\varphi_2'(1;1)(1-p_2,p_2) \geq 0.$$

Hence, necessary optimality condition (4.6) is fulfilled.

CONCLUSION

Vectorization is a powerful tool for the investigation of set optimization problems. The main advantages are the derivation of optimality conditions for nonconvex set optimization problems, like Karush-Kuhn-Tucker conditions and multiplier-free optimality conditions in a finite dimensional setting. The used subproblems are parameterized approximation problems of a well investigated branch of applied mathematics. Since these subproblems have a rich mathematical structure, this vectorization approach has the potential for further results in set optimization.

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