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RELAXED LAGRANGIAN DUALITY IN CONVEX INFINITE OPTIMIZATION: REVERSE STRONG DUALITY AND OPTIMALITY

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Abstract. We associate with each convex optimization problem posed on some locally convex space with an infinite index set T, and a given non-empty family $\mathscr{H}$ formed by finite subsets of T, a suitable Lagrangian-Haar dual problem. We provide reverse $\mathscr{H}$-strong duality theorems, $\mathscr{H}$-Farkas type lemmas, and optimality theorems. Special attention is addressed to infinite and semi-infinite linear optimization problems.

Keywords. Convex infinite programming; Haar duality; Lagrangian duality; Optimality.

1. INTRODUCTION

In a recent paper on convex infinite optimization [1], we provided reducibility, zero duality gaps, and strong duality theorems for a new type of Lagrangian-Haar duality associated with families of finite sets of indices. More precisely, given an optimization problem

$$(\mathrm{P}) \qquad \inf f(x) \quad \text{s.t. } f_t(x) \leq 0, \ t \in T, \tag{1.1}$$

such that X is a locally convex Hausdorff topological vector space, T is an arbitrary infinite index set, and $\{f;\ f_t, t \in T\}$ are convex proper functions on X, as well as a family $\mathscr{H}$ of non-empty finite subsets of the index set T, we consider the $\mathscr{H}$-dual problem

$$(\mathrm{D}_{\mathscr{H}}) \qquad \sup_{H \in \mathscr{H},\ \mu \in \mathbb{R}_+^H} \inf_{x \in X} \left\{ f(x) + \sum_{t \in H} \mu_t f_t(x) \right\}, \tag{1.2}$$

where $\mu \in \mathbb{R}_+^H$ stands for $(\mu_t)_{t \in H} \in \mathbb{R}_+^H$, with the rule $0 \times (+\infty) = 0$. When $\mathscr{H}$ is the family $\mathscr{F}(T)$ of all non-empty finite subsets of T, one obtains the standard Lagrangian-Haar dual of (P),

$$(\mathrm{D}) \qquad \sup_{H \in \mathscr{F}(T),\ \mu \in \mathbb{R}_+^H} \inf_{x \in X} \left\{ f(x) + \sum_{t \in H} \mu_t f_t(x) \right\}. \tag{1.3}$$

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As in [1], this paper pays particular attention to the families $\mathscr{H}_1 := \{\{t\},\ t \in T\}$ of singletons and (when $T = \mathbb{N}$) $\mathscr{H}_{\mathbb{N}} := \{\{1,\ldots,m\},\ m \in \mathbb{N}\}$ of sets of initial natural numbers. The dual pair $(\mathrm{P}) - (\mathrm{D}_{\mathscr{H}_{\mathbb{N}}})$ has been used in [2] in the framework of convex semi-infinite programming (CSIP), where $X = \mathbb{R}^n$. More precisely, [2] gives a sufficient condition for the optimal value of a SIP problem (P) with $T = \mathbb{N}$ to be the limit, as $m \longrightarrow \infty$, of the optimal values of the sequence of ordinary convex programs $(\mathrm{P}_m)_{m\in\mathbb{N}}$ which results of replacing T by $\{1,\ldots,m\}$ in (P). This assumption on T is not as strong as it can seem at first sight as, if T is an uncountable topological space which contains a countable dense subset S and the mapping $t \longmapsto f_t(x)$ is continuous on T for any $x \in \bigcap_{t\in T} \operatorname{dom} f_t$, then (P) is equivalent to the countable subproblem which results of replacing T by S in (P). In the particular case of linear semi-infinite programming (LSIP), we can write

$$(\mathrm{P}) \qquad \inf \langle c^*, x\rangle \quad \text{s.t.} \quad \langle a_t^*, x\rangle \le b_t, \ \ t \in T, \tag{1.4}$$

with $\{c^*;\ a_t^*, t \in T\} \subset \mathbb{R}^n$ and $\{b_t, t \in T\} \subset \mathbb{R}$, where, in most applications, T is a convex body (i.e., a compact convex set with non-empty interior) in some Euclidean space and the mapping $t \longmapsto (a_t^*, b_t)$ is continuous on T. Then, T can be replaced by any dense subset S to get an equivalent countable LSIP problem.

There exists a wide literature on the dual pair (P)-(D); see e.g., the works [3, 4, 5, 6, 7, 8, 9]. Most of them focused on constraint qualifications and/or duality theorems, and some of them made use, in order to obtain optimality conditions, of suitable versions of the celebrated Farkas' Lemma that have been reviewed in [10].

The duality theorems for the pair (P)-($\mathrm{D}_{\mathscr{H}}$) provide conditions guaranteeing a zero duality gap, i.e., $\inf(\mathrm{P}) = \sup(\mathrm{D}_{\mathscr{H}})$ (see, [1, Theorem 6.1]). Other duality theorems in [1] are strong in the sense that the optimal value of ($\mathrm{D}_{\mathscr{H}}$) is attained, situation represented by the equation $\inf(\mathrm{P}) = \max(\mathrm{D}_{\mathscr{H}})$ (see, [1, Theorems 5.1-5.3]). Similarly, the reverse duality theorems, in the third section of this paper, are duality theorems where the optimal value of (P) is attained, situation represented by the equation $\min(\mathrm{P}) = \sup(\mathrm{D}_{\mathscr{H}})$. Reverse (also called converse) duality theorems for the classical Lagrange dual problem, that is, for $\mathscr{H} = \mathscr{F}(T)$, in convex infinite programming (CIP in short), can be found in [6, Theorem 3.3] and [7, Theorem 3]. Section 4 provides *ad hoc* Farkas-type results oriented to obtain, in Section 5, optimality conditions which are expressed in terms of multipliers associated to the indices belonging to the elements of $\mathscr{H}$.

2. Preliminaries

Let X be a locally convex Hausdorff topological vector space, and suppose that its topological dual X^*, with null element 0_{X^*}, is endowed with the weak*-topology. We denote by $\overline{A}$ and $\operatorname{ri} A$ the closure and the relative interior of a set $A \subset X$, and by $\operatorname{co} A$ its convex hull. For a set $\emptyset \neq A \subset X$, by the convex cone generated by A we mean $\operatorname{cone} A := \mathbb{R}_+(\operatorname{co} A) = \{\mu x : \mu \in \mathbb{R}_+,\ x \in \operatorname{co} A\}$, by $\operatorname{span} A$ its linear span, and by A_∞ the recession cone of a convex set A. The negative polar of $\emptyset \neq A \subset X$ is the convex cone $A^- := \{x^* \in X^* : \langle x^*, x\rangle \le 0, \forall x \in A\}$. The lineality space of a convex cone $K \subset X$ is $\operatorname{lin} K = K \cap (-K)$.

The w^*-closure of a set $\mathbb{A} \subset X^*$ is also denoted by $\overline{\mathbb{A}}$. If $\mathbb{A} \subset X^* \times \overline{\mathbb{R}}$, then $\overline{\mathbb{A}}$ denotes the closure of $\mathbb{A}$ w.r.t. the product topology. A set $\mathbb{A} \subset X^* \times \mathbb{R}$ is said to be w^*-*closed* (respectively, w^*-closed convex) regarding another subset $\mathbb{B} \subset X^* \times \mathbb{R}$ if $\bar{\mathbb{A}} \cap \mathbb{B} = \mathbb{A} \cap \mathbb{B}$ (respectively, $(\overline{\operatorname{co}}\mathbb{A}) \cap \mathbb{B} = \mathbb{A} \cap \mathbb{B}$), see [11] (respectively, [12]).

A function $h : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ is proper if its epigraph $\operatorname{epi} h$ is non-empty and never takes the value $-\infty$; it is convex if $\operatorname{epi} h$ is convex; it is lower semicontinuous (lsc, in brief) if $\operatorname{epi} h$ is closed; and it is upper semicontinuous (usc, in brief) if $-h$ is lsc. For a proper function h, we denote by $[h \leq 0] := \{x \in X : h(x) \leq 0\}$ its lower level set of 0, and by $\operatorname{dom} h$, $\overline{h}$, ∂h, and h^* its domain, its lsc envelope, its Fenchel subdifferential, and its Legendre-Fenchel conjugate, respectively. We also denote by $\Gamma(X)$ the class of lsc proper convex functions on X. By δ_A we denote the indicator function of $A \subset X$, with $\delta_A \in \Gamma(X)$ whenever $A \neq \emptyset$ is closed and convex.

We need to recall some basic facts about convex analysis recession. Given $h \in \Gamma(X)$, the recession cone of the closed convex set $\operatorname{epi} h$ is the epigraph of the so-called *recession function* h_∞ of h: $(\operatorname{epi} h)_\infty = \operatorname{epi} h_\infty$. The recession function h_∞ coincides with the support function of the domain of the conjugate h^* of h (e.g., [13, Theorem 6.8.5]):

$$h_\infty = (\delta_{\operatorname{dom} h^*})^*. \tag{2.1}$$

From (2.1),

$$[h_\infty \leq 0] = (\operatorname{dom} h^*)^- = \{x \in X : \langle x^*, x\rangle \leq 0, \forall x^* \in \operatorname{dom} h^*\}, \tag{2.2}$$

which is called the *recession cone* of the function h and provides the common recession cone to all the non-empty sublevel sets $[h \leq r]$. Given $\{h_1, \cdots, h_m\} \subset \Gamma(X)$ such that $\bigcap_{1\leq k\leq m} \operatorname{dom} h_k \neq \emptyset$, by [14, Proposition 3.2.3] (whose proof is independent of the dimension of X), one has for all $\mu \in \mathbb{R}^m_+$:

$$\left(\sum_{k=1}^{m} \mu_k h_k\right)_\infty = \sum_{k=1}^{m} \mu_k (h_k)_\infty. \tag{2.3}$$

2.1. **Classical Lagrange CIP duality.** The *support* of $\lambda : T \to \mathbb{R}$ is the set $\operatorname{supp} \lambda := \{t \in T : \lambda_t \neq 0\}$. Let $\mathbb{R}^{(T)}$ be the *space of generalized finite sequences* formed by all real-valued functions on T with finite support, i.e.,

$$\mathbb{R}^{(T)} := \{\lambda : T \to \mathbb{R}_+ \text{ such that } \operatorname{supp} \lambda \text{ is finite}\},$$

with positive cone $\mathbb{R}^{(T)}_+ := \{\lambda \in \mathbb{R}^{(T)} : \lambda_t \geq 0, \forall t \in T\}$. We can associate to each $\lambda \in \mathbb{R}^{(T)}_+$ the function $\sum_{t\in T} \lambda_t f_t :\ X \to \mathbb{R} \cup \{+\infty\}$ such that

$$\left(\sum_{t\in T} \lambda_t f_t\right)(x) = \begin{cases} \sum\limits_{t\in \operatorname{supp}\lambda} \lambda_t f_t(x), & \text{if } \operatorname{supp}\lambda \neq \emptyset, \\ 0, & \text{if } \operatorname{supp}\lambda = \emptyset. \end{cases}$$

So, we can reformulate (D) in (1.3) as

$$\text{(D)} \qquad \sup_{\lambda\in\mathbb{R}^{(T)}_+} \inf_{x\in X} \left\{ f(x) + \left(\sum_{t\in T} \lambda_t f_t\right)(x) \right\}.$$

It is known that the function $\varphi : X^* \to \overline{\mathbb{R}}$ such that

$$\varphi(x^*) := \inf_{\lambda\in\mathbb{R}^{(T)}_+} \left(f + \sum_{t\in T} \lambda_t f_t\right)^* (x^*)$$

and the set

$$\mathscr{A} := \bigcup_{\lambda\in\mathbb{R}^{(T)}_+} \operatorname{epi} \left(f + \sum_{t\in T} \lambda_t f_t\right)^* \subset X^* \times \mathbb{R}$$

are both convex, and $\operatorname{epi}\overline{\varphi} = \overline{\mathscr{A}}$ (see, e.g., [1, 6, 7]).

We denote the feasible set of (P) by

$$E := \bigcap_{t\in T} [f_t \leq 0].$$

Then,

$$-\infty \leq (f+\delta_E)^*(x^*) \leq \varphi(x^*) \leq f^*(x^*) \leq +\infty, \ \forall x^* \in X^*.$$

Taking $x^* = 0_{X^*}$, one obtains the weak duality for the pair $(\mathrm{P})-(\mathrm{D})$:

$$-\infty \leq \inf_X f \leq \sup(\mathrm{D}) \leq \inf(\mathrm{P}) \leq +\infty.$$

2.2. **Relaxed Lagrange CIP duality.** Let $\mathscr{H}$ be a non-empty family of non-empty finite subsets of T, that is, $\emptyset \neq \mathscr{H} \subset \mathscr{F}(T)$, with associated dual problem $(\mathrm{D}_{\mathscr{H}})$ as in (1.2). Obviously,

$$\sup(\mathrm{D}_{\mathscr{H}}) \leq \sup(\mathrm{D}_{\mathscr{F}(T)}) = \sup(\mathrm{D}) \leq \inf(\mathrm{P}). \tag{2.4}$$

Let us define the sets

$$E_{\mathscr{H}} := \bigcap_{H\in\mathscr{H}, t\in H} [f_t \leq 0],$$

$$\mathscr{A}_{\mathscr{H}} := \bigcup_{H\in\mathscr{H}, \mu\in\mathbb{R}_+^H} \operatorname{epi}\left(f + \sum_{t\in H} \mu_t f_t\right)^*,$$

and the function $\varphi_{\mathscr{H}} : X^* \to \mathbb{R}$ such that

$$\varphi_{\mathscr{H}} := \inf_{H\in\mathscr{H}, \mu\in\mathbb{R}_+^H} \left(f + \sum_{t\in H} \mu_t f_t\right)^*.$$

Obviously, $\mathscr{A}_{\mathscr{H}} \subset \mathscr{A}$ and $\varphi_{\mathscr{H}} \geq \varphi$.

Definition 2.1. (i) A family $\mathscr{H} \subset \mathscr{F}(T)$ is said to be *covering* if $\bigcup_{H\in\mathscr{H}} H = T$.
(ii) A family $\mathscr{H} \subset \mathscr{F}(T)$ is said to be *directed* if, for each $H, K \in \mathscr{H}$, there exists $L \in \mathscr{H}$ such that $H \cup K \subset L$.

The families $\mathscr{F}(T)$ and $\mathscr{H}_{\mathbb{N}}$ are both covering and directed families, whereas $\mathscr{H}_1$ is just covering.

As shown in [1, Proposition 3.2], for each directed covering family $\mathscr{H} \subset \mathscr{F}(T)$, one has

$$\mathscr{A}_{\mathscr{H}} = \mathscr{A}_{\mathscr{F}(T)} = \mathscr{A}. \tag{2.5}$$

Consequently,

$$\varphi_{\mathscr{H}} = \varphi_{\mathscr{F}(T)} = \varphi, \text{ and } \sup(\mathrm{D}_{\mathscr{H}}) = \sup(\mathrm{D}_{\mathscr{F}(T)}) \equiv \sup(\mathrm{D}). \tag{2.6}$$

Let $\mathscr{H} \subset \mathscr{F}(T)$ be a covering family. Then, $E_{\mathscr{H}} = E$ and, according to [1, Lemma 5.2], $\{f;\ f_t, t\in T\} \subset \Gamma(X)$ entails

$$(\varphi_{\mathscr{H}})^* = f + \delta_E, \tag{2.7}$$

and if, additionnally, $E \cap (\operatorname{dom} f) \neq \emptyset$, then

$$\operatorname{epi}(f+\delta_E)^* = \overline{\operatorname{co}}\mathscr{A}_{\mathscr{H}} = \overline{\operatorname{co}}\left(\bigcup_{H\in\mathscr{H}, \mu\in\mathbb{R}_+^H} \operatorname{epi}\left(f + \sum_{t\in H} \mu_t f_t\right)^*\right).$$

Moreover, by [1, Theorem 5.1], $\mathscr{H}$-strong duality holds at a given $x^* \in X^*$, i.e.,

$$(f+\delta_E)^*(x^*) = \min_{H\in\mathscr{H},\, \mu\in\mathbb{R}_+^H} \left(f + \sum_{t\in H} \mu_t f_t \right)^* (x^*), \tag{2.8}$$

if and only if $\mathscr{A}_{\mathscr{H}}$ is w^*-closed convex regarding $\{x^*\}\times\mathbb{R}$.

2.3. **The $\mathscr{H}$-dual problem as a limit.** It is easy to see that the mapping $\mathscr{F}(T) \supset \mathscr{H} \longmapsto \sup(\mathrm{D}_{\mathscr{H}}) \in \overline{\mathbb{R}}$ is non-decreasing w.r.t. the inclusion $\subset$ in $\mathscr{F}(T)$. Consequently, if the family $\mathscr{H} \subset \mathscr{F}(T)$ is directed, we can express $\sup(\mathrm{D}_{\mathscr{H}})$ as the limit of a net as follows:

$$\sup(\mathrm{D}_{\mathscr{H}}) = \sup_{H\in\mathscr{H}} \sup(\mathrm{D}_H) = \lim_{H\in\mathscr{H}} \sup(\mathrm{D}_H).$$

If, moreover, $\mathscr{H}$ is covering, then

$$\sup(\mathrm{D}) = \lim_{H\in\mathscr{H}} \sup(\mathrm{D}_H). \tag{2.9}$$

In particular, if $T=\mathbb{N}$, we consider the countable program

$$(\mathrm{P}_{\mathbb{N}}) \quad \inf f(x) \text{ s.t. } f_k(x) \le 0, k\in\mathbb{N}, \tag{2.10}$$

and the sequence of finite subproblems

$$(\mathrm{P}_m) \quad \inf f(x) \text{ s.t. } f_k(x) \le 0,\ k\in\{1,\cdots,m\},\ m\in\mathbb{N}, \tag{2.11}$$

whose ordinary Lagrangian dual problems are

$$(\mathrm{D}_m) \qquad \sup_{\mu\in\mathbb{R}_+^m} \inf_{x\in X} \left\{ f(x) + \sum_{k=1}^m \mu_k f_k(x) \right\},\ m\in\mathbb{N}. \tag{2.12}$$

From (2.9), the Lagrangian-Haar dual of $(\mathrm{P}_{\mathbb{N}})$,

$$(\mathrm{D}_{\mathbb{N}}) \qquad \sup_{\lambda\in\mathbb{R}_+^{(\mathbb{N})}} \inf_{x\in X} \left\{ f(x) + \sum_{k\in\mathbb{N}} \lambda_k f_k(x) \right\}, \tag{2.13}$$

and its $\mathscr{H}_{\mathbb{N}}$-dual Lagrange problem $(\mathrm{D}_{\mathscr{H}_{\mathbb{N}}})$ can be expressed as limits in this way:

$$\sup(\mathrm{D}_{\mathbb{N}}) = \sup(\mathrm{D}_{\mathscr{H}_{\mathbb{N}}}) = \lim_{m\to\infty} \sup(\mathrm{D}_m). \tag{2.14}$$

Corollary 3.3 below provides a sufficient condition for the primal counterpart of (2.14):

$$\inf(\mathrm{P}_{\mathbb{N}}) = \lim_{m\to\infty} \inf(\mathrm{P}_m).$$

3. $\mathscr{H}$-REVERSE STRONG DUALITY

Let us go back to the general convex infinite optimization problem (P) in (1.1). Along this section, we assume that $\{f;\ f_t, t\in T\} \subset \Gamma(X)$ and $E\cap \operatorname{dom} f \neq \emptyset$, meaning that $\inf(\mathrm{P}) \neq +\infty$.

Definition 3.1. Given a covering family $\mathscr{H}\subset\mathscr{F}(T)$, we say that *$\mathscr{H}$-reverse strong duality* holds if

$$\min(\mathrm{P}) = \sup(\mathrm{D}_{\mathscr{H}}),$$

equivalently, that there exists $\bar{x}\in E\cap\operatorname{dom} f$ such that

$$f(\bar{x}) = \sup(\mathrm{D}_{\mathscr{H}}) \in \mathbb{R}.$$

We first show that $\mathscr{H}$-reverse strong duality can be described in terms of subdifferentiability of the function $\varphi_{\mathscr{H}}$.

Recall that the subdifferential of a function $g : X^* \to \overline{\mathbb{R}}$ at a point $a^* \in X^*$ is given by

$$\partial g(a^*) := \begin{cases} \{x \in X \, : \, g(x^*) \geq g(a^*) + \langle x^* - a^*, x\rangle \, , \forall x^* \in X^*\} \, , & \text{if } g(a^*) \in \mathbb{R}, \\ \emptyset, & \text{if } g(a^*) \notin \mathbb{R}. \end{cases}$$

We have

$$x \in \partial g(a^*) \Leftrightarrow g(a^*) + g^*(x) = \langle a^*, x\rangle \, . \tag{3.1}$$

Lemma 3.1. *Let $\mathscr{H}$ be a covering family. Then, $\mathscr{H}$-reverse strong duality holds if and only if $\varphi_{\mathscr{H}}$ is subdifferentiable at 0_{X^*}. In such a case, one has $\partial\varphi_{\mathscr{H}}(0_{X^*}) = \mathrm{sol(P)}$, where $\mathrm{sol(P)}$ is the optimal solution set of* (P).

Proof. Let $x \in \partial\varphi_{\mathscr{H}}(0_{X^*})$. Since we are assuming that $\mathscr{H}$ is covering, we conclude from (2.7) and (3.1) that

$$(f + \delta_E)(x) = (\varphi_{\mathscr{H}})^*(x) = -\varphi_{\mathscr{H}}(0_{X^*}) \in \mathbb{R}.$$

Then $x \in E$ and

$$\inf(\mathrm{P}) \leq f(x) = -\varphi_{\mathscr{H}}(0_{X^*}) = \sup(\mathrm{D}_{\mathscr{H}}) \leq \inf(\mathrm{P}).$$

Consequently, if $\varphi_{\mathscr{H}}$ is subdifferentiable at 0_{X^*}, then $\mathscr{H}$-reverse strong duality holds and $\partial\varphi_{\mathscr{H}}(0_{X^*}) \subset \mathrm{sol(P)}$.

Assume now that $\mathscr{H}$-reverse strong duality holds. There exists $x \in E \cap (\mathrm{dom}\, f)$ such that

$$(\varphi_{\mathscr{H}})^*(x) = f(x) = \sup(\mathrm{D}_{\mathscr{H}}) = -\varphi_{\mathscr{H}}(0_{X^*}) \in \mathbb{R}, \tag{3.2}$$

that means $x \in \partial\varphi_{\mathscr{H}}(0_{X^*})$ and the first part of Lemma 3.1 is proved with, in addition, the inclusion $\partial\varphi_{\mathscr{H}}(0_{X^*}) \subset \mathrm{sol(P)}$. It remains to prove that if $\mathscr{H}$-reverse strong duality holds, then $\mathrm{sol(P)} \subset \partial\varphi_{\mathscr{H}}(0_{X^*})$. Now, for each $x \in \mathrm{sol(P)}$, we have (3.2). So, $\varphi_{\mathscr{H}}(0_{X^*}) + (\varphi_{\mathscr{H}})^*(x) = 0$, that means $x \in \partial\varphi_{\mathscr{H}}(0_{X^*})$. □

In favorable circumstances, we know that $\varphi_{\mathscr{H}}$ is a convex function. For instance, when the covering family $\mathscr{H}$ is also directed, by (2.5) and (2.6), $\mathscr{A}_{\mathscr{H}} = \mathscr{A}$ and $\varphi_{\mathscr{H}} = \varphi$, respectively, implying the convexity of both $\mathscr{A}_{\mathscr{H}}$ and $\varphi_{\mathscr{H}}$. Another important example is furnished by

$$\varphi_{\mathscr{H}_1} = \inf_{(t,\mu) \in T \times \mathbb{R}_+} (f + \mu f_t)^*,$$

which is convex under the assumptions (a), (b), (c) of Corollary 3.1 below (see [1, Remark 5.5]). In order to propose a tractable subdifferentiability criterion when $\varphi_{\mathscr{H}}$ is convex, we need to recall some facts about quasicontinuous convex functions and convex analysis recession.

Definition 3.2. A convex function $g : X^* \to \overline{\mathbb{R}}$ is said to be $\tau(X^*, X)$-quasicontinuous ([15], [16]), where τ is the Mackey topology on X^*, if the following four properties are satisfied:

(1) $\mathrm{aff}(\mathrm{dom}\, g)$ is $\tau(X^*, X)$-closed (or $\sigma(X^*, X)$-closed),
(2) $\mathrm{aff}(\mathrm{dom}\, g)$ is of finite codimension,
(3) the $\tau(X^*, X)$-relative interior of $\mathrm{dom}\, g$, say $\mathrm{ri}(\mathrm{dom}\, g)$, is non-empty,
(4) the restriction of g to $\mathrm{aff}(\mathrm{dom}\, g)$ is $\tau(X^*, X)$-continuous on $\mathrm{ri}(\mathrm{dom}\, g)$.

Lemmas 3.2, 3.3, 3.4 below will be used in the sequel.

Lemma 3.2 ([15, Proposition 5.4]). *Let $h \in \Gamma(X)$. The conjugate function h^* is $\tau(X^*,X)$-quasicontinuous if and only if h is weakly inf-locally compact; that is to say $[h \leq r]$ is weakly locally compact for each $r \in \mathbb{R}$.*

Lemma 3.3 ([17, Theorem II.4]). *A convex function $g : X^* \to \overline{\mathbb{R}}$ majorized by a $\tau(X^*,X)$-quasicontinuous one is $\tau(X^*,X)$-quasicontinuous, too.*

Lemma 3.4 ([17, Theorem III.3]). *Let $g : X^* \to \overline{\mathbb{R}}$ be a $\tau(X^*,X)$-quasicontinuous convex function such that $g(0_{X^*}) \neq -\infty$ and $\overline{\operatorname{cone}} \operatorname{dom} g$ is a linear subspace of X^*. Then $\partial g(0_{X^*})$ is the sum of a non-empty weakly compact convex set and a finite dimensional linear subspace of X.*

We define the recession cone of (P) by setting

$$(\mathrm{P})_\infty := \bigcap_{t \in T} [(f_t)_\infty \leq 0] \cap [f_\infty \leq 0].$$

For the theorem and the corollaries below, recall that $\inf(\mathrm{P}) \neq +\infty$ as $E \cap \operatorname{dom} f \neq \emptyset$.

Theorem 3.1 ($\mathscr{H}$-reverse strong duality). *Let $\mathscr{H}$ be a covering family such that $\varphi_{\mathscr{H}}$ is convex $\tau(X^*,X)$-quasicontinuous and $(\mathrm{P})_\infty$ is a linear subspace of X. Then $\mathscr{H}$-reverse strong duality holds:*

$$\min(\mathrm{P}) = \sup(\mathrm{D}_{\mathscr{H}}) \in \mathbb{R}.$$

Moreover, $\operatorname{sol}(\mathrm{P})$ *is the sum of a weakly compact convex set and a finite dimensional linear subspace of X.*

Proof. One has

$$\varphi_{\mathscr{H}}(0_{X^*}) = -\sup(\mathrm{D}_{\mathscr{H}}) \geq -\inf(\mathrm{P}) > -\infty$$

(the last strict inequality holds as $E \cap \operatorname{dom} f \neq \emptyset$). In order to apply Lemma 3.4 to the convex function $\varphi_{\mathscr{H}}$, we have to prove that $\overline{\operatorname{cone}} \operatorname{dom} \varphi_{\mathscr{H}}$ is a linear subspace. We have

$$\begin{aligned}\overline{\operatorname{cone}} \operatorname{dom} \varphi_{\mathscr{H}} &= (\operatorname{dom} \varphi_{\mathscr{H}})^{--} \\ &= \{x^* \in X^* : \langle x^*, x \rangle \leq 0, \forall x \in (\operatorname{dom} \varphi_{\mathscr{H}})^{-}\}.\end{aligned}$$

Therefore, $\overline{\operatorname{cone}} \operatorname{dom} \varphi_{\mathscr{H}}$ is a linear subspace if and only if $(\operatorname{dom} \varphi_{\mathscr{H}})^{-}$ is a linear subspace. Now,

$$\operatorname{dom} \varphi_{\mathscr{H}} = \bigcup_{H \in \mathscr{H}} \bigcup_{\mu \in \mathbb{R}_+^H} \operatorname{dom} \left(f + \sum_{t \in H} \mu_t f_t \right)^*$$

and we can write

$$
\begin{aligned}
(\operatorname{dom}\varphi_{\mathscr{H}})^- &= \bigcap_{H\in\mathscr{H}}\bigcap_{\mu\in\mathbb{R}_+^H}\left(\operatorname{dom}\left(f+\sum_{t\in H}\mu_t f_t\right)^*\right)^- \\
&= \bigcap_{H\in\mathscr{H}}\bigcap_{\mu\in\mathbb{R}_+^H}\left[\left(f+\sum_{t\in H}\mu_t f_t\right)_\infty \le 0\right] \text{ (by (2.2))} \\
&= \bigcap_{H\in\mathscr{H}}\bigcap_{\mu\in\mathbb{R}_+^H}\left[\left(f_\infty+\sum_{t\in H}\mu_t (f_t)_\infty\right) \le 0\right] \text{ (by (2.3))} \\
&= \bigcap_{H\in\mathscr{H}}\left[\left(\sup_{\mu\in\mathbb{R}_+^H}\left(f_\infty+\sum_{t\in H}\mu_t (f_t)_\infty\right)\right) \le 0\right] \\
&= \bigcap_{H\in\mathscr{H}}\left[\left(f_\infty+\sup_{\mu\in\mathbb{R}_+^H}\sum_{t\in H}\mu_t (f_t)_\infty\right) \le 0\right] \\
&= \bigcap_{H\in\mathscr{H}}\left[\left(f_\infty+\delta_{[\sup_{t\in H}(f_t)_\infty\le 0]}\right) \le 0\right] \\
&= \bigcap_{H\in\mathscr{H}}\bigcap_{t\in H}[(f_t)_\infty\le 0]\cap[f_\infty\le 0] \\
&= \bigcap_{t\in T}[(f_t)_\infty\le 0]\cap[f_\infty\le 0] = (\mathrm{P})_\infty,
\end{aligned}
$$

the penultimate equality coming from the fact that $\mathscr{H}$ is covering. We conclude the proof of Theorem 3.1 with Lemmas 3.1 and 3.4. □

Remark 3.1. Note that if $X = X^* = \mathbb{R}^n$, then the function $\varphi_{\mathscr{H}}$, when convex, is automatically $\tau(X^*,X)$-quasicontinuous since any extended real-valued convex function on $\mathbb{R}^n$ with non-empty domain is quasicontinuous (e.g., [18, Theorem 10.1]).

Corollary 3.1 ($\mathscr{H}_1$-reverse strong duality)**.** *Assume that* (P) *satisfies the following conditions:*
(a) $\operatorname{dom} f \subset \bigcap_{t\in T}\operatorname{dom} f_t$.
(b) *T is a convex and compact subset of some locally convex topological vector space.*
(c) *$T \ni t \mapsto f_t(x)$ is concave and usc on T for each $x \in \bigcap_{t\in T}\operatorname{dom} f_t$.*
(d) *There exists $(\bar t,\bar\mu)\in T\times\mathbb{R}_+$ such that $f+\bar\mu f_{\bar t}$ is weakly inf-locally compact.*
(e) *$(\mathrm{P})_\infty$ is a linear subspace.*
Then,

$$\min(\mathrm{P}) = \sup_{(t,\mu)\in T\times\mathbb{R}_+}\inf_{x\in X}\left\{f(x)+\mu f_t(x)\right\}\in\mathbb{R}.$$

Proof. From the first three assumptions and [1, Remark 5.5], we obtain that $\varphi_{\mathscr{H}_1}$ is convex. Moreover, $\varphi_{\mathscr{H}_1} = \inf_{(t,\mu)\in T\times\mathbb{R}_+}(f+\mu f_t)^*$ is majorized by the function $(f+\bar\mu f_{\bar t})^*$, which is $\tau(X^*,X)$-quasicontinuous by Lemma 3.2 as, by (d), $f+\bar\mu f_{\bar t}\in\Gamma(X)$ is weakly inf-locally compact. So, by Lemma 3.3, $\varphi_{\mathscr{H}_1}$ is $\tau(X^*,X)$-quasicontinuous, and we conclude the proof by applying Theorem 3.1 with $\mathscr{H}=\mathscr{H}_1$ thanks to (e). □

The following result recovers a variant of the reverse duality theorem of [6, Theorem 3.3].

Corollary 3.2 ($\mathscr{F}(T)$-reverse strong duality)**.** *Assume that* $E \cap \operatorname{dom} f \neq \emptyset$ *and that the two following conditions are satisfied:*

(f) $\exists \lambda \in \mathbb{R}_+^{(T)}$ *such that* $f + \sum_{t\in T} \lambda_t f_t$ *is weakly inf-locally compact.*
(e) $(\mathrm{P})_\infty$ *is a linear subspace.*

Then

$$\min(\mathrm{P}) = \sup(\mathrm{D}) = \sup_{\lambda\in\mathbb{R}_+^{(T)}} \inf_{x\in X} \left\{ f(x) + \sum_{t\in T} \lambda_t f_t(x) \right\} \in \mathbb{R}.$$

Proof. Condition (f) amounts to

$$\exists H \in \mathscr{F}(T), \exists \mu \in \mathbb{R}_+^H \text{ such that } f + \sum_{t\in H} \mu_t f_t \text{ weakly inf-locally compact.}$$

Moreover, $\varphi_{\mathscr{F}(T)}$ is majorized by $(f+\sum_{t\in H}\mu_t f_t)^*$, which is $\tau(X^*,X)$-quasicontinuous by Lemma 3.2. By Lemma 3.3, $\varphi_{\mathscr{F}(T)}$ is then $\tau(X^*,X)$-quasicontinuous. Taking $\mathscr{H} = \mathscr{F}(T)$ in Theorem 3.1, we obtain, by (2.5) and (2.6), that

$$\min(\mathrm{P}) = \sup(\mathrm{D}_{\mathscr{H}}) = \sup(\mathrm{D}).$$

The proof is complete. □

We finally consider the countable case when $T = \mathbb{N}$. Let $(\mathrm{P}_\mathbb{N})$, (P_m), $(\mathrm{D}_\mathbb{N})$, and (D_m) be as in (2.10), (2.11), (2.12), and (2.13), respectively.

Corollary 3.3 ($\mathscr{H}_\mathbb{N}$-reverse strong duality)**.** *Assume* $\inf(\mathrm{P}_\mathbb{N}) \neq +\infty$ *and the two conditions below are satisfied:*

(g) $\exists\, (N,\mu) \in \mathbb{N}\times\mathbb{R}_+^N$ *such that* $f + \sum_{k=1}^N \mu_k f_k$ *is weakly inf-locally compact,*
(e) $(\mathrm{P})_\infty$ *is a linear subspace.*

Then

$$\min(\mathrm{P}_\mathbb{N}) = \lim_{m\to\infty} \inf(\mathrm{P}_m) = \lim_{m\to\infty} \sup(\mathrm{D}_m) = \sup(\mathrm{D}_\mathbb{N}).$$

Moreover, the optimal solution set of $(\mathrm{P}_\mathbb{N})$ *is the sum of a weakly compact convex set and a finite dimensional linear subspace.*

Proof. Since the covering family $\mathscr{H}_\mathbb{N}$ is directed, we know that $\varphi_{\mathscr{H}_\mathbb{N}}$ is a convex function. Moreover, $\varphi_{\mathscr{H}_\mathbb{N}}$ is majorized by $\left(f+\sum_{k=1}^N \mu_k f_k\right)^*$, which is $\tau(X^*,X)$-quasicontinuous by Lemma 3.2. By Lemma 3.3, $\varphi_{\mathscr{H}_\mathbb{N}}$ is then $\tau(X^*,X)$-quasicontinuous and, by [1, Formula (5.6)], $\sup(\mathrm{D}_\mathbb{N}) = \lim_{m\to\infty}\sup(\mathrm{D}_m)$. Applying Theorem 3.1 with $\mathscr{H} = \mathscr{H}_\mathbb{N}$, we obtain

$$\min(\mathrm{P}_\mathbb{N}) = \sup(\mathrm{D}_\mathbb{N}) = \sup_{m\in\mathbb{N}} \sup(\mathrm{D}_m) = \lim_{m\to\infty} \sup(\mathrm{D}_m) \leq \lim_{m\to\infty} \inf(\mathrm{P}_m) \leq \min(\mathrm{P}_\mathbb{N}),$$

and the proof is complete. □

Remark 3.2. We now comment conditions (a) $-$ (g) when $X = \mathbb{R}^n$, that is, in CSIP. Conditions (d), (f), and (g) are obviously satisfied while condition (e) is equivalent [19, Exercise 8.15] to

(h) $f_\infty(x) > 0, \forall x \in \left[(0^+E)\cap M^\perp\right] \diagdown \{0_n\}$,

where $M = \{x \in \operatorname{lin}(0^+E) : f_\infty(x) = 0 = f_\infty(-x)\}$. So, Corollary 3.2 is, in the CSIP setting, equivalent to [2, Theorem 3.2] (see also [19, Theorem 8.8(i)]). Analogously, [2, Corollary 4.2] is the CSIP version of Corollary 3.3.

If (P) is the LSIP problem in (1.4), we can write $f(x) = \langle c^*, x\rangle$ and $f_t(x) = \langle a_t^*, x\rangle - b_t$, $t \in T$. Then since all functions have full domain, (a) trivially holds. Moreover, since

$$(\mathrm{P})_\infty = \bigcap_{t\in T} [a_t^* \leq 0] \cap [c^* \leq 0],$$

condition (e) can be expressed as follows:

(e$'$) $\{x \in \mathbb{R}^n : \langle c^*, x\rangle \leq 0;\ \langle a_t^*, x\rangle \leq 0, \forall t \in T\}$ is a linear subspace.

Taking into account that a convex cone K is a subspace if and only if $-K \subset K$, (e$'$) is equivalent to

(e$''$) $[\langle c^*, x\rangle \leq 0;\ \langle a_t^*, x\rangle \leq 0, \forall t \in T] \Longrightarrow [\langle c^*, x\rangle = 0 = \ \langle a_t^*, x\rangle, \forall t \in T]$.

Moreover, condition (e$'$) can be reformulated in terms of the data as

(e$'''$) The pointed cone of $\overline{\mathrm{cone}}\left(\{c^*; a_t^*, t \in T\} \times \mathbb{R}_+\right)$ (i.e., its intersection with the orthogonal subspace to its lineality) is a half-line in $\mathbb{R}^{n+1}$ [19, Theorem 5.13(ii)] (or, more precisely, the half-line $\mathbb{R}_+ (0_n, 1)$ [20, page 155]).

In the same vein, since $\operatorname{dom} f = \mathbb{R}^n$, $f_\infty = \langle c^*, \cdot\rangle$, $0^+E = \bigcap_{t\in T} [a_t^* \leq 0]$, and

$$M^\perp = \{x \in \mathbb{R}^n : \langle c^*, x\rangle = 0 = \langle a_t^*, x\rangle, \forall t \in T\}^\perp = \operatorname{span}\{c^*; a_t^*, t \in T\},$$

condition (h) can be expressed as

(h$'$) $\langle c^*, x\rangle > 0, \forall x \in \left(\bigcap_{t\in T} [a_t^* \leq 0]\right) \cap \operatorname{span}\{c^*; a_t^*, t \in T\} \setminus \{0_n\}$.

Example 3.1. Consider the linear semi-infinite programming problem

$$\begin{array}{lll} (\mathrm{P}) & \inf\limits_{x\in\mathbb{R}^2} & f(x) = \langle c^*, x\rangle \\ & \text{s.t.} & -tx_1 + (t-1)x_2 + t - t^2 \leq 0,\ t \in [0,1], \end{array}$$

with $c^* \in \mathbb{R}_+^2 \setminus \{(0,0)\}$ (see [1, Example 3.1]). According to Remark 3.2, (a), (d), (f), and (g) hold independently of the data. Condition (b) holds because $[0,1] \subset \mathbb{R}$ is compact and convex and (c) because $t \longmapsto -tx_1 + (t-1)x_2 + t - t^2$ is concave on $\mathbb{R}$ for any $x \in \mathbb{R}^2$. Regarding (e), the set in (e$'$)

$$\left\{x \in \mathbb{R}^2 : \langle c^*, x\rangle \leq 0; -tx_1 + (t-1)x_2 \leq 0, \forall t \in [0,1]\right\} = \left\{x \in \mathbb{R}_+^2 : \langle c^*, x\rangle \leq 0\right\}$$

is $\{(0,0)\}$ when c^* belongs to the interior $\mathbb{R}_{++}^2$ of $\mathbb{R}_+^2$ and a positive axis when c^* belongs to its boundary. Hence, (e) only holds for $c^* \in \mathbb{R}_{++}^2$. Observe that the cone in (e$''$) is

$$\operatorname{cone}\left\{\begin{pmatrix} c_1^* \\ c_2^* \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}\right\} \times \mathbb{R}_+,$$

and its pointed cone is

$$\mathbb{R}_+ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \left(\text{resp., cone}\left\{\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}, \text{cone}\left\{\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}\right),$$

when $c^* \in \mathbb{R}_{++}^2$ ($c^* \in \mathbb{R}_{++}(1,0), c^* \in \mathbb{R}_{++}(1,0)$, respectively). So, we obtain again that (e) only holds for $c^* \in \mathbb{R}_{++}^2$. Regarding condition (h), if $c^* \in \mathbb{R}_{++}^2$, since $\bigcap_{t\in[0,1]} [a_t^* \leq 0] = \mathbb{R}_+^2$ and $\operatorname{span}\{c^*; a_t^*, t \in T\} = \mathbb{R}^2$, (h) holds; otherwise, $\operatorname{span}\{c^*; a_t^*, t \in T\}$ is a positive axis and (h) fails, otherwise. Thus, (e) and (h) hold or not simultaneously.

In conclusion, by Corollary 3.1, $\mathscr{H}_1$-reverse strong duality holds whenever $c^* \in \mathbb{R}^2_{++}$ while, by Corollary 3.2, $\mathscr{F}(T)$-reverse strong duality holds whenever $c^* \in \mathbb{R}^2_{++}$. Observe that, from the direct computations carried out in [1, Example 3.1], $\mathscr{H}_1$-reverse strong duality actually holds for all $c^* \in \mathbb{R}^2_+ \setminus \{(0,0)\}$.

Example 3.2. The countable linear semi-infinite programming problem

$$\begin{array}{lll}(\mathrm{P}_{\mathbb{N}}) & \inf\limits_{x\in\mathbb{R}^2} & x_2 \\ & s.t. & x_1 + k(k+1)x_2 \geq 2k+1,\ k\in\mathbb{N},\end{array}$$

violates the assumptions of Corollaries 3.1, 3.2, and 3.3, as (b) and (c) obviously fail, as well as (e) and (h). In fact, (e$'$) and (e$''$) fail because

$$\left\{x\in\mathbb{R}^2 : x_2 \leq 0, -x_1 - k(k+1)x_2 \leq 0,\ k\in\mathbb{N}\right\} = \mathbb{R}_+ \times \{0\}$$

is not a linear subspace and the pointed cone of

$$\overline{\mathrm{cone}}\{(0,1);(-1,-k(k+1)),k\in\mathbb{N}\}\times\mathbb{R}_+ = \left\{x\in\mathbb{R}^3 : x_1 \leq 0, x_3 \geq 0\right\}$$

is not a half-line, respectively, while (h) fails because x_2 vanishes on an edge of

$$\left(0^+E\right)\cap M^{\perp} = 0^+E\cap\mathbb{R}^2 = \mathrm{cone}\{(-2,1),(1,0)\}.$$

So, we cannot apply the mentioned corollaries to conclude that $\mathscr{H}$-reverse strong duality holds for $\mathscr{H} = \mathscr{H}_1, \mathscr{H}_{\mathbb{N}}, \mathscr{F}(T)$. Actually, $\mathscr{H}$-reverse strong duality does not hold for these three families because the feasible set of $(\mathrm{P}_{\mathbb{N}})$ is

$$E = \mathrm{co}\left(\left\{\left(k,\frac{1}{k}\right),k\in\mathbb{N}\right\}\cup\left\{x\in\mathbb{R}^2 : x_1+2x_2=3, x_1\leq 1\right\}\right),$$

which implies $\inf(\mathrm{P}_{\mathbb{N}}) = 0$ with $\mathrm{sol}(\mathrm{P}_{\mathbb{N}}) = \emptyset$, while $\sup(D) = -\infty$, which in turn implies $\sup(D_{\mathscr{H}}) = -\infty$ for any $\mathscr{H}$ such that $\emptyset \neq \mathscr{H} \subset \mathscr{F}(T)$, by (2.4).

4. $\mathscr{H}$−FARKAS LEMMA

We now establish some new versions of Farkas lemma relative to a given family $\mathscr{H} \subset \mathscr{F}(T)$. These results assert the equivalence between some inclusion (i) of the solution set E of $\{f_t(x) \leq 0, t\in T\}$ into certain set involving f and some condition (ii) involving $\{f;\ f_t, t\in T\}$ and $\mathscr{H}$. We first provide a Farkas-type result relative to the family $\mathscr{H}_1$ without assuming the lower semicontinuity of the involved functions. Stronger results (characterizations of Farkas lemma) will be then obtained under the lower semicontinuity (or even continuity) assumption.

Proposition 4.1 ($\mathscr{H}_1$-Farkas lemma)**.** *Assume conditions* (a),(b),(c) *in Corollary 3.1 altogether with the generalized Slater condition:*

$$\exists \bar{x}\in\mathrm{dom}\, f:\quad f_t(\bar{x}) < 0,\ \forall t\in T.$$

Then, for any $\alpha\in\mathbb{R}$, the following statements are equivalent:
(i) $[f_t(x)\leq 0, \forall t\in T] \implies f(x)\geq\alpha$.
(ii) *There exist $\bar{t}\in T$ and $\bar{\mu}\in\mathbb{R}_+$ such that*

$$f(x) + \bar{\mu} f_{\bar{t}}(x) \geq \alpha,\quad \forall x\in X. \tag{4.1}$$

Proof. We observe first that (i) is equivalent to $\inf(\mathrm{P}) \geq \alpha$, where (P) is the CIP in (1.1). So, it follows from [1, Theorem 5.3] that $\inf(\mathrm{P}) = \max(\mathrm{D}_{\mathscr{H}_1}) \geq \alpha$; i.e., (i) is equivalent to

$$\max_{(t,\mu)\in T\times\mathbb{R}} \inf_{x\in\operatorname{dom} f} \{f(x) + \mu f_t(x)\} \geq \alpha.$$

In other words, there exists $(\bar{t}, \bar{\mu}) \in T \times \mathbb{R}_+$ satisfying (4.1), which is (ii), and we are done. □

Observe that statement (i) means that E is contained in the reverse convex set $\{x \in X : f(x) \geq \alpha\}$ while (ii) would be the same replacing the infinite family $\{f_t, t \in T\}$ by the singleton one $\{f_{\bar{t}}\}$, so that Proposition 4.1 characterizes when an inequality $f(x) \geq \alpha$ is consequence of some single constraint $f_t(x) \leq 0$.

The following two propositions provide, under the lower semicontinuity assumption, a characterization in terms of $\mathscr{A}_{\mathscr{H}}$ (statement (I)) of the Farkas lemma (statement (II)) relative to an arbitrary non-empty covering family $\mathscr{H} \subset \mathscr{F}(T)$.

Proposition 4.2 (Characterization of $\mathscr{H}$-Farkas lemma)**.** *Let $\mathscr{H} \subset \mathscr{F}(T)$ be a covering family. Assume that $\{f;\ f_t, t \in T\} \subset \Gamma(X)$, $E \cap (\operatorname{dom} f) \neq \emptyset$, and consider the following statements:*

(I) *$\mathscr{A}_{\mathscr{H}}$ is w^*-closed convex regarding $\{0_{X^*}\} \times \mathbb{R}$.*

(II) *For $\alpha \in \mathbb{R}$, the next two conditions are equivalent:*

(i) $[f_t(x) \leq 0, \forall t \in T] \implies f(x) \geq \alpha$,

(ii) *there exist $H \in \mathscr{H}$ and $\mu \in \mathbb{R}_+^H$ such that*

$$f(x) + \sum_{t\in H} \mu_t f_t(x) \geq \alpha, \forall x \in X. \tag{4.2}$$

Then, $[(\mathrm{I}) \implies (\mathrm{II})]$, and the converse implication, $[(\mathrm{II}) \implies (\mathrm{I})]$, holds when $\inf(\mathrm{P}) \in \mathbb{R}$.

Proof. By the characterization of $\mathscr{H}$-strong duality at a point in (2.8), applied to $x^* = 0_{X^*}$, one obtains that (I) is equivalent to

$$\inf(\mathrm{P}) = \max(\mathrm{D}_{\mathscr{H}}), \tag{4.3}$$

which is itself equivalent to the existence of $H \in \mathscr{H}$ and $\mu \in \mathbb{R}_+^H$ such that

$$\inf(\mathrm{P}) = \inf_{x\in X} \left(f(x) + \sum_{t\in H} \mu_t f_t(x) \right).$$

Since (i) is equivalent to $\inf(\mathrm{P}) \geq \alpha$, it now follows that $[(\mathrm{I}) \implies (\mathrm{II})]$.

Conversely, if $\inf(\mathrm{P}) \in \mathbb{R}$ and (II) holds, then just take $\alpha = \inf(\mathrm{P})$. As (II) holds, it follows that there are $H \in \mathscr{H}$ and $\mu \in \mathbb{R}_+^H$ such that (4.2) holds, and

$$\sup(\mathrm{D}_{\mathscr{H}}) \geq \inf_{x\in X} \left(f(x) + \sum_{t\in H} \mu_t f_t(x) \right) \geq \alpha = \inf(\mathrm{P}).$$

In other words, $\sup(\mathrm{D}_{\mathscr{H}}) = \inf(\mathrm{P})$, $\sup(\mathrm{D}_{\mathscr{H}})$ is attained at $H \in \mathscr{H}$ and $\mu \in \mathbb{R}_+^H$, meaning that (4.3) holds, which is (I), and the proof is complete. □

Remark 4.1. In the special case when $\mathscr{H} = \mathscr{F}(T)$, the condition (ii) in Proposition 4.2 reads as

(ii′) there exists $\lambda \in \mathbb{R}_+^{(T)}$ such that $f(x) + \sum_{t\in T} \lambda_t f_t(x) \geq \alpha$, for all $x \in X$,

and Proposition 4.2 goes back to the Farkas lemma given in [3, Theorem 2] under a slightly different qualification condition. So, Proposition 4.2 is a variant of [3, Theorem 2].

Let us get back to the linear case, where

$$f(x) = \langle c^*, x\rangle, \; f_t(x) = \langle a_t^*, x\rangle - b_t, t \in T, \tag{4.4}$$

with $\{c^*; a_t^*, t \in T\} \subset X^*$ and $\{b_t, t \in T\} \subset \mathbb{R}$. Then, $\mathscr{A}_{\mathscr{H}} = \{(c^*, 0)\} + \mathscr{K}_{\mathscr{H}}$ (see [1, (4.4)]), where

$$\mathscr{K}_{\mathscr{H}} = \bigcup_{H \in \mathscr{H}} \operatorname{cone}\left(\{(a_t^*, b_t),\, t \in H\} + \{0_{X^*}\} \times \mathbb{R}_+\right).$$

In particular,

$$\mathscr{K}_{\mathscr{H}_1} \quad = \bigcup_{t \in T} \operatorname{cone}\{(a_t^*, b_t + \varepsilon) : \varepsilon \geq 0\}$$

and, by [1, Proposition 4.1],

$$\mathscr{K}_{\mathscr{F}(T)} = \operatorname{cone}\left(\{(a_t^*, b_t),\, t \in T\} + \{0_{X^*}\} \times \mathbb{R}_+\right).$$

For instance, for the LSIP problem in Example 3.1,

$$\mathscr{K}_{\mathscr{H}_1} \quad = \bigcup_{t \in [0,1]} \operatorname{cone}\left\{\left(-t, t-1, t^2 - t + \varepsilon\right) : \varepsilon \geq 0\right\}$$

while $\mathscr{K}_{\mathscr{F}(T)}$ is (see [1, Example 4.1]) the union of the origin with the epigraph of the convex function

$$\psi(x) := \begin{cases} \frac{x_1 x_2}{x_1 + x_2}, & x \in \mathbb{R}_-^2 \diagdown \{0_2\}, \\ +\infty, & \text{else.} \end{cases}$$

We finish this section with a characterization, in terms of $\mathscr{K}_{\mathscr{H}}$, of the Farkas lemma (statement (II) below) relative to an arbitrary non-empty covering family $\mathscr{H} \subset \mathscr{F}(T)$.

Proposition 4.3 ($\mathscr{H}$-Farkas lemma for linear infinite systems)**.** *Consider the linear functions* $\{f; f_t, t \in T\}$ *defined in* (4.4)*, and suppose that* $\inf(\mathrm{P})$ *is finite and that* $\mathscr{H}$ *is a covering family. Given* $c^* \in X^*$*, the following statements are equivalent:*

(I) $\overline{\operatorname{co}}(\mathscr{K}_{\mathscr{H}}) \cap (\{-c^*\} \times \mathbb{R}_+) = \mathscr{K}_{\mathscr{H}} \cap (\{-c^*\} \times \mathbb{R}_+)$.

(II) *For* $\alpha \in \mathbb{R}$*, the following statements are equivalent:*

(i) $[\langle a_t^*, x\rangle \leq b_t, \forall t \in T] \Longrightarrow \langle c^*, x\rangle \geq \alpha$.

(ii) *There exist* $H \in \mathscr{H}$ *and* $\mu \in \mathbb{R}_+^H$ *such that* $\sum_{t \in H} \mu_t a_t^* = -c^*$ *and* $-\sum_{t \in H} \mu_t b_t \geq \alpha$.

Proof. When $\mathscr{H}$ is a covering family and $E \neq \emptyset$, according to [1, Corollary 5.3], one has

$$\left(\inf(\mathrm{P}) = \max(\mathrm{D}_{\mathscr{H}})\right) \Longleftrightarrow \left((\overline{\operatorname{co}}\, \mathscr{K}_{\mathscr{H}}) \cap (\{-c^*\} \times \mathbb{R}_+) = \mathscr{K}_{\mathscr{H}} \cap (\{-c^*\} \times \mathbb{R}_+)\right). \tag{4.5}$$

The rest of the proof is similar to that of Proposition 4.2, using (2.8) and (4.5). □

5. $\mathscr{H}$-Optimality Conditions

In this section, we establish the optimality conditions for the problem (P) associated with some family $\mathscr{H} \subset \mathscr{F}(T)$. We shall represent by $\operatorname{sol}(\mathrm{D}_{\mathscr{H}})$ the set of optimal solutions of $(\mathrm{D}_{\mathscr{H}})$. In particular, when $\mathscr{H} = \mathscr{F}(T)$, one obtains the classical KKT conditions involving finitely many multipliers and, when $\mathscr{H} = \mathscr{H}_1$, optimality conditions involving a unique multiplier.

Theorem 5.1 (Primal-dual $\mathscr{H}-$optimality condition)**.** *Let* $\bar{x} \in E \cap (\operatorname{dom} f)$, $H \in \mathscr{H}$, *and* $\mu \in \mathbb{R}_+^H$. *Then, the following statements are equivalent:*
(i) $\bar{x} \in \operatorname{sol}(\mathrm{P})$, $(H,\mu) \in \operatorname{sol}(\mathrm{D}_{\mathscr{H}})$, *and* $\inf(\mathrm{P}) = \sup(\mathrm{D}_{\mathscr{H}})$.
(ii) $f(\bar{x}) = \inf_X \left(f + \sum_{t\in H} \mu_t f_t \right)$, *and* $\mu_t f_t(\bar{x}) = 0$, *for all* $t \in H$.
(iii) $0_{X^*} \in \partial \left(f + \sum_{t\in H} \mu_t f_t \right)(\bar{x})$, *and* $\mu_t f_t(\bar{x}) = 0$, *for all* $t \in H$.

Proof. $[(\mathrm{i}) \Rightarrow (\mathrm{ii})]$ We have

$$\inf_X \left(f + \sum_{t\in H} \mu_t f_t \right) = \sup(\mathrm{D}_{\mathscr{H}}) = \inf(\mathrm{P}) = f(\bar{x}),$$

and

$$f(\bar{x}) = \inf_X \left(f + \sum_{t\in H} \mu_t f_t \right) \leq f(\bar{x}) + \sum_{t\in H} \mu_t f_t(\bar{x}) \leq f(\bar{x}).$$

Hence, $\sum_{t\in H} \mu_t f_t(\bar{x}) = 0$ and (ii) holds.

$[(\mathrm{ii}) \Rightarrow (\mathrm{iii})]$ We have

$$\left(f + \sum_{t\in H} \mu_t f_t \right)(\bar{x}) = f(\bar{x}) = \inf_X \left(f + \sum_{t\in H} \mu_t f_t \right).$$

Thus, $\bar{x} \in \operatorname{argmin} \left(f + \sum_{t\in H} \mu_t f_t \right)$ or, equivalently, $0_{X^*} \in \partial \left(f + \sum_{t\in H} \mu_t f_t \right)(\bar{x})$.

$[(\mathrm{iii}) \Rightarrow (\mathrm{i})]$ Now we write

$$\inf(\mathrm{P}) \leq f(\bar{x}) = \left(f + \sum_{t\in H} \mu_t f_t \right)(\bar{x}) = \inf_X \left(f + \sum_{t\in H} \mu_t f_t \right) \leq \sup(\mathrm{D}_{\mathscr{H}}) \leq \inf(\mathrm{P}),$$

and (i) holds. □

Corollary 5.1 (1st $\mathscr{H}-$optimality condition for (P))**.** *Assume that* $\inf(\mathrm{P}) = \max(\mathrm{D}_{\mathscr{H}})$ *and let* $\bar{x} \in E \cap (\operatorname{dom} f)$. *Then, the following statements are equivalent:*
(i) $\bar{x} \in \operatorname{sol}(\mathrm{P})$.
(ii) *For each* $(H,\mu) \in \operatorname{sol}(\mathrm{D}_{\mathscr{H}})$, *we have*

$$0_{X^*} \in \partial \left(f + \sum_{t\in H} \mu_t f_t \right)(\bar{x}), \text{ and } \mu_t f_t(\bar{x}) = 0, \ \forall t \in H. \tag{5.1}$$

(iii) *There exists* $(H,\mu) \in \operatorname{sol}(\mathrm{D}_{\mathscr{H}})$ *such that (5.1) is fulfilled.*

Proof. $[(\mathrm{i}) \Rightarrow (\mathrm{ii})]$ is just $[(\mathrm{i}) \Rightarrow (\mathrm{iii})]$ in Theorem 5.1.
$[(\mathrm{ii}) \Rightarrow (\mathrm{iii})]$ is due to the assumption $\operatorname{sol}(\mathrm{D}_{\mathscr{H}}) \neq \emptyset$.
$[(\mathrm{iii}) \Rightarrow (\mathrm{i})]$ follows from $[(\mathrm{iii}) \Rightarrow (\mathrm{i})]$ in Theorem 5.1. □

Corollary 5.2 (2nd $\mathscr{H}-$optimality condition for (P))**.** *Let* $\mathscr{H} \subset \mathscr{F}(T)$ *be a covering family. Assume that* $\{f;\ f_t, t \in T\} \subset \Gamma(X)$ *and* $E \cap (\operatorname{dom} f) \neq \emptyset$. *Assume further that* $\mathscr{A}_{\mathscr{H}}$ *is* w^**-closed convex regarding* $\{0_{X^*}\} \times \mathbb{R}$. *Then* $\bar{x} \in \operatorname{sol}(\mathrm{P})$ *if and only if there exist* $H \in \mathscr{H}$ *and* $\mu \in \mathbb{R}_+^H$ *such that* (5.1) *holds.*

Proof. Taking $x^* = 0_{X^*}$ in (2.8) one has $\inf(\mathrm{P}) = \max(\mathrm{D}_{\mathscr{H}})$. Corollary 5.1 concludes the proof.
□

Remark 5.1. When $\mathscr{H}=\mathscr{F}(T)$, the conclusion of Corollary 5.2 is that $\bar{x}\in \operatorname{sol}(\mathrm{P})$ if and only if there exists $\lambda\in\mathbb{R}_+^{(T)}$ such that

$$0_{X^*}\in\partial\left(f+\sum_{t\in T}\lambda_t f_t\right)(\bar{x}) \text{ and } \lambda_t f_t(\bar{x})=0,\forall t\in T,$$

which recalls us about the optimality condition given in [3, Theorem 3] under the assumption that both the sets $\mathscr{K}_{\mathscr{F}(T)}$ and $\operatorname{epi} f^*+\overline{\mathscr{K}_{\mathscr{F}(T)}}$ are w^*-closed.

Corollary 5.3 ($\mathscr{H}-$**optimality condition for linear** (P)). *Let* (P) *be linear with* $E\neq\emptyset$. *Let* $\mathscr{H}$ *be a covering family. Assume that* $\mathscr{K}_{\mathscr{H}}$ *is weak***-closed convex regarding* $\{-c^*\}\times\mathbb{R}$. *Then* $\bar{x}\in\operatorname{sol}(\mathrm{P})$ *if and only if there exist* $H\in\mathscr{H}$ *and* $\mu\in\mathbb{R}_+^H$ *such that*

$$\sum_{t\in H}\mu_t a_t^*=-c^* \text{ and } \sum_{t\in H}\mu_t b_t=-\langle c^*,\bar{x}\rangle. \tag{5.2}$$

Proof. By [1, Corollaty 5.3], one has $\inf(\mathrm{P})=\max(\mathrm{D}_{\mathscr{H}})$. In the linear case one has (5.1) $\Leftrightarrow$ (5.2). We conclude the proof with Corollary 5.1. □

Corollary 5.4 (**Optimality condition for** ($\mathrm{D}_{\mathscr{H}}$)). *Assume that* $\min(\mathrm{P})=\sup(\mathrm{D}_{\mathscr{H}})\neq+\infty$, *and let* $H\in\mathscr{H}$ *and* $\mu\in\mathbb{R}_+^H$. *Then, the following statements are equivalent:*
(i) $(H,\mu)\in\operatorname{sol}(\mathrm{D}_{\mathscr{H}})$.
(ii) *For each* $\bar{x}\in\operatorname{sol}(\mathrm{P})$, *(5.1) holds.*
(iii) *There exists* $\bar{x}\in\operatorname{sol}(\mathrm{P})$ *such that (5.1) is fulfilled.*

Proof. [(i) $\Rightarrow$ (ii)] follows from [(i) $\Rightarrow$ (iii)] in Theorem 5.1.
[(ii) $\Rightarrow$ (iii)] is due to the assumption $\operatorname{sol}(\mathrm{P})\neq\emptyset$.
[(iii) $\Rightarrow$ (i)] follows from [(iii) $\Rightarrow$ (i)] in Theorem 5.1. □

We finish by revisiting again Example 3.1, with $\mathscr{H}=\mathscr{H}_1$. For $c^*\in\mathbb{R}^2_{++}$, let us check the fulfilment of (5.2) at $\bar{x}=\left(\left(\frac{c_2^*}{c_1^*+c_2^*}\right)^2,\left(\frac{c_1^*}{c_1^*+c_2^*}\right)^2\right)$. Taking $H=\{\bar{t}\}$, with $\bar{t}=\frac{c_1^*}{c_1^*+c_2^*}\in\,]0,1[$, and $\mu\in\mathbb{R}_+^{([0,1])}$ such that $\mu_{\bar{t}}=c_1^*+c_2^*>0$ and $\mu_t=0$ for all $t\in[0,1]\diagdown\{\bar{t}\}$, one has

$$\sum_{t\in H}\mu_t a_t^*=(c_1^*+c_2^*)\left(-\frac{c_1^*}{c_1^*+c_2^*},-\frac{c_2^*}{c_1^*+c_2^*}\right)=-c^*$$

and

$$\sum_{t\in H}\mu_t b_t=(c_1^*+c_2^*)\left(\left(\frac{c_1^*}{c_1^*+c_2^*}\right)^2-\frac{c_1^*}{c_1^*+c_2^*}\right)=-\frac{c_1^*c_2^*}{c_1^*+c_2^*}=-\langle c^*,\bar{x}\rangle,$$

so that $\bar{x}\in\operatorname{sol}(\mathrm{P})$ (recall that $\mathscr{K}_{\mathscr{H}_1}$ is closed). Moreover, $(H,\mu)\in\operatorname{sol}(\mathrm{D}_{\mathscr{H}})$ by Corollary 5.4 as

$$\partial\left(c^*+\sum_{t\in H}\mu_t a_t^*\right)=\left\{c^*+(c_1^*+c_2^*)\left(-\frac{c_1^*}{c_1^*+c_2^*},-\frac{c_2^*}{c_1^*+c_2^*}\right)\right\}=\{(0,0)\}$$

and the complementarity condition $\mu_t f_t(\bar{x})=0$, for all $t\in T$, holds.

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