J. Appl. Numer. Optim. 3 (2021), No. 3, pp. 533-554
Available online at http://jano.biemdas.com
https://doi.org/10.23952/jano.3.2021.3.08

DUAL THREE-OPERATOR SPLITTING ALGORITHMS FOR SOLVING COMPOSITE MONOTONE INCLUSION WITH APPLICATIONS TO CONVEX MINIMIZATION

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Abstract. In this paper, we study a monotone inclusion problem involving the mixtures of composite and parallel-sum type monotone operators with one of them being a cocoercive operator. Since the resolvent of the composite operator does not have a closed-form solution, the exact three-operator splitting algorithm could not be directly applied. As a result, it is meaningful to propose an effective iterative algorithm to solve this resolvent operator. Based on the primal-dual idea, we first solve the resolvent of the composite operators under suitable conditions. Furthermore, we present two iterative algorithms to solve the composite monotone inclusion problem, and prove their convergence based on the inexact three-operator splitting algorithm. As an application, a corresponding composite convex optimization problem is solved by two novel approaches. Finally, some numerical experiments are investigated on the image deblurring problems to demonstrate the efficiency of the proposed algorithms.

Keywords. Forward-backward splitting algorithm; Operator splitting algorithm; Primal-dual; Three-operator splitting algorithm; Total variation.

1. INTRODUCTION

In recent years, with the rapid development of image processing, medical image reconstruction, and machine learning, the applications of convex optimization models are under spotlight. However, owing to the enormous scale and complexity of modern datasets, these optimization models are usually not smooth, the size is large, and the traditional optimization methods encounter difficulties. As a result, the composite monotone inclusion scheme becomes an increasingly critical method that plays a key role in solving various complicated models of convex optimization problems. Simultaneously, there exist considerable results on the composite monotone inclusions in the literature. Inspired by the seminal work in [1], Briceno-Arias and Combettes [2] first proposed a primal-dual method based on a forward-backward-forward approach, a new method of solving composite monotone inclusions, to solve the monotone+skew model in [2]. Subsequently, Combettes and Pesquet [3] extended this method to a wider range of the composite monotone inclusion with the mixtures of composite, Lipschitzian, and parallel-sum type monotone operators. Furthermore, in [4], Boţ and Hendrich studied the convergence

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Received November 11, 2021; Accepted December 2, 2021.

behavior of the primal-dual scheme of Combettes and Pesquet [3]. A convergence rate for the partial primal-dual gap function related to a primal-dual pair of optimization problems was also demonstrated by them, who took full advantage of conjugate duality techniques. If one of the composite monotone inclusion operators, instead of Lipschitzian operators, is cocoercive, then the primal-dual approach with the forward-backward-forward scheme is not enough. As a result, Vu [5] showed a new splitting algorithm for solving the composite monotone inclusions involving cocoercive operators, which depends on forward-backward splitting algorithms. Different from the approach in [5], a new variable metric forward-backward splitting algorithm was assumed to solve the composite monotone inclusions involving cocoercive operators by Combettes and Vu [6], who combined the new approach and the primal-dual method investigated in [3]. Followed by the work of [2, 6, 7], Vu [8] put forward a variable metric extension of the forward-backward-forward algorithm for solving the composite monotone inclusion with a monotone Lipschitzian operator. Based on the primal-dual theory, Bo̧t and Hendrich [9] demonstrated a Douglas-Rachford type primal-dual method for solving the inclusions with the mixtures of composite and parallel-sum type monotone operators. And Condat [10] revealed a primal-dual splitting method to solve the convex optimization problem involving Lipschitzian, proximable, and linear composite terms. For other related works in this direction, we refer to [11, 12, 13, 14].

In this paper, we propose a monotone inclusion problem. We could employ the three-operator splitting algorithm [15] to solve it, however, we need to compute its exact resolvent operator. In view of the fact that the computation of its resolvent operator is usually intractable, it is meaningful to propose an effective iterative algorithm to solve this resolvent. Based on the primal-dual idea, we first solve the resolvent of the composite operators under suitable qualification. Furthermore, we present two iterative algorithms to solve the composite monotone inclusion problem, and prove their convergence based on the inexact three-operator splitting algorithm. We also display the efficiency of the proposed iterative algorithms in the context of image deblurring.

The paper is organized as follows. In Section 2, we present our monotone inclusion problem, which involves the mixtures of composite and parallel-sum type monotone operators with one of them being a cocoercive operator, and recall some useful definitions and lemmas. In Section 3, we present an effective approach to compute the resolvent operator under suitable conditions, and establish its convergence rate. In Section 4, a dual three-operator splitting algorithm is proposed to solve the inclusion problem. In Section 5, we employ the proposed iterative algorithms to solve the convex optimization problem. In Section 6, we display several numerical experiments to illustrate the performance of our proposed algorithms in image deblurring. Finally, we give some conclusions in Section 7, the last section.

2. Problem Statement and Preliminaries

2.1. **Problem statement.** Let H be a real Hilbert space. Let $A : H \to 2^H$ be a maximally monotone operator, and let $C : H \to H$ be a μ-cocoercive operator with some $\mu > 0$. Let $m \geq 1$ be an integer. For each $i = 1, 2, \cdots, m$, let G_i be a real Hilbert space and $r_i \in G_i$. Let $B_i : G_i \to 2^{G_i}$ be a maximally monotone operator, and let $D_i : G_i \to 2^{G_i}$ be maximally monotone and ν_i-strongly monotone, where $\nu_i \geq 0$. Let $L_i : H \to G_i$ be a nonzero bounded linear operator.

In this paper, we focus on solving the following monotone inclusion problem:

$$\text{find } x \in H, \quad \text{such that} \quad 0 \in Ax + \sum_{i=1}^{m} L_i^*((B_i \Box D_i)(L_i x - r_i)) + Cx. \tag{2.1}$$

Problem (2.1) reduces to the three-operator monotone inclusion problem in [15] when $L_i = D_i = I, m = 1$, and $r_i = 0$, where I is the identity operator. Indeed, it reads $0 \in Ax + Bx + Cx$. The three-operator splitting algorithm proposed by Davis and Yin [15] reads

$$\begin{cases} x_B^k = J_{\gamma B}(z^k), \\ x_A^k = J_{\gamma A}(2x_B^k - z^k - \gamma C x_B^k), \\ z^{k+1} = z^k + \lambda_k (x_A^k - x_B^k). \end{cases}$$

Based on [15, Remark 3.4], an inexact three operator splitting algorithm was shown in [16], which is a vital part of our proposed algorithms. In the case that D_i is monotone, both D_i^{-1} and C are Lipschitzian, a primal-dual methods with forward-backward-forward schemes were investigated in [3], which is a generalization form of [2]. In addition to this algorithm, a variable metric froward-backward-forward splitting algorithm [8] was proposed to solve this problem. Subsequently, a new algorithm, a primal-dual method with the forward-backward approach, was demonstrated in [5] for the case that w_i is added to the front of L_i^* and $\sum_{i=1}^m w_i = 1$. On the other hand, a variable metric forward-backward splitting algorithm was investigated in [6]. Furthermore, in [9], the authors solved (2.1) for the case when D_i is maximally monotone and $C = 0$. Inspired and motivated by these algorithms, we consider the primal-dual approach with the inexact three-operator splitting algorithm [16] to solve monotone inclusion problem (2.1).

Before presenting the main results, we first make the following assumptions, which will be used in the sequel

(1) $\sum_{i=1}^m L_i^*(B_i \Box D_i)(L_i x - r_i)$ is maximal monotone;
(2) The solution set of monotone inclusion problem (2.1) is nonempty.

2.2. **Preliminaries.** Throughout this paper, $\mathbb{N}$ is assumed to be the set of nonnegative integers, and H, G_i is assumed to be a real Hilbert space. The inner product and the associated norms of H are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. The symbols $\rightharpoonup$ and $\to$ denote, respectively, the weak and strong convergence. The class of proper, lower semicontinuous, and convex function from H to $(-\infty, +\infty]$ is denoted by $\Gamma_0(H)$, and $Fix(T)$ denotes the set of fixed points of T.

In the following, let $A : H \to 2^H$ be a set-valued operator, where 2^H denotes the power set of H, and let I be the identity operator on H.

We denote the following symbols:

(1) the set of zeros of A is $zerA := \{x \in H \mid 0 \in Ax\}$;
(2) the domain of A is $domA := \{x \in H \mid Ax \neq \varnothing\}$;
(3) the range of A is $ranA := \{y \in H \mid \exists x \in H : y \in Ax\}$;
(4) the graph set A is $graA := \{(x,y) \in H^2 \mid y \in Ax\}$;
(5) the *resolvent* of A is $J_A = (I + A)^{-1}$.

The following definitions and lemmas are essential for our main results, and they can be found in the monograph [17]. Let us first recall the definition of maximal monotone mappings.

Definition 2.1. (Maximal monotone mapping) Let $A : H \to 2^H$ be a set-valued mapping. Then A is said to be *monotone* if $\langle x - y, u - v \rangle \geq 0$ for all $(x,u) \in graA$ and $(y,v) \in graA$. A is said to be *maximal monotone* if there exists no monotone operator $B : H \to 2^H$ such that $graB$ properly contains $graA$.

Definition 2.2. (Resolvent operator) Let $A : H \to 2^H$ be a set-valued operator. The resolvent of a monotone operator A with index $\lambda > 0$ is defined as $J_{\lambda A} = (I + \lambda A)^{-1}$.

Let $f \in \Gamma_0(H)$. Then ∂f is maximal monotone, where ∂f represents the *subdifferential* of f defined by

$$\partial f : H \to 2^H : x \mapsto \{u \in H : \forall y \in H, \langle y - x, u\rangle + f(x) \leq f(y)\}.$$

Recall that the proximity operator of $f \in \Gamma_0(H)$ is defined as follows

$$prox_f(u) = \arg\min_x \left\{ \frac{1}{2}\|x-u\|^2 + f(x) \right\}.$$

It is known that $prox_f = J_{\partial f}$.

Definition 2.3. (Nonexpansive and α-averaged) Let D be a nonempty subset of H, and let $T : D \to H$ be a mapping. Then T is said to be *nonexpansive* if it is Lipschitz continuous with constant 1, i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in D$. For any $\alpha \in (0,1)$, T is said to be *α-averaged* if there exists a nonexpansive mapping $R : D \to H$ such that $T = (1-\alpha)I + \alpha R$.

The class of α-averaged mappings is usually denoted by $\mathscr{A}(\alpha)$. The following two lemmas give some useful characterizations of firmly nonexpansive mappings and α-averaged mappings, respectively.

Lemma 2.1. *Let D be a nonempty subset of H, and let $T : D \to H$ be a mapping. Then the following statements are equivalent:*

(1) *T is firmly nonexpansive;*
(2) *$2T - I$ is nonexpansive;*
(3) *$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle$, for all $x, y \in H$.*

If T is firmly nonexpansive, then it follows from Lemma 2.1 that $T \in \mathscr{A}(1/2)$.

Lemma 2.2. *Let D be a nonempty subset of H, $T : D \to H$ be nonexpansive, and $\alpha \in (0,1)$. Then the following are equivalent:*

(1) *T is α-averaged;*
(2) *$(1 - \frac{1}{\alpha})I + (\frac{1}{\alpha})T$ is nonexpansive;*
(3) *for all $x \in D$ and $y \in D$, $\|Tx - Ty\|^2 \leq \|x - y\|^2 - \frac{1-\alpha}{\alpha}\|(I - T)x - (Id - T)y\|^2$.*

Lemma 2.3. *Let $A : H \to 2^H$ be a maximal monotone operator, and let $\gamma \in (0, +\infty)$. Then $J_{\gamma A} : H \to H$ and $I - J_{\gamma A} : H \to H$ are firmly nonexpansive and maximally monotone.*

A cocoercive mapping is also called an *inverse strongly monotone mapping* (see, e.g., [18]).

Definition 2.4. (Cocoercive operator) A single-valued mapping $A : H \to H$ is said to be *β-cocoercive* with $\beta \in (0, +\infty)$ if $\beta\|Ax - Ay\|^2 \leq \langle Ax - Ay, x - y\rangle$ for all $x, y \in H$.

Let us recall the definition of uniformly monotone mappings.

Definition 2.5. (Uniformly monotone operator) A set-valued mapping $A : H \to 2^H$ is said to be *uniformly monotone of a modulus ϕ* : $[0, +\infty] \to [0, +\infty]$ if ϕ is a nondecreasing function with $\phi(0) = 0$ such that, for all $x, y \in H$,

$$u \in Ax,\ v \in Ay \implies \langle u - v, x - y\rangle \geq \phi(\|x - y\|).$$

If $\phi \equiv \beta(\cdot)^2 > 0$, then mapping A is said to be *strongly monotone*.

Lemma 2.4. *Let $f : H \to (-\infty, +\infty]$ be proper and convex. Then the following assertions hold:*

(1) *if f is strictly convex, then ∂f is strictly monotone.*

(2) *if f is strongly convex with constant $\beta \in (0, +\infty)$, then ∂f is strongly monotone with constant β.*

Lemma 2.5. (Baillon-Haddad theorem) *Let $f : H \to \mathbb{R}$ be a convex differentiable function with $\frac{1}{\beta}$-Lipschitz continuous gradient for some $\beta \in (0, +\infty)$. Then ∇f is β-cocoercive.*

Definition 2.6. (Demiregular) A set-valued mapping $A : H \to 2^H$ is said to be *demiregular* at $x \in dom(A)$ if, for all $u \in Ax$ and all sequences $(x^k, u^k) \in gra(A)$ with $x^k \rightharpoonup x$ and $u^k \to u$, $x^k \to x$.

The following two definition are crucial to the analysis of our algorithms.

Definition 2.7. [19] (Parallel sum) Let $A, B : H \to 2^H$. Define the parallel sum of two set-valued operators as $A \Box B : H \to 2^H$, $A \Box B = (A^{-1} + B^{-1})^{-1}$. If A and B are monotone, then $A \Box B$ are monotone. However, if A and B are maximally monotone, this property is generally not true for $A \Box B$.

Definition 2.8. (Infimal convolution) Let $f, g : H \to R$. Then the infimal convolution of two functions f and g are defined as $f \Box g : H \to R :\ x \mapsto \inf_{y \in H}(f(y) + g(x - y))$.

3. Computing the Resolvent

As described before, it is important to compute the resolvent operator $J_{\sum_{i=1}^m L_i^*(B_i \Box D_i)(L_i x - r_i)}$ when we can solve problem (2.1). Therefore, we first discuss how to compute resolvent operator $J_{\sum_{i=1}^m L_i^*(B_i \Box D_i)(L_i x - r_i)}$ in this section. We then establish its convergence rate.

This problem is presented as follows.

Problem 3.1. Let $m \geq 1$ be an integer and $u \in H$. Let H and $\{G_i\}_{i=1}^m$ be real Hilbert spaces. For every $i \in 1, \cdots, m$, let $r_i \in G_i$, $B_i : G_i \to 2^{G_i}$ be maximally monotone, and let $D_i : G_i \to 2^{G_i}$ be maximally monotone and v_i-strongly monotone, where $v_i \geq 0$. Let $L_i : H \to G_i$ be a nonzero bounded linear operator. The problem is to solve the resolvent of composite operator:

$$J_{\sum_{i=1}^m L_i^*(B_i \Box D_i)(L_i x - r_i)}(u). \tag{3.1}$$

To solve problem 3.1, the following lemma demonstrates that monotone inclusion (3.3) could be solved via its dual monotone inclusion.

Lemma 3.1. *For monotone inclusion problem (3.1), let $x = J_{\sum_{i=1}^m L_i^*(B_i \Box D_i)(L_i x - r_i)}(u)$. Then, the following assertions hold.*

(i) *There exists $v_i \in G_i, i = 1, 2, \cdots, m$ such that*

$$0 \in L_i(u - \sum_{j=1}^m L_j^* v_j) - B_i^{-1} v_i - D_i^{-1} v_i - r_i, \quad v_i \in 1, 2, \cdots, m. \tag{3.2}$$

Monotone inclusion (3.2) is called the dual monotone inclusion of problem (3.1). The solution set of (3.2) is denoted by Ω.

(ii) *Let $(v_1, \cdots, v_m)$ be a solution of (3.2). Then $x = u - \sum_{i=1}^m L_i^* v_i$.*

Proof. (i) The resolvent operator (3.1) is equivalent to the following monotone inclusion

$$0 \in x - u + \sum_{i=1}^{m} L_i^*(B_i \Box D_i)(L_i x - r_i). \tag{3.3}$$

It follows from monotone inclusion (3.3) that x is a solution of (3.1). Then

$$\begin{aligned}
&\Leftrightarrow \quad \exists v_i \in G_i, (\forall i \in 1,\cdots,m) \quad v_i \in (B_i \Box D_i)(L_i x - r_i)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad x = u - \sum_{i=1}^{m} L_i^* v_i\\
&\Leftrightarrow \quad \exists v_i \in G_i, (\forall i \in 1,\cdots,m) \quad L_i x - r_i \in B_i^{-1} v_i + D_i^{-1} v_i\\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad x = u - \sum_{i=1}^{m} L_i^* v_i\\
&\Rightarrow \quad 0 \in L_i(u - \sum_{i=1}^{m} L_i^* v_i) - B_i^{-1} v_i - D_i^{-1} v_i - r_i, \quad \forall i \in 1,\cdots,m.
\end{aligned}$$

(ii) Let $(v_i,\cdots,v_m)$ be a solution of (3.2), and let $x = u - \sum_{i=1}^{m} L_i^* v_i$. Then, for any $i \in \{1,\cdots,m\}$,

$$\begin{aligned}
& 0 \in L_i x - B_i^{-1} v_i - D_i^{-1} v_i - r_i\\
\Leftrightarrow\quad & L_i x - r_i \in B_i^{-1} v_i + D_i^{-1} v_i\\
\Leftrightarrow\quad & v_i \in (B_i \Box D_i)(L_i x - r_i)\\
\Rightarrow\quad & 0 \in x - u + \sum_{i=1}^{m} L_i^*(B_i \Box D_i)(L_i x - r_i),
\end{aligned}$$

which means that x is a solution of monotone inclusion problem (3.1). □

The following Lemma is crucial for presenting our main results.

Lemma 3.2. *Let $G = G_1 \oplus G_2 \oplus \cdots \oplus G_m$ be the Hilbert direct sum of the Hilbert spaces $\{G_i\}_{1\le i\le m}$. The scalar product and the associated norm defined in G are*

$$\langle \cdot,\cdot \rangle : \langle x, y \rangle \mapsto \sum_{i=1}^{m} \langle x_i, y_i \rangle,$$

and

$$\|\cdot\| : x \mapsto \sqrt{\sum_{i=1}^{m} \|x_i\|^2},$$

where $x = (x_i)_{1\le i\le m} \in G$ and $y = (y_i)_{1\le i\le m} \in G$. Define the following operators

$$\begin{aligned}
&A : G \to 2^G : \ v \mapsto (B_i^{-1} v_i)_{1\le i\le m},\\
&D : G \to G : \ v \mapsto (r_i + D_i^{-1} v_i)_{1\le i\le m},\\
&L : H \to G : \ x \mapsto (L_i x)_{1\le i\le m},
\end{aligned}$$

and

$$B : G \to G : \ v \mapsto L(L^* v - u) + D.$$

Then A is maximally monotone, D is $\min_{1\leq i\leq m}\{v_i\}$-cocoercive, and $L^*v = \sum_{i=1}^m L_i^* v_i, v\in G$. In addition, B is β-cocoercive, where $\beta = \frac{1}{\frac{1}{\min_{1\leq i\leq m}\{v_i\}}+\|L\|^2}$.

Proof. Since $B_i : G_i \to 2^{G_i}$ is maximally monotone for any $i\in\{1,2,\cdots,m\}$, then B_i^{-1} is also maximally monotone. Therefore, A is maximally monotone.

Next, we prove that D is $\min_{1\leq i\leq m}\{v_i\}$-cocoercive. Let $z\in G, v\in G$. It follows that

$$\begin{aligned}\langle z-v, Dz-Dv\rangle &= \sum_{i=1}^m \langle z_i - v_i, D_i^{-1}z_i - D_i^{-1}v_i\rangle \\ &\geq \sum_{i=1}^m v_i\|D_i^{-1}z_i - D_i^{-1}v_i\|^2 \\ &\geq \min_{1\leq i\leq m}\{v_i\}\|Dz-Dv\|^2,\end{aligned}$$

where the first inequality comes from the fact that D_i^{-1} is v_i-cocoercive. Therefore, D is $\min_{1\leq i\leq m}\{v_i\}$-cocoercive. Furthermore, letting $y\in G$ and $x\in G$, we have

$$\langle y, Lx\rangle = \sum_{i=1}^m \langle y_i, L_i x\rangle = \langle \sum_{i=1}^m L_i^* y_i, x\rangle.$$

Then $L^*y = \sum_{i=1}^m L_i^* y_i$. Since $L(L^*v-u)$ is the gradient operator of function $\frac{1}{2}\|L^*v-u\|^2$, it follows from Baillon-Haddad theorem that $L(L^*v-u)$ is $\frac{1}{\|L\|^2}$-cocoercive. In view of [17, Proposition 4.12], we conclude that B is β-cocoercive, where

$$\beta = \frac{1}{\frac{1}{\min_{1\leq i\leq m}\{v_i\}}+\|L\|^2}.$$

Based on the operator introduced in Lemma 3.2, we can reformulate monotone inclusion (3.1) as follows:

$$\text{find} \quad v\in G, \qquad \text{such that} \quad 0\in Av+Bv. \tag{3.4}$$

According to Lemma 3.2, we know that A is maximally monotone, and B is β-cocoercive. Then, we can employ the forward-backward splitting algorithm to solve monotone inclusion (3.4). Hence, we obtain the following main theorem for solving resolvent operator (3.1). □

Theorem 3.1. *Let $u\in H$, and let $m\geq 1$ be an integer. Let H and $\{G_i\}_{i=1}^m$ be real Hilbert spaces. For every $i\in\{1,\cdots,m\}$, let $r_i\in G_i$, $B_i : G_i\to 2^{G_i}$ be maximally monotone, $D_i : G_i \to 2^{G_i}$ be maximally monotone and v_i-strongly monotone with $v_i\geq 0$. Let $L_i : H\to G_i$ be a nonzero bounded linear operator. For any $v_i^0\in G_i$, define the following iteration sequences:*

$$\left\lfloor\begin{array}{l} x^k = u - \sum_{i=1}^m L_i^* v_i^k, \\ for\ i=1,\cdots,m \\ \left\lfloor\begin{array}{l} y_i^k = v_i^k - \gamma_k(-L_i x^k + D_i^{-1}v_i^k + r_i), \\ v_i^{k+1} = v_i^k + \lambda_k(J_{\gamma_k B_i^{-1}}(y_i^k) - v_i^k), \end{array}\right.\end{array}\right. \tag{3.5}$$

where $\{\gamma_k\}\subset(0,2\beta), \{\lambda_k\}\subset(0,\frac{1}{\alpha_k})$, and $\alpha_k = \frac{2\beta}{4\beta-\gamma_k}$. Then, the following conclusions hold.
(i) *For any $v\in\Omega, \lim_{n\to+\infty}\|v_k - v\|$ exists.*

If $\{\lambda_k\}$ satisfies one of the following conditions:

(a) $0 \le \underline{\lambda} \le \lambda_k \le \frac{1}{\alpha_k} - \tau$, *where* $\tau \in (0, \frac{1}{\alpha_k} - \underline{\lambda})$;

(b) $\sum_{k=0}^{+\infty} \lambda_k(\frac{1}{\alpha_k} - \underline{\lambda}) = +\infty$ *and* $\sum_{k=0}^{+\infty} |\gamma_{k+1} - \gamma_k| < +\infty$,

then

(ii) $\lim_{k\to+\infty} \|v_i^k - J_{\gamma_k B_i^{-1}}(v_i^k - \gamma_k(-L_i(u - \sum_{i=1}^m L_i^* v_i^k) + D_i^{-1} v_i^k + r_i))\| = 0$, $\forall i \in \{1,2,\cdots,m\}$;

(iii) $\{v_i^k\}$ *converges weakly to* v_i, *for any* $i \in \{1,2,\cdots,m\}$. *In addition,* $x = u - \sum_{i=1}^m L_i^* v_i^k$.

Furthermore, suppose that (a) *is satisfied and* $0 < \underline{\gamma} \le \gamma_k \le 2\beta - \varepsilon$, *where* $\varepsilon \in (0, 2\beta - \underline{\gamma})$, *or* (b) *is satisfied,* $\lambda_k \ge \underline{\lambda} > 0$, *and* $0 < \underline{\gamma} \le \gamma_k \le 2\beta - \varepsilon$, *where* $\varepsilon \in (0, 2\beta - \underline{\gamma})$. *Then*

(iv) $L_i(\sum_{i=1}^m L_i^* v_i^k - u) + D_i^{-1} v_i^k + r_i \to L_i(\sum_{i=1}^m L_i^* v_i - u) + D_i^{-1} v_i + r_i$ *as* $k \to +\infty$, *for any* $i \in \{1,2,\cdots,m\}$;

(v) $x_k \to x$ *as* $k \to +\infty$.

Proof. It follows from the notations introduced in Lemma 3.2 that iterative scheme (3.5) can be rewritten as

$$\begin{cases} y^k = v^k - \gamma_k B v^k, \\ v^{k+1} = v^k + \lambda_k(J_{\gamma_k A}(y^k) - v^k). \end{cases}$$

As we pointed out that dual monotone inclusion (3.2) is equivalent to (3.1). Therefore, the conclusions of (i), (ii), (iii) and (iv) come directly from [20, Theorem 3.5]. Finally, we prove (v). Let $v \in \Omega$. In view of the schwaritz inequality and the fact that D is monotone, one has

$$\begin{aligned} \|x^k - x\|^2 &= \langle u - L^* v^k - (u - L^* v), u - L^* v^k - (u - L^* v)\rangle \\ &= \langle LL^* v - LL^* v^k, v - v^k\rangle \\ &\le \langle L(L^* v^k - u) - L(L^* v - u), v^k - v\rangle + \langle v^k - v, Dv^k - Dv\rangle \\ &= \langle v^k - v, Bv^k - Bv\rangle \\ &\le \|v^k - v\|\|Bv^k - Bv\|. \end{aligned}$$

Following (i) and (ii), we can conclude that $x^k \to x$ as $k \to +\infty$. This completes the proof. □

Further, we also give the convergence rate of the algorithm.

Theorem 3.2. *(Convergence rate) Let the assumptions be as in Theorem 3.1, and let* $\gamma_k = \gamma$, $v_0 \in G$. *Suppose that the following conditions hold:*

(a) $\overline{\beta} = \|1 - \gamma L^* L\| + \frac{\gamma}{\min_{1\le i\le m}\{v_i\}} < 1$;

(b) $0 < \lambda_k < \frac{1}{1-\overline{\beta}}$.

Then,

(i) $\|x_n - x^*\| \le \frac{\|L\|\hat{\beta}^n}{1-\hat{\beta}}\|v_0 - v_1\|$ *and* $\|x_n - x^*\| = \mathscr{O}(\hat{\beta}^n)$, *where* $\hat{\beta} = 1 - (1-\bar{\beta})\lambda_k$ *and* $\bar{\beta} \in (0,1)$.

(ii) $\|x_n - x^*\| \le \frac{\|L\|}{1-\hat{\beta}}\|v_n - v_{n+1}\|$.

Proof. According to (3.4), let $T = J_{\gamma A}(I - \gamma B)$. By [17, Proposition 26.1], one has $v_{n+1} = Tv_n$. Observe that $\bar{\beta} \in (0,1)$. It follows from the Lemma 2.3 that $J_{\gamma A}$ is nonexpansive. Combining

(3.5) and the product space technique, one has

$$\begin{aligned}\|Tv-Tw\| &= \|J_{\gamma A}(v-\gamma Bv)-J_{\gamma A}(w-\gamma Bw)\|\\ &\leq \|v-\gamma Bv-(w-\gamma Bw)\|\\ &= \|v-\gamma(L(L^*v-u)+Dv)-(w-\gamma(L(L^*w-u)+Dw))\|\\ &= \|(1-\gamma LL^*)v-(1-\gamma LL^*)w-\gamma Dv+\gamma Dw\|\\ &\leq \|1-\gamma LL^*\|\|v-w\|+\gamma\|Dv-Dw\|\\ &\leq (\|1-\gamma LL^*\|+\frac{\gamma}{\min_{1\leq i\leq m}\{v_i\}})\|v-w\|\\ &= \bar{\beta}\|v-w\|,\end{aligned}$$

where the last inequality comes from the fact that D is $\min_{1\leq i\leq m}\{v_i\}$-cocoercive and $\frac{1}{\min_{1\leq i\leq m}\{v_i\}}$-Lipschitz continuous. So, by (i), T is Lipschitz continuous with constant $\bar{\beta}\in[0,1)$. Furthermore, in view of (3.5), we let $T^{'}=(1-\lambda_k)I+\lambda_k T$. Then, for any v,w,

$$\begin{aligned}\|T^{'}v-T^{'}w\| &= \|(1-\lambda_k)v+\lambda_k Tv-((1-\lambda_k)w+\lambda_k Tw)\|\\ &\leq (1-\lambda_k)\|v-w\|+\lambda_k\|Tv-Tw\|\\ &\leq (1-(1-\bar{\beta})\lambda_k)\|v-w\|.\end{aligned}$$

Therefore, it follows from (ii) that $T^{'}$ is $(1-(1-\bar{\beta})\lambda_k)$-Lipschitz continuous. For the sake of convenience, let $\hat{\beta}=1-(1-\bar{\beta})\lambda_k$, $\hat{\beta}\in(0,1)$. In view of [17, Theorem 1.50 (iv)], we obtain the priori error estimate

$$\|v_n-v^*\|\leq\frac{\hat{\beta}^n}{1-\hat{\beta}}\|v_0-v_1\|.$$

Furthermore, it follows from (3.5) that

$$\begin{aligned}\|x_n-x^*\| &= \|u-\sum_{i=1}^m L_i^*v_i^n-u+\sum_{i=1}^m L_i^*v_i^*\|\\ &\leq \frac{\|L\|\hat{\beta}^n}{1-\hat{\beta}}\|v_0-v_1\|.\end{aligned}$$

That is, we have the convergence rate $\|x_n-x^*\|=\mathscr{O}(\hat{\beta}^n)$. According to the [17, Theorem 1.50 (v)], we also obtain the posteriori error estimate

$$\|v_n-v^*\|\leq\frac{1}{1-\hat{\beta}}\|v_n-v_{n+1}\|.$$

Thus,

$$\begin{aligned}\|x_n-x^*\| &\leq \frac{\|L\|}{1-\hat{\beta}}\|v_n-v_{n+1}\|\\ &\leq \|1-\gamma LL^*\|\|x-y\|+\gamma\|Dx-Dy\|\\ &\leq (\|1-\gamma LL^*\|+\frac{\gamma}{\min_{1\leq i\leq n}\{v_i\}})\|x-y\|\\ &= \bar{\beta}\|x-y\|,\end{aligned}$$

where the last inequality comes from the fact that D is $\min_{1\leq i\leq m}\{v_i\}$-cocoercive and $\frac{1}{\min_{1\leq i\leq m}\{v_i\}}$-Lipschitz. So, T is Lipschitz continuous with constant $\bar{\beta}\in[0,1)$. Therefore, it follows from the [17, Theorem 1.50] that

$$\|x_n-x^*\|\leq\frac{\bar{\beta}^n}{1-\bar{\beta}}\|x_0-x_1\|.$$

That is, we have the convergence rate $\|x_n-x^*\|=\mathscr{O}(\bar{\beta}^n)$. This completes the proof. □

4. Dual Three-Operator Splitting Algorithms

In this section, we focus on illustrating our main iterative algorithms to solve composite monotone inclusion (2.1). To solve monotone inclusion Problem 3.1, we consider the inexact three-operator splitting algorithm. The following inexact three-operator iteration sequence can be found in [16].

$$\begin{cases} x_B^k = J_{\gamma B}(z^k)+e_B^k; \\ x_A^k = J_{\gamma A}(2x_B^k-z^k-\gamma(Cx_B^k+e_C^k))+e_A^k; \\ z^{k+1}=z^k+\lambda_k(x_A^k-x_B^k). \end{cases}$$

Therefore, combining the inexact three-operator splitting algorithm and composite monotone inclusion (2.1), we obtain the following iterative sequence:

$$\begin{cases} x_B^k = J_{\gamma\sum_{i=1}^m L_i^*(B_i\square D_i)(L_i\cdot-r_i)}(z^k)+e_B^k; \\ x_A^k = J_{\gamma A}(2x_B^k-z^k-\gamma(Cx_B^k+e_C^k))+e_A^k; \\ z^{k+1}=z^k+\lambda_k(x_A^k-x_B^k), \end{cases} \tag{4.1}$$

for each $k\geq 0$. Although resolvent operator $J_{\gamma\sum_{i=1}^m L_i^*(B_i\square D_i)(L_i\cdot-r_i)}$ usually does not have an explicit form of the solution, we can construct an effective algorithm to solve it based on Theorem 4.1. In details, the iterative sequence $\{x_B^k\}$ can be obtained via the following way. Let $k\geq 0$ and $v_0\in G$. For each $j_k\geq 0$, define

$$\begin{cases} x^{j_k}=z^k-\sum_{i=1}^m L_i^* v_i^{j_k}; \\ v_i^{j_{k+1}}=v_i^{j_k}+\bar{\lambda}_k(J_{\gamma_k(\gamma B_i)^{-1}}(v_i^{j_k}-\gamma_k(-L_i x^{j_k}+(\gamma D_i)^{-1}v_i^{j_k}+r_i))-v_i^{j_k}). \end{cases} \tag{4.2}$$

If $\{x^{j_k}\}$ converges, then stop. Therefore, let $x_B^k=x^{J_k}$, where J_k is the iteration numbers when $\{x^{j_k}\}$ converges. Combining (4.1) and (4.2), we obtain the following iterative algorithm to solve composite monotone inclusion problem (2.1).

Next, we prove the convergence of Algorithm 1.

Theorem 4.1. *Let $A:H\to 2^H$ be maximal monotone. Let $C:H\to H$ be μ-cocoercive for some $\mu>0$. For any $i=1,\cdots,m$, let $B_i:G_i\to 2^{G_i}$ be maximally monotone, $D_i:G_i\to 2^{G_i}$ be maximally monotone and v_i-strongly monotone, where $v_i\geq 0$, and let $L_i:H\to G_i$ be a nonzero bounded linear operator. Assume that $zer(A+\sum_{i=1}^m L_i^*(B_i\square D_i)(L_i\cdot-r_i)+C)$ is nonempty and the following conditions are satisfied:*

(a) *$\gamma\in(0,2\mu\varepsilon)$ and $\varepsilon\in(0,1)$;*

(b) *For some $\underline{\lambda}>0,\lambda_k\in(\underline{\lambda},\frac{1}{\alpha})$ with $\sum_{k=0}^{+\infty}\lambda_k(\frac{1}{\alpha}-\lambda_k)=+\infty$, where $\alpha=\frac{1}{2-\varepsilon}$;*

Let $\alpha_k=\frac{2\beta}{4\beta-\gamma_k}$. If $\{\bar{\lambda}_k\}$ and $\{\gamma_k\}$ satisfy one of the following conditions:

Algorithm 1 The first type of the dual three-operator splitting algorithm for solving (2.1)

Input: For any z^0, v_0, choose the parameters $\{\lambda_k\}$, $\{\gamma_k\}$, γ and $\{\bar{\lambda}_k\}$.
(Outer iteration step) For $k = 0,1,2,\cdots$, do
1: (Inner iteration step) For $j_k = 0,1,2,\cdots$,
$x^{j_k} = z^k - \sum_{i=1}^m L_i^* v_i^{j_k}$,
$v_i^{j_k+1} = v_i^{j_k} + \bar{\lambda}_k(\gamma J_{\frac{\gamma_k}{\gamma}B_i^{-1}}(\frac{1}{\gamma}v_i^{j_k} - \frac{\gamma_k}{\gamma}(-L_i x^{j_k} + \frac{1}{\gamma}D_i^{-1}v_i^{j_k} + r_i)) - v_i^{j_k})$;
Stop the inner iteration until a given stopping criterion is met. Let $x_B^k = x^{J_k}$.
2: $x_A^k = J_{\gamma A}(2x_B^k - z^k - \gamma(Cx_B^k + e_C^k)) + e_A^k$;
3: $z^{k+1} = z^k + \lambda_k(x_A^k - x_B^k)$.
Stop the outer iteration until a given stopping criterion is met.
Output: x_B^k, x_A^k and z^{k+1}.

(d) $0 < \underline{\lambda} \leq \bar{\lambda}_k \leq \frac{1}{\alpha_k} - \tau$, *where* $\tau \in (0, \frac{1}{\alpha_k} - \underline{\lambda})$ *and* $0 < \underline{\gamma} \leq \gamma_k \leq 2\beta - \varepsilon$, *where* $\varepsilon \in (0, 2\beta - \underline{\gamma})$.

(e) $\sum_{k=0}^{+\infty} \bar{\lambda}_k(\frac{1}{\alpha_k} - \bar{\lambda}_k) = +\infty$, $\sum_{k=0}^{+\infty} |\gamma_{k+1} - \gamma_k| < +\infty$ *and* $\bar{\lambda}_k \geq \underline{\lambda} > 0, 0 < \underline{\gamma} < \gamma_k \leq 2\beta - \varepsilon$, *where* $\varepsilon \in (0, 2\beta - \underline{\gamma})$.

(f) $\{e_A^k\}$ *and* $\{e_C^k\}$ *are two absolutely summable sequences in* H, *i.e.,* $\sum_{k=0}^{+\infty} \|e_A^k\| < +\infty$ *and* $\sum_{k=0}^{+\infty} \|e_C^k\| < +\infty$,

then the following conclusions hold:

(1) $\{x_B^k\}$ *and* $\{x_A^k\}$ *converge weakly to a solution of the composite monotone inclusion (2.1);*

(2) *if one of the following conditions holds:*

(g) *A is uniformly monotone on every nonempty bounded subset of domA;*

(h) *B is uniformly monotone on every nonempty bounded subset of domB;*

(i) *C is demiregular at every point* $x \in zer(A + \sum_{i=1}^m L_i^*(B_i \Box D_i)(L_i \cdot - r_i) + C - z)$, *then* $\{x_A^k\}$ *and* $\{x_B^k\}$ *converge strongly to a point in* $zer(A + \sum_{i=1}^m L_i^*(B_i \Box D_i)(L_i \cdot - r_i) + C - z)$.

Proof. For any fixed $k \geq 0$, let $z_B^k = J_{\gamma \sum_{i=1}^m L_i^*(B_i \Box D_i)(L_i \cdot - r_i)} z^k$. In view of Theorem 4.1, one has that $\{x^{j_k}\}$ converges strongly to resolvent $J_{\gamma \sum_{i=1}^m L_i^*(B_i \Box D_i)(L_i \cdot - r_i)} z^k$, that is, $x^{j_k} \to J_{\gamma \sum_{i=1}^m L_i^*(B_i \Box D_i)(L_i \cdot - r_i)} z^k$ as $j_k \to +\infty$. Then, for any $k^q, q > 1$, there exists j_{k_0} for $j_k > j_{k_0}$ such that $\|x^{j_k} - z_B^k\| \leq \frac{1}{k^q}$. Thus the error between $\{x_B^k\}$ and $J_{\gamma \sum_{i=1}^m L_i^*(B_i \Box D_i)(L_i \cdot - r_i)} z^k$ satisfies

$$\sum_{k=0}^{+\infty} \|x_B^k - J_{\gamma \sum_{i=1}^m L_i^*(B_i \Box D_i)(L_i \cdot - r_i)} z^k\| \leq \sum_{k=0}^{+\infty} \frac{1}{k^q} < +\infty.$$

From [16, Theorem 3.1], we can conclude the proof of this theorem immdediately. $\square$

It is interesting that if we exchange the order of A and B in the inexact three-operator splitting algorithm, we could obtain a different algorithm to solve composite monotone inclusion problem (2.1). In three-operator inclusion problem [15], let C be the same as in (2.1), $A = \sum_{i=1}^m L_i^*(B_i \Box D_i)(L_i \cdot - r_i)$, and $B = A$, where C, $\sum_{i=1}^m L_i^*(B_i \Box D_i)(L_i \cdot - r_i)$ and A are defined in (2.1). Therefore, for each $k \geq 1$, we have the following iterative sequence:

$$\begin{cases} x_A^k = J_{\gamma A}(z^k) + e_A^k; \\ x_B^k = J_{\gamma \sum_{i=1}^m L_i^*(B_i \square D_i)(L_i \cdot - r_i)}(2x_A^k - z^k - \gamma(Cx_A^k + e_C^k)) + e_B^k; \\ z^{k+1} = z^k + \lambda_k(x_B^k - x_A^k). \end{cases} \tag{4.3}$$

Similar to the derivation of Algorithm 1, for resolvent operator $J_{\gamma \sum_{i=1}^m L_i^*(B_i \square D_i)(L_i \cdot - r_i)}$, we can still construct an effective iterative algorithm based on Theorem 4.1. The detailed iteration sequence $\{x_B^k\}$ is given by the following way. Let $k \geq 1$ and $v_0 \in G$. For each $j_k \geq 0$,

$$\begin{cases} x^{j_k} = 2x_A^k - z^k - \gamma(Cx_A^k + e_C^k) - \sum_{i=1}^m L_i^* v_i^{j_k}; \\ v_i^{j_{k+1}} = v_i^{j_k} + \bar{\lambda}_k(J_{\gamma_k(\gamma B_i)^{-1}}(v_i^{j_k} - \gamma_k(-L_i(x^{j_k}) + (\gamma D_i)^{-1} v_i^{j_k} + r_i)) - v_i^{j_k}). \end{cases} \tag{4.4}$$

If $\{x^{j_k}\}$ converges, then stop. Therefore, let $x_B^k = x^{J_k}$, where J_k is the iteration numbers when $\{x^{j_k}\}$ converges. Combining (4.3) and (4.4), we see that the iterative algorithm is obtained as follows:

Algorithm 2 The second type of the dual three-operator splitting algorithm for solving (2.1)

Input: For any $z^0, y_0 \in H$, choose parameters $\{\lambda_k\}$, $\{\gamma_k\}$, γ and $\{\bar{\lambda}_k\}$.
(Outer iteration step) For $k = 0, 1, 2, \cdots$, do
1: $x_A^k = J_{\gamma A}(z^k) + e_A^k$;
2: (Inner iteration step) For $j_k = 0, 1, 2, \cdots$,
$x^{j_k} = 2x_A^k - z^k - \gamma(Cx_A^k + e_C^k) - \sum_{i=1}^m L_i^* v_i^{j_k}$;
$v_i^{j_{k+1}} = v_i^{j_k} + \bar{\lambda}_k(\gamma J_{\frac{\gamma_k}{\gamma} B_i^{-1}}(\frac{1}{\gamma} v_i^{j_k} - \frac{\gamma_k}{\gamma}(-L_i x^{j_k} + \frac{1}{\gamma} D_i^{-1} v_i^{j_k} + r_i)) - v_i^{j_k})$.
Stop the inner iteration until a given stopping criterion is met. Let $x_B^k = x^{J_k}$.
3: $z^{k+1} = z^k + \lambda_k(x_B^k - x_A^k)$.
Stop the outer iteration until a given stopping criterion is met.
Output: x_A^k, x_B^k and z^{k+1}.

Also, we can obtain the convergence of Algorithm 2. Since the proof is similar to Theorem 4.1, we omit it here.

Theorem 4.2. *Let $A : H \to 2^H$ be maximal monotone. Let $C : H \to H$ be μ-cocoercive for some $\mu > 0$. For any $i = 1, \cdots, m$, let $B_i : G_i \to 2^G$ be maximally monotone, $D_i : G_i \to 2^{G_i}$ be maximally monotone and ν_i-strongly monotone, where $\nu_i \geq 0$, and let $L_i : H \to G_i$ be a nonzero bounded linear operator. Assume that $zer(A + \sum_{i=1}^m L_i^*(B_i \square D_i)(L_i \cdot - r_i) + C)$ is nonempty, and the following conditions are satisfied:*

(a) $\gamma \in (0, 2\mu\varepsilon)$ *and* $\varepsilon \in (0, 1)$.

(b) *For some* $\underline{\lambda} > 0, \lambda_k \in (\underline{\lambda}, \frac{1}{\alpha})$ *with* $\sum_{k=0}^{+\infty} \lambda_k(\frac{1}{\alpha} - \lambda_k) = +\infty$, *where* $\alpha = \frac{1}{2-\varepsilon}$.

Let $\alpha_k = \frac{2\beta}{4\beta - \gamma_k}$. *If* $\{\bar{\lambda}_k\}$ *and* $\{\gamma_k\}$ *satisfy one of the following conditions:*

(d) $0 < \underline{\lambda} \leq \bar{\lambda}_k \leq \frac{1}{\alpha_k} - \tau$, *where* $\tau \in (0, \frac{1}{\alpha} - \underline{\lambda})$ *and* $0 < \underline{\gamma} \leq \gamma_k \leq 2\beta - \varepsilon$, *where* $\varepsilon \in (0, 2\beta - \underline{\gamma})$*;*

(e) $\sum_{k=0}^{+\infty}\bar{\lambda}_k(\frac{1}{\alpha_k}-\bar{\lambda}_k)=+\infty$, $\sum_{k=0}^{+\infty}|\gamma_{k+1}-\gamma_k|<+\infty$ *and* $\bar{\lambda}_k\geq\underline{\lambda}>0, 0<\underline{\gamma}<\gamma_k\leq 2\beta-\varepsilon$, *where* $\varepsilon\in(0,2\beta-\underline{\gamma})$;

(f) $\{e_A^k\}$ *and* $\{e_C^k\}$ *are two absolutely summable sequences in* H, *i.e.,* $\sum_{k=0}^{+\infty}\|e_A^k\|<+\infty$ *and* $\sum_{k=0}^{+\infty}\|e_C^k\|<+\infty$,

then the following conclusion hold:

(1) $\{x_B^k\}$ *and* $\{x_A^k\}$ *converge weakly to a solution of the composite monotone inclusion (2.1);*

(2) *if one of the following conditions holds:*

(g) A *is uniformly monotone on every nonempty bounded subset of* $domA$;

(h) B *is uniformly monotone on every nonempty bounded subset of* $domB$;

(i) C *is demiregular at every point* $x\in zer(A+\sum_{i=1}^m L_i^*(B_i\Box D_i)(L_i\cdot-r_i)+C-z)$, *then* $\{x_A^k\}$ *and* $\{x_B^k\}$ *converge strongly to a point in* $zer(A+\sum_{i=1}^m L_i^*(B_i\Box D_i)(L_i\cdot-r_i)+C-z)$.

5. Applications to Convex Optimization Problems

In this section, we employ the results obtained in the previous section to solve the following convex optimization problem:

$$\min_{x\in H} f(x)+\sum_{i=1}^m((g_i\Box q_i)(L_ix-r_i))+h(x), \tag{5.1}$$

where $f\in\Gamma_0(H)$, $q_i\in\Gamma_0(G_i)$, $r_i\in G_i$, and q_i are functions in $\Gamma_0(G_i)$ with v_i-strongly convex with $v_i\geq 0$, $h:H\to\mathbb{R}$ is convex differentiable with a $\frac{1}{\mu}$-Lipschitz continuous gradient, and $L_i:H\to G_i$ is a nonzero bounded linear operator.

Then the first-order optimality condition of problem (5.1) reduces to (2.1). More precisely, by Fermat's lemma, let x^* be a solution of problem (5.1). Then, under qualification conditions,

$$0\in\partial f(x^*)+\sum_{i=1}^m L_i^*(\partial g_i\Box\partial q_i)(L_ix^*-r_i)+\nabla h(x^*). \tag{5.2}$$

Therefore, it is sufficient to take $A=\partial f$, $B_i=\partial g_i$, $D_i=\partial q_i$, $C=\nabla h$ in monotone inclusion problem (2.1). Due to the equivalence of problems (5.2) and (2.1), we employ the theoretical results presented in Section 4 to solve optimization problem (5.1). Therefore, we develop two efficient iterative algorithms. The first one is demonstrated in Algorithm 3.

Algorithm 3 The first type of the inexact three-operator splitting algorithm for solving (5.1)

Input: For any z^0, v_0, choose the parameters $\{\lambda_k\}$, $\{\gamma_k\}$, γ and $\{\bar{\lambda}_k\}$.

(Outer iteration step) For $k=0,1,2,\cdots$, do

1: (Inner iteration step) For $j_k=0,1,2,\cdots$,

$x^{j_k}=z^k-\sum_{i=1}^m L_i^*v_i^{j_k}$,

$v_i^{j_{k+1}}=v_i^{j_k}+\bar{\lambda}_k(\gamma prox_{\frac{\gamma_k}{\gamma}g_i^*}(\frac{1}{\gamma}v_i^{j_k}-\frac{\gamma_k}{\gamma}(-L_ix^{j_k}+\frac{1}{\gamma}\nabla q_i^*(v_i^{j_k})+r_i))-v_i^{j_k})$;

Stop the inner iteration until a given stopping criterion is met. Let $x^k=x^{J_k}$.

2: $y^k=prox_{\gamma f}(2x^k-z^k-\gamma(\nabla h(x^k)+e_C^k))+e_A^k$;

3: $z^{k+1}=z^k+\lambda_k(y^k-x^k)$.

Stop the outer iteration until a given stopping criterion is met.

Output: x^k, y^k and z^{k+1}.

As a direct application of Theorem 4.1, we obtain the convergence of Algorithm 3 in the following theorem.

Theorem 5.1. *Let f be function in $\Gamma_0(H)$ and $z \in H$. Let $h : H \to \mathbb{R}$ be convex differentiable with a $\frac{1}{\mu}$-Lipschitz continuous gradient. For any $i = 1, \cdots, m$, let g_i be a function in $\Gamma_0(G_i)$, and let q_i be function in $\Gamma_0(G_i)$ with v_i- strongly convex, where $v_i \geq 0$. Let $L_i : H \to G_i$ be a nonzero bounded linear operator and $r_i \in G_i$. Assume that $zer(\partial f + \sum_{i=1}^{m} L_i^*(\partial g_i \square \partial q_i)(L_i \cdot - r_i) + \nabla h)$ is nonempty. Let $\{(x^k, y^k)\}$ be the sequence generated by Algorithm 3, where the iterative parameters satisfy the conditions of Theorem 4.1. Then the following conclusions hold:*

(1) *$\{x^k\}$ and $\{y^k\}$ converge weakly to a solution of the convex minimization problem (5.1);*

(2) *if one of the following conditions holds:*

(g) *f is uniformly convex on every nonempty bounded subset of $dom\partial f$;*

(h) *g is uniformly convex on every nonempty bounded subset of $dom\partial g$;*

(i) *∇h is demiregular at every point of $zer(\partial f + \sum_{i=1}^{m} L_i^*(\partial g_i \square \partial q_i)(L_i \cdot - r_i) + \nabla h - z)$,*

then $\{x^k\}$ and $\{y^k\}$ converge strongly to a solution of the problem (5.1).

Proof. Since the subdifferential of a proper, convex, and lower semi-continuous function is a maximal monotone operator, then ∂f, ∂g_i and ∂q_i are maximal monotone. On the other hand, by Lemma 2.4 and Lemma 2.5, ∇h is μ-cocoercive, and ∂q_i is v_i-strongly convex. Therefore, we can draw from Theorem 4.1 the conclusions of Theorem 5.1 immediately. □

Similarly, we can exchange the order of ∂f and ∂g in the inexact three-operator splitting algorithm (Algorithm 3), and obtain a different algorithm to solve convex optimization problem (5.1). The detail of the iterative algorithm is illustrated as follows:

Algorithm 4 The second type of the inexact three-operator splitting algorithm for solving (5.1)

Input: For any $z^0, y_0 \in H$, choose the parameters $\{\lambda_k\}$, $\{\gamma_k\}$, γ and $\{\bar{\lambda}_k\}$.

For $k = 0, 1, 2, \cdots$, do

1: $x^k = prox_{\gamma f}(z^k) + e_A^k$,

2: Inner iteration: for $j_k = 0, 1, 2, \cdots$,

$x^{j_k} = 2x^k - z^k - \gamma(\nabla h(x^k) + e_C^k) - \sum_{i=1}^{m} L_i^* v_i^{j_k}$;

$v_i^{j_{k+1}} = v_i^{j_k} + \bar{\lambda}_k(\gamma prox_{\frac{\gamma_k}{\gamma} g_i^*}(\frac{1}{\gamma} v_i^{j_k} - \frac{\gamma_k}{\gamma}(-L_i x^{j_k} + \frac{1}{\gamma}\nabla q_i^*(v_i^{j_k}) + r_i)) - v_i^{j_k})$.

Stop the inner iteration until a given stopping criterion is met. Let $y^k = x^{J_k}$.

3: $z^{k+1} = z^k + \lambda_k(y^k - x^k)$.

Stop the outer iteration until a given stopping criterion is met.

Output: x^k, y^k and z^{k+1}.

Similar to Theorem 5.1, we can also prove the convergence of Algorithm 4. The following Theorem comes directly from Theorem 4.2.

Theorem 5.2. *Let f be a function in $\Gamma_0(H)$ and $z \in H$. Let $h : H \to \mathbb{R}$ be convex differentiable with a $\frac{1}{\mu}$-Lipschitz continuous gradient. For any $i = 1, \cdots, m$, let g_i be a function in $\Gamma_0(G_i)$, and let q_i be a function in $\Gamma_0(G_i)$ with v_i- strongly convex, where $v_i \geq 0$. Let $L_i : H \to G_i$ be a nonzero bounded linear operator and $r_i \in G_i$. Assume that $zer(\partial f + \sum_{i=1}^{m} L_i^*(\partial g_i \square \partial q_i)(L_i \cdot - r_i) + \nabla h)$ is nonempty. Let $\{(x^k, y^k)\}$ be the sequence generated by Algorithm 4, where the iterative parameters satisfy the conditions of Theorem 4.1. Then the following conclusions hold:*

(1) $\{x^k\}$ *and* $\{y^k\}$ *converge weakly to a solution of the convex minimization problem (5.1);*

(2) *if one of the following conditions holds:*

(g) f *is uniformly convex on every nonempty bounded subset of* $dom\partial f$*;*

(h) g *is uniformly convex on every nonempty bounded subset of* $dom\partial g$*;*

(i) ∇h *is demiregular at every point of* $zer(\partial f + \sum_{i=1}^{m} L_i^*(\partial g_i \Box \partial q_i)(L_i \cdot -r_i) + \nabla h - z)$*, then* $\{x^k\}$ *and* $\{y^k\}$ *converge strongly to a solution of problem (5.1).*

Remark 5.1. (1) In [21], Yan proposed a primal-dual three-operator (PD3O) algorithm to solve the problem (5.1) with $m = 1$ and $r_i = 0$, and the algorithm required $\gamma_k \in (0, \frac{1}{\|L\|^2})$. In contrast to Algorithm 4, we relax to $\gamma_k \in (0, \frac{2}{\|L\|^2})$. Therefore, we provide wider selection of parameter γ_k than Yan [21]. Moreover, we use relaxation strategies, which are absent in the PD3O. Compared with the PD3O, Algorithm 4 is a two-loop iterative algorithm, while the former is a fully splitting algorithm without the inner iteration.

(2) Algorithm 4 extends the corresponding algorithm in Tang et al. [22] in two folds:

(a) we use the relaxation strategies to accelerate the algorithm. The algorithm in [22] is a special case of Algorithm 4 when $r_i = 0$, $m = 1$, and $\bar{\lambda}_k = \lambda_k = 1$;

(b) we prove the weak convergence of Algorithm 4 in infinite-dimensional Hilbert spaces, while the corresponding algorithm in [22] was only proved in finite-dimensional Hilbert spaces.

(3) Algorithm 3 is a new iterative algorithm for solving the minimization of the sum of three convex functions problem (5.1), which is not covered by existing algorithms. It is an interesting topic if we can prove the theoretical convergence of Algorithm 3 when the number of inner iteration is one.

6. Numerical Experiments

In this section, we demonstrate the efficiency of the proposed iterative algorithms in the context of image deblurring. All the experiments are accomplished by Matlab and on a standard HP laptop with Intel(R) Core(TM) i7-6500U 2.50GHz CPU and 8GB RAM.

We aim at solving the following constrained total variation (TV) minimization problem, which is widely used in image deblurring problem:

$$\min_{x\in C} \frac{1}{2}\|Ax-b\|_2^2 + \mu\|x\|_{TV}, \tag{6.1}$$

where $\mu > 0$ is the regularization parameter, $\|x\|_{TV}$ is the total variation term, $A \in R^{m\times n}$ is the blurring matrix, b is the recorded blurred image data, x is the original image which is unknown, and C is a nonempty closed convex set, which represents a prior information of the deblurring images. Here, we choose C as a nonnegative constraint set, i.e., $C = \{x \in R^n : x_i \geq 0, i = 1, 2, \cdots, n\}$. Constrained optimization problem (6.1) is equivalent to the unconstrained optimization problem:

$$\min \frac{1}{2}\|Ax-b\|_2^2 + \mu\|x\|_{TV} + \delta_C(x), \tag{6.2}$$

where $\delta_C(x)$ denotes the indicator function, which is 0 if $x \in C$; or $+\infty$, otherwise.

Since the total variation $\|x\|_{TV}$ including the isotropic total variation and anisotropic total variation can be represented by a combination of a convex function φ with a first order difference matrix L, i.e., $\|x\|_{TV} = \varphi(Lx)$ (see, for example [23, 24, 25]), then optimization problem (6.2) is a special case of composite optimization problem (5.1). For the proposed Algorithms

3 and 4, there are four parameters, which are listed in Table 1. The parameters λ_k and $\bar{\lambda}_k$ are usually called relaxation parameters.

TABLE 1. The selection range of parameters for Algorithms 3 and 4.

λ_k	γ	$\bar{\lambda}_k$	γ_k
$(0, 2-\varepsilon)$	$(0, \frac{2}{\|A\|^2}\varepsilon)$	$(0, \frac{4-\lambda\|L\|^2}{2})$	$(0, \frac{2}{\|L\|^2})$

In Table 1, $\|A\|$ is estimated via the power iteration method and $\|L\|^2 = 8$ for the total variation according to the estimation made in [23]. Since the inexact three-operator splitting algorithm is a fixed point algorithm, we do not use primal-dual gaps as stopping criteria. We define the relative error between two successive iterative sequences which is less than a prescribed tolerance value as the stopping criteria. That is,

$$Rel = \frac{\|z^{k+1} - z^k\|}{\|z^k\|} \leq \varepsilon,$$

where ε is a given small number. To evaluate the quality of the deblurring images, the signal-to-noise ratio (SNR), and the normalized mean square distance (NMSD), two evaluation indicators commonly used in image processing, are employed, which are defined by

$$SNR = 10log\frac{\|x\|^2}{\|x - x_r\|^2}$$

and

$$NMSD = \frac{\|x_r - x\|}{\|x\|},$$

where x_r is the deblurring image, and x is the clean image. Generally, the larger the SNR of the deblurring image, the better the deblurring image quality is. However, unlike SNR, the closer the value of NMSD is to 0, the better the image deblurring effect is.

We compare the proposed algorithms with PD3O [21], PDFP [26], and Condat [10] in solving deblurring problem (6.2). All methods are applied to the images of "Barbara", "Goldhill", "Boat", and "Aerial", which the pixels of each image are 512×512. In the overall experiments, in order to verify the rationality and effectiveness of the proposed algorithms, we consider the test images "Barbara" and "Aerial" that go through a $5*5$ mean filter followed by random normally distributed noise with mean zero and variance 225, while the test images "Goldhill" and "Boat" adopt a $5*5$ mean filter and a random normally distributed noise with mean zero and variance 400. Meanwhile, all the methods are compared by setting the same parameters and the regularization parameter $\mu = 3.8$ is used. Generally speaking, the parameter γ should be chosen as large as possible to get a fast convergence, we choose $\gamma = \frac{1.5}{L}$ in all iterative algorithms, where L is the Lipschitz constant of ∇h. Without loss of generality, we stop the iterative algorithms when the relative errors are less than 10^{-3}, 10^{-4} and 10^{-5} respectively or the maximum number of iteration exceeds 40000. Next, we verify the impacts of the inner iterations for Algorithms 3 and 4, and observe that our algorithms have advantages over other algorithms with the same parameters.

First, we show how the proposed iterative algorithms are affected by the inner iterations. A Gauss-Seidel idea is adopted to update the initial value of the inner iterations. More precisely, we use the sequence obtained by solving the previous subproblem as the initial value of the

FIGURE 6.1. (a) The original Barbara with the size of 512×512. (b) The original Goldhill with the size of 512×512. (c) The original Boat with the size of 512×512. (d) The original Aerial with the size of 512×512.

TABLE 2. Numerical results of Algorithms 3 and 4 for different choices of inner iterations for "Barbara" image.

Inner iterations	Method	$\varepsilon = 10^{-3}$			$\varepsilon = 10^{-4}$			$\varepsilon = 10^{-5}$		
		NMSD	*SNR(dB)*	*Iter*	*NMSD*	*SNR(dB)*	*Iter*	*NMSD*	*SNR(dB)*	*Iter*
4	Algorithm 3	0.1308	17.6665	22	0.1320	17.5912	69	0.1322	17.5781	172
	Algorithm 4	0.1309	17.6619	22	0.1320	17.5908	69	0.1322	17.5780	173
6	Algorithm 3	0.1307	17.6774	20	0.1319	17.5930	66	0.1322	17.5782	170
	Algorithm 4	0.1307	17.6721	20	0.1319	17.5925	66	0.1322	17.5782	170
8	Algorithm 3	0.1306	17.6839	19	0.1319	17.5938	65	0.1322	17.5782	170
	Algorithm 4	0.1306	17.6782	19	0.1319	17.5933	65	0.1322	17.5782	170
10	Algorithm 3	0.1306	17.6844	19	0.1319	17.5940	65	0.1322	17.5782	170
	Algorithm 4	0.1306	17.6844	18	0.1319	17.5935	65	0.1322	17.5782	170

next subproblem. The inner iteration numbers are selected as 4, 6, 8, and 10 respectively, and we illustrate the numerical experiment results listed in Table 2 for the "Barbara" image. In this case, we choose $\lambda_k = 1$ and $\bar{\lambda}_k = 1$. Similar to Yan [21] and Chen [26], we set $\gamma_k = \frac{1}{8}$. It follows from Table 2 that demonstrates the performance of Algorithms 3 and 4 with different choices of inner iterations, which not only report the SNR and NMSD, but also illustrate the outer iteration number when $Rel \leq \varepsilon$. It can be noted from Table 2 that the performance of Algorithms 3 and 4 is nearly the same no matter how many inner iterations chosen. More specifically, we observe that the SNR will increase a little, while the outer iteration number will have a slight reduction as the number of inner iteration increases when the relative error is less than 10^{-3}. However, as the accuracy of relative error is higher, the SNR, NMSD, and outer iteration numbers are not affected by the inner iteration. This confirms that the usefulness of Algorithms 3 and 4 is not greatly affected by the inner iterations.

To further verify the effectiveness of the Algorithms 3 and 4, Table 3 gives an insight into the performance of the proposed algorithms when compared with the other state-of-the-art algorithms. In this experiment, in order to compare with other algorithms, we choose the same parameters that $\lambda_k = 1$, $\bar{\lambda}_k = 1$, $\gamma_k = \frac{1}{8}$ and the inner iterative number of proposed algorithms be four. Without loss of generality, the experiment will be carried out under various types of images which are shown in Figure 1. It follows from Table 2 that the proposed Algorithms 3 and 4 which are compared with other algorithms lead to the improvement in the convergence rate for the "Barbara" image. More precisely, we observe that the Condat algorithm has the highest SNR, but it has the most iterative numbers and the slowest convergence rate when the relative error is less than 10^{-3}. However, our algorithms have the least number of outer iterations compared to the other three algorithms, and our SNR is higher than PDFP and PD3O. Simultaneously, it can be noted that the NMSD and SNR of the five algorithms are a little different when the relative error is less than 10^{-4} or 10^{-5}, but the iterative numbers of our algorithms are significantly less than the other three algorithms, which implies that our algorithms possess much faster convergence speed with the increase of the accuracy. Similarly, the same experimental analysis results are also shown in Table 3 for different types of other images.

TABLE 3. Numerical results of the compared algorithms.

Image	Method	$\varepsilon = 10^{-3}$			$\varepsilon = 10^{-4}$			$\varepsilon = 10^{-5}$		
		NMSD	*SNR(dB)*	*Iter*	*NMSD*	*SNR(dB)*	*Iter*	*NMSD*	*SNR(dB)*	*Iter*
"Barbara"	PDFP [26]	0.1315	17.6215	34	0.1321	17.5818	103	0.1322	17.5770	251
	PD3O [21]	0.1315	17.6236	34	0.1321	17.5819	103	0.1322	17.5770	251
	Condat [10]	0.1308	17.6687	41	0.1319	17.5920	130	0.1321	17.5787	302
	Algorithm 3	0.1308	17.6665	22	0.1320	17.5912	69	0.1322	17.5781	172
	Algorithm 4	0.1309	17.6619	22	0.1320	17.5908	69	0.1322	17.5780	173
"Goldhill"	PDFP [26]	0.0935	20.5878	35	0.0938	20.5531	98	0.0939	20.5493	227
	PD3O [21]	0.0934	20.5896	35	0.0938	20.5532	98	0.0939	20.5493	227
	Condat [10]	0.0930	20.6309	44	0.0937	20.5628	126	0.0939	20.5512	270
	Algorithm 3	0.0928	20.6484	21	0.0937	20.5672	60	0.0939	20.5512	142
	Algorithm 4	0.0929	20.6423	21	0.0937	20.5666	60	0.0939	20.5511	142
"Boat"	PDFP [26]	0.1178	18.5763	37	0.1188	18.5003	105	0.1190	18.4888	247
	PD3O [21]	0.1178	18.5793	37	0.1188	18.5006	105	0.1190	18.4888	247
	Condat [10]	0.1168	18.6532	48	0.1186	18.5216	141	0.1190	18.4924	329
	Algorithm 3	0.1168	18.6536	26	0.1186	18.5175	79	0.1190	18.4906	198
	Algorithm 4	0.1169	18.6464	26	0.1186	18.5167	79	0.1190	18.4906	198
"Aerial"	PDFP [26]	0.0941	20.5323	31	0.0949	20.4520	93	0.0951	20.4379	226
	PD3O [21]	0.0940	20.5364	31	0.0949	20.4523	93	0.0951	20.4379	226
	Condat [10]	0.0933	20.6036	38	0.0947	20.4741	124	0.0950	20.4417	304
	Algorithm 3	0.0935	20.5847	22	0.0948	20.4663	72	0.0951	20.4395	190
	Algorithm 4	0.0935	20.5847	21	0.0948	20.4655	72	0.0951	20.4394	190

Figure 6.2 to Figure 6.5 show the deblurring images by five types of algorithms in different images when $\varepsilon = 10^{-5}$, $\lambda = \frac{1}{8}$ and the number of inner iterations of our proposed algorithms be four. We observe visually well deblurring images via the proposed Algorithms 3 and 4.

FIGURE 6.2. (a) The blurring image of “Barbara”. (b) The deblurring image by Algorithm 3. (c) The deblurring image by Algorithm 4. (d) The deblurring image by PDFP. (e) The deblurring image by PD3O. (f) The deblurring image by Condat.

FIGURE 6.3. (a) The blurring image of “Goldhill”. (b) The deblurring image by Algorithm 3. (c) The deblurring image by Algorithm 4. (d) The deblurring image by PDFP. (e) The deblurring image by PD3O. (f) The deblurring image by Condat.

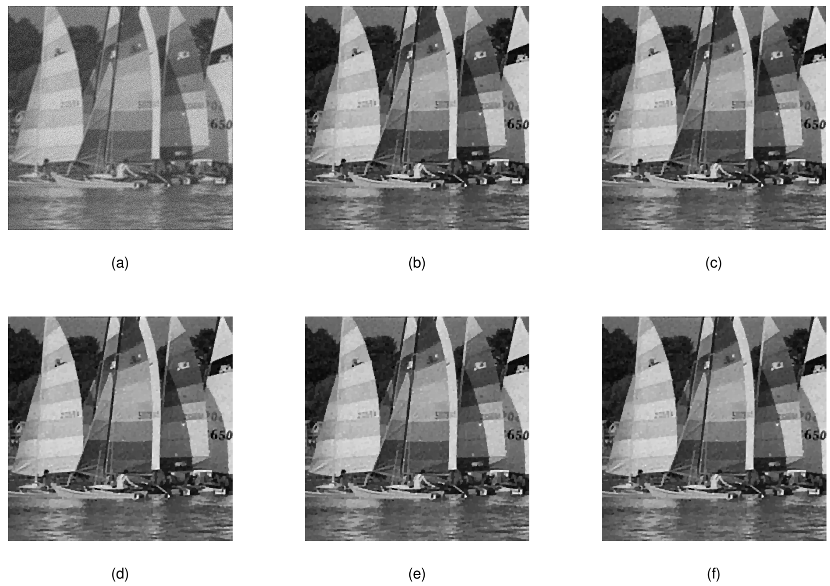

FIGURE 6.4. (a) The blurring image of "Boat". (b) The deblurring image by Algorithm 3. (c) The deblurring image by Algorithm 4. (d) The deblurring image by PDFP. (e) The deblurring image by PD3O. (f) The deblurring image by Condat.

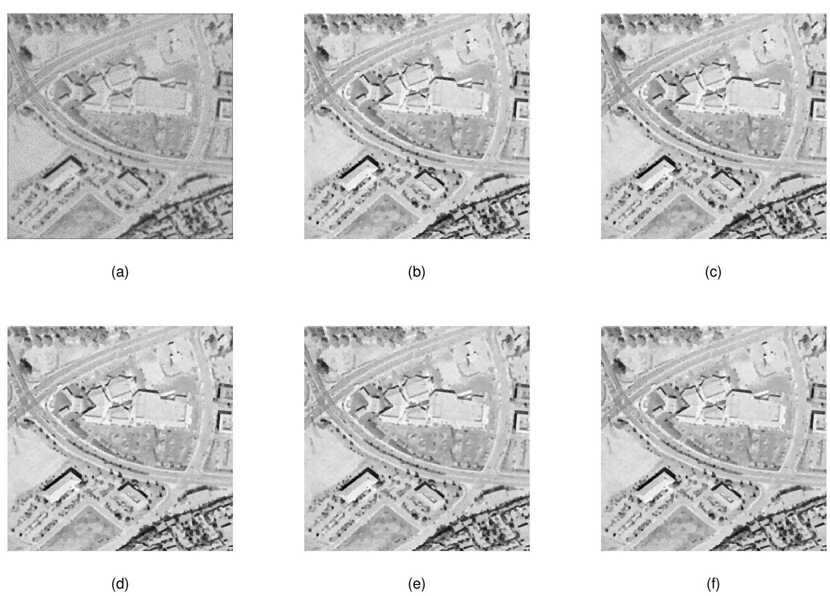

FIGURE 6.5. (a) The blurring image of "Aerial". (b) The deblurring image by Algorithm 3. (c) The deblurring image by Algorithm 4. (d) The deblurring image by PDFP. (e) The deblurring image by PD3O. (f) The deblurring image by Condat.

7. Conclusion

In this paper, we generalized the three-operator splitting algorithm proposed by Davis and Yin [15] to solve a general form of composite monotone inclusion problem (2.1). In particular, we proposed an efficient iterative algorithm for solving the resolvent of operator $\sum_{i=1}^{m} L_i^*(B_i \square D_i)(L_i \cdot -r_i)$ in (3.1), which was combined with the forward-backward splitting algorithm and the dual approach for the convex optimization problem. Subsequently, based on the inexact three-operator splitting algorithm introduced in [16], we investigated several iterative algorithms for solving the composite monotone inclusion problem (2.1), and also demonstrated a convergence rate analysis. Furthermore, we employed the proposed iterative algorithms to solve composite convex optimization problem (5.1), which has been received considerable interest in the fields of variational image restoration and image reconstruction. Finally, we illustrated the efficiency of the proposed Algorithm 3 and Algorithm 4 via numerical experiments on image deblurring with total variation regularization.

Acknowledgments
This work was supported by the National Natural Science Foundations of China (12061045 and 11661056).

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