A NOTE ON MINIMAX OPTIMIZATION PROBLEMS WITH AN INFINITE NUMBER OF CONSTRAINTS

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Abstract. In this paper, employing some advanced tools of variational analysis and generalized differentiation, we establish necessary conditions for locally optimal solutions of minimax optimization problems with infinitely many constraints. Sufficient conditions for such solutions to the considered problem are also provided by introducing generalized convex functions defined in terms of the limiting subdifferential for locally Lipschitz functions. In addition, some duality results for minimax optimization problems with infinitely many constraints are also provided. Furthermore, we derive necessary and sufficient conditions for a weak Pareto solution to the multiobjective optimization problem.

Keywords. Duality; Limiting subdifferential; Minimax problem; Optimality conditions; Semi-infinite optimization.

1. INTRODUCTION

Let X denote the Asplund space (i.e., Banach spaces whose separable subspaces have separable duals), Ω be a nonempty locally closed subset of X, and let T be an arbitrary (possibly *infinite*) index set. Consider the following minimax optimization problem

$$\min_{x \in F} \max_{k \in K} f_k(x), \qquad (P)$$

where the feasible set $F := \{x \in \Omega \mid g_t(x) \leq 0, \ t \in T\}$ is defined by an infinite number of inequality constraints, and the functions f_k, $k \in K := \{1, \ldots, l\}$, and g_t, $t \in T$ are locally Lipschitz on X. Hence, problem (P) becomes a minimax semi-infinite optimization problem. Hereafter, we use the notation $f := (f_1, \ldots, f_l)$, and $g_T := (g_t)_{t \in T}$. In what follows, Ω is always assumed to be SNC at the point under consideration. Fundamental theoretical aspects and a wide range of applications of the semi-infinite optimization have been studied intensively by many researchers recently; see, e.g., [1, 2, 3, 4, 5, 6, 7] and the references therein. Minimax optimization problems have been the subject of immense interest in the past few years. Recently, many researchers studied optimality conditions and duality theorems for minimax optimization problems; see, e.g., [8, 9, 10] and the references therein.

In this paper, we focus on the theoretical aspects for problem (P). By employing some advanced tools of variational analysis and generalized differentiation, we establish necessary conditions for locally optimal solutions of problem (P), under the constraint qualification (CQ). Then, we present sufficient conditions for such solutions for problem (P), by means of introducing generalized convex functions defined in terms of the limiting subdifferential for locally Lipschitz functions. In addition, we introduce a dual problem to the primal one and examine weak, strong and converse duality relations under generalized convexity assumptions. Finally, as an application, we obtain optimality conditions for a weak Pareto solution of a class of semi-infinite multiobjective optimization problems by the obtained results for problem (P).

The rest of the paper is organized as follows. Section 2 provides some preliminaries. In Section 3, we present some results on semi-infinite minimax optimization problems, including necessary conditions for locally optimal solutions and sufficient conditions for globally optimal solutions. Section 4 is devoted to investigating duality relations for semi-infinite minimax optimization problems. The applications are given in Section 5.

2. PRELIMINARIES

In this section, we use the following notations and preliminary results that will be used throughout the paper; see, e.g., [11, 12]. Unless otherwise specified, all spaces under consideration are assumed to be Asplund. The canonical pairing between space X and its topological dual X^* is denoted by $\langle \cdot, \cdot \rangle$, while the symbol $\| \cdot \|$ stands for the norm in the considered space. As usual, the polar cone of a set $\Omega \subset X$ is defined by

$$\Omega^\circ := \{x^* \in X^* \mid \langle x^*, x \rangle \leq 0 \text{ for all } x \in \Omega\}.$$

Given a multifunction $\Psi : X \rightrightarrows X^*$, we denote by

$$\underset{x \to \bar{x}}{\text{Limsup}}\, \Psi(x) := \left\{ x^* \in X^* \mid \exists x_k \to \bar{x} \text{ and } x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in \Psi(x_k) \text{ for all } k \in \mathbb{N} := \{1, 2, \ldots\} \right\}$$

the Painlevé–Kuratowski upper/outer limit of Ψ at $\bar{x}$, where the notation $\xrightarrow{w^*}$ indicates the convergence in the weak* topology of X^*.

Given $\Omega \subset X$, and $\bar{x} \in \Omega$, define the Fréchet/regular normal cone to Ω at $\bar{x}$ by

$$\widehat{N}(\bar{x}; \Omega) := \left\{ x^* \in X^* \,\middle|\, \limsup_{x \xrightarrow{\Omega} \bar{x}} \frac{\langle x^*, x - \bar{x} \rangle}{\|x - \bar{x}\|} \leq 0 \right\},$$

where $x \xrightarrow{\Omega} \bar{x}$ means that $x \to \bar{x}$ with $x \in \Omega$. If $\bar{x} \notin \Omega$, we put $\widehat{N}(\bar{x}; \Omega) := \emptyset$.

The Mordukhovich/limiting normal cone $N(\bar{x}; \Omega)$ to Ω at $\bar{x} \in \Omega \subset X$ is obtained from regular normal cones by taking the sequential Painlevé–Kuratowski upper limits as

$$N(\bar{x}; \Omega) := \underset{x \xrightarrow{\Omega} \bar{x}}{\text{Limsup}}\, \widehat{N}(x; \Omega).$$

If $\bar{x} \notin \Omega$, we put $N(\bar{x}; \Omega) := \emptyset$.

For an extended real-valued function $f : X \to \overline{\mathbb{R}} := [-\infty, \infty]$, its epigraph is defined by

$$\text{epi}\, f := \{(x, \mu) \in X \times \mathbb{R} \mid \mu \geq f(x)\}.$$

The Mordukhovich/limiting subdifferential of f at $\bar{x} \in X$ with $|f(\bar{x})| < \infty$ is defined by

$$\partial f(\bar{x}) := \{x^* \in X^* \mid (x^*, -1) \in N\left((\bar{x}, f(\bar{x})); \operatorname{epi} f\right)\}.$$

If $|f(\bar{x})| = \infty$, then one puts $\partial f(\bar{x}) := \emptyset$.

Considering the indicator function $\delta(\cdot; \Omega)$ defined by $\delta(x; \Omega) := 0$ for $x \in \Omega$ and by $\delta(x; \Omega) := \infty$ otherwise, we have a relation between the Mordukhovich normal cone and the limiting sub-differential of the indicator function as follows (see [11, Proposition 1.79]):

$$\partial \delta(\bar{x}; \Omega) = N(\bar{x}; \Omega) \text{ for all } \bar{x} \in \Omega. \tag{2.1}$$

The nonsmooth version of Fermat's rule is formulated as follows (see [11, Proposition 1.114]): If $\bar{x} \in X$ is a local minimizer of $f : X \to \overline{\mathbb{R}}$, then

$$0 \in \partial f(\bar{x}). \tag{2.2}$$

A set $\Omega \subset X$ is *sequentially normally compact* (SNC) at $\bar{x} \in \Omega$ if for any sequences

$$x_k \xrightarrow{\Omega} \bar{x} \text{ and } x_k^* \xrightarrow{\omega^*} 0 \text{ with } x_k^* \in \widehat{N}(x_k; \Omega),$$

one has $\|x_k^*\| \to 0$ as $k \to \infty$. Obviously, this SNC property is automatically satisfied in finite dimensional spaces.

A function $\psi : X \to \mathbb{R}$ is said to be *sequentially normally compact* (SNC) at $\bar{x} \in X$ if gph ψ is SNC at $(\bar{x}, \psi(\bar{x}))$. According to [11, Corollary 1.69(i)], ψ is SNC at $\bar{x} \in X$ if it is Lipschitz continuous around $\bar{x}$.

Lemma 2.1. [12, Theorem 2.16] *Let Ω_1, $\Omega_2 \subset X$ be such that $\bar{x} \in \Omega_1 \cap \Omega_2$, and that at least one of $\{\Omega_1, \Omega_2\}$ is SNC at this point. If the normal qualification condition*

$$N(\bar{x}; \Omega_1) \cap (-N(\bar{x}; \Omega_2)) = \{0\}$$

is satisfied, then we have the inclusion

$$N(\bar{x}; \Omega_1 \cap \Omega_2) \subset N(\bar{x}; \Omega_1) + N(\bar{x}; \Omega_2).$$

For establishing optimality conditions, the following lemma, which is related to the limiting subdifferential calculus, is very useful.

Lemma 2.2. [11, Theorem 3.36 and Theorem 3.46]

(i) *Let $\phi_i : X \to \overline{\mathbb{R}}$, $i = 1, 2, \ldots, m$, $m \geq 2$ be lower semicontinuous around $\bar{x} \in X$, and let all but one of these functions be Lipschitz continuous around $\bar{x}$. Then*

$$\partial (\phi_1 + \phi_2 + \ldots + \phi_m)(\bar{x}) \subset \partial \phi_1(\bar{x}) + \partial \phi_2(\bar{x}) + \ldots + \partial \phi_m(\bar{x}). \tag{2.3}$$

(ii) *Let $\phi_i : X \to \overline{\mathbb{R}}$, $i = 1, 2, \ldots, m$, $m \geq 2$ be lower semicontinuous around $\bar{x}$ for $i \in I_{\max}(\bar{x})$ and be upper semicontinuous at $\bar{x}$ for $i \notin I_{\max}(\bar{x})$. Suppose that each ϕ_i, $i = 1, \ldots, m$, is Lipschitz continuous around $\bar{x}$. Then we have the inclusion*

$$\partial (\max \phi_i)(\bar{x}) \subset \bigcup \left\{ \partial \left(\sum_{i \in I_{\max}(\bar{x})} \lambda_i \phi_i \right) (\bar{x}) \; \middle| \; (\lambda_1, \ldots, \lambda_m) \in \Lambda(\bar{x}) \right\},$$

where the equality holds and the maximum functions are lower regular at $\bar{x}$ if each ϕ_i is lower regular at this point and sets $I_{\max}(\bar{x})$ and $\Lambda(\bar{x})$ are defined as follows:

$$I_{\max}(\bar{x}) := \{i \in \{1, \ldots, m\} \mid \phi_i(\bar{x}) = (\max \phi_i)(\bar{x})\},$$

$$\Lambda(\bar{x}) := \left\{ (\lambda_1, \ldots, \lambda_m) \mid \lambda_i \geq 0, \ \sum_{i=1}^{m} \lambda_i = 1, \ \lambda_i \left(\phi_i(\bar{x}) - (\max \phi_i)(\bar{x}) \right) = 0 \right\}.$$

3. OPTIMALITY CONDITIONS

In this section, we first establish necessary optimality conditions for locally optimal solutions of problem (P). Then, by introducing concepts of generalized convexity, we show sufficient conditions for such solutions for problem (P). Below, we give the definition of solutions for problem (P).

Definition 3.1. Let $\phi(x) := \max_{k \in K} f_k(x)$, $x \in X$. A point $\bar{x} \in F$ is called a locally optimal solution of problem (P) if and only if there is a neighborhood U of $\bar{x}$ such that

$$\phi(\bar{x}) \leq \phi(x) \text{ for all } x \in U \cap F.$$

Now, denote by $\mathbb{R}^{(T)}$ the linear space of generalized finite sequences $\lambda = (\lambda_t)_{t \in T}$ such that $\lambda_t \in \mathbb{R}$ for each $t \in T$ but only finitely many λ_t differ from zero. In other words,

$$\mathbb{R}^{(T)} := \{\lambda = (\lambda_t)_{t \in T} \mid \lambda_t = 0 \text{ for all } t \in T \text{ but only finitely many } \lambda_t \neq 0\}.$$

The nonnegative cone of $\mathbb{R}^{(T)}$ is denoted by

$$\mathbb{R}_+^{(T)} := \{\lambda = (\lambda_t)_{t \in T} \in \mathbb{R}^{(T)} \mid \lambda_t \geq 0, \ t \in T\}.$$

With $\lambda \in \mathbb{R}^{(T)}$, its supporting set, denoted by $T(\lambda) := \{t \in T \mid \lambda_t \neq 0\}$, is a finite subset of T.

Definition 3.2. One says that the constraint qualification (CQ) is satisfied at $\bar{x} \in \Omega$ if there do not exist $\lambda_t \geq 0$, $t \in T(\lambda)$, such that

$$0 \in \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + N(\bar{x}; \Omega).$$

Now, we establish Karush–Kuhn–Tucker necessary conditions for locally optimal solutions of problem (P).

Theorem 3.1. *Let the* (CQ) *be satisfied at $\bar{x} \in \Omega$. If $\bar{x}$ is a locally optimal solution of problem* (P), *then there exist $\alpha_k \geq 0$, $k \in K$ with $\sum\limits_{k \in K} \alpha_k = 1$, $\lambda \in \mathbb{R}_+^{(T)}$, such that*

$$0 \in \sum_{k \in K} \alpha_k \partial f_k(\bar{x}) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + N(\bar{x}; \Omega),$$

$$\alpha_k \left(f_k(\bar{x}) - \max_{k \in K} f_k(\bar{x}) \right) = 0, \ k \in K, \tag{3.1}$$

$$g_t(\bar{x}) = 0, \ t \in T(\lambda).$$

Proof. Let $\bar{x}$ be a locally optimal solution of problem (P), equivalently, $\bar{x}$ is a locally optimal solution of the following unconstrained optimization problem

$$\min_{x \in X} \phi(x) + \delta(x; F), \tag{3.2}$$

where $\phi(x) := \max_{k \in K} f_k(x)$. Invoking now the nonsmooth version of Fermat's rule (2.2) to problem (3.2), we have

$$0 \in \partial \left(\phi + \delta(\cdot; F)\right)(\bar{x}). \tag{3.3}$$

Since the function $\delta(\cdot; F)$ is l.s.c around $\bar{x}$ and the function ϕ is Lipschitz continuous around $\bar{x}$, it follows from the sum rule (2.3) applied to (3.3) and from the relation in (2.1) that

$$0 \in \partial \phi(\bar{x}) + N(\bar{x}; F). \tag{3.4}$$

Applying further the formula for the limiting subdifferential of maximum functions and the limiting subdifferential sum rule for Lipschitz functions (Lemma 2.2), we have

$$\partial \phi(\bar{x}) = \partial \left(\max_{k \in K} f_k \right)(\bar{x})$$
$$\subset \left\{ \sum_{k \in K(\bar{x})} \alpha_k \partial f_k(\bar{x}) \mid \alpha_k \geq 0, \alpha_k \left(f_k(\bar{x}) - \max_{k \in K} f_k(\bar{x}) \right) = 0, k \in K(\bar{x}), \sum_{k \in K(\bar{x})} \alpha_k = 1 \right\}, \tag{3.5}$$

where $K(\bar{x}) := \{ k \in K \mid f_k(\bar{x}) = \phi(\bar{x}) \} \neq \emptyset$.

On the other hand, by letting

$$\tilde{\Omega} := \{ x \in X \mid g_t(x) \leq 0, \ t \in T \},$$

we have $F := \tilde{\Omega} \cap \Omega$. The (CQ) being fulfilled at $\bar{x}$ entails that there do not exist $\lambda_t \geq 0, t \in T(\lambda)$ such that

$$0 \in \sum_{t \in T} \lambda_t \partial g_t(\bar{x}).$$

Hence, we conclude by [12, Exercise 3.74] that

$$N(\bar{x}; \tilde{\Omega}) \subset \left\{ \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) \mid \lambda_t \geq 0, t \in T(\lambda) \right\}.$$

As the (CQ) is satisfied at $\bar{x}$, we obtain $g_t(\bar{x}) = 0, t \in T(\lambda)$. If it is not valid, then $g_t(\bar{x}) < 0$, $t \in T(\lambda)$, such that $\bar{x}$ is an interior point of feasible set, and there exist $\lambda_t \geq 0, t \in T(\lambda)$ hold $0 \in \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + N(\bar{x}; \Omega)$. This contradicts the (CQ) condition. Due to the fact that Ω is assumed to be SNC at $\bar{x}$, we apply Lemma 2.1 to obtain

$$N(\bar{x}; F) = N(\bar{x}; \tilde{\Omega} \cap \Omega) \subset N(\bar{x}; \tilde{\Omega}) + N(\bar{x}; \Omega). \tag{3.6}$$

It follows from (3.4), (3.5), and (3.6) that

$$0 \in \left\{ \sum_{k \in K(\bar{x})} \alpha_k \partial f_k(\bar{x}) \mid \alpha_k \geq 0, \alpha_k \left(f_k(\bar{x}) - \max_{k \in K} f_k(\bar{x}) \right) = 0, k \in K(\bar{x}), \sum_{k \in K(\bar{x})} \alpha_k = 1 \right\}$$
$$+ \left\{ \sum_{t \in T(\lambda)} \lambda_t \partial g_t(\bar{x}) \mid \lambda_t \geq 0, t \in T(\lambda) \right\} + N(\bar{x}; \Omega). \tag{3.7}$$

Since $\lambda \in \mathbb{R}_+^{(T)}$, put $\lambda_t := 0$ for $t \in T \backslash T(\lambda)$, $\alpha_k := 0$ for $k \in K \backslash K(\bar{x})$. Now, it is clear that (3.7) implies (3.1), and that completes the proof. □

The following example shows that the conclusion of Theorem 3.1 may fail to hold if the (CQ) is not satisfied at the point under consideration. In other words, the following example asserts the importance of the (CQ) imposed in Theorem 3.1. This means that the (CQ) in Theorem 3.1 is essential.

Example 3.1. Let $f : \mathbb{R} \to \mathbb{R}^2$ be defined by $f := (f_1, f_2)$, where $f_1(x) := f_2(x) := x - x^3$, and let $g_t := \mathbb{R} \to \mathbb{R}$ be given by $g_t(x) = tx^2$, $x \in \mathbb{R}$, $t \in T := [1, 2]$. We consider problem (P) with $K := \{1, 2\}$ and $\Omega := (-\infty, 0]$. Then $F := \{0\}$ and thus, $\bar{x} := 0$ is a locally optimal solution of (P). Since $N(\bar{x}; \Omega) := [0, \infty)$ and $\partial g_t(\bar{x}) = 0$ for all $t \in T$, we have

$$0 \in \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + N(\bar{x}; \Omega),$$

where $\lambda_t \geq 0$, $t \in T(\lambda)$. Hence the (CQ) is not satisfied at $\bar{x}$. Actually, Karush–Kuhn–Tucker conditions (3.1) fail to hold by choosing $\alpha \in \mathbb{R}_+^2$ with $\alpha_1 + \alpha_2 = 1$.

Now, we introduce the concept of the generalized convexity to obtain sufficient conditions for (globally) optimal solutions of problem (P). This is inspired by [10, 13, 14, 15].

Definition 3.3. We say that (f, g_T) is *generalized convex* on Ω at $\bar{x} \in \Omega$ if, for any $x \in \Omega$, $\xi_k \in \partial f_k(\bar{x})$, $k \in K$ and any $\eta_t \in \partial g_t(\bar{x})$, $t \in T$, there exists $v \in N(\bar{x}; \Omega)^\circ$ such that

$$f_k(x) - f_k(\bar{x}) \geq \langle \xi_k, v \rangle, k \in K,$$
$$g_t(x) - g_t(\bar{x}) \geq \langle \eta_t, v \rangle, t \in T.$$

The following theorem describes sufficient optimality conditions for (globally) optimal solutions of problem (P).

Theorem 3.2. *Let $\bar{x} \in F$ satisfy Karush–Kuhn–Tucker conditions* (3.1). *If (f, g_T) is generalized convex on Ω at $\bar{x}$, then $\bar{x}$ is a (globally) optimal solution of problem* (P).

Proof. Let $\phi(x) := \max_{k \in K} f_k(x)$. Since $\bar{x} \in F$ satisfies condition (3.1), there exist $\alpha_k \geq 0$, $k \in K$ with $\sum_{k \in K} \alpha_k = 1$, $\lambda \in \mathbb{R}_+^{(T)}$, and $\xi_k \in \partial f_k(\bar{x})$, $k \in K$, $\eta_t \in \partial g_t(\bar{x})$, $t \in T$, such that

$$-\left(\sum_{k \in K} \alpha_k \xi_k + \sum_{t \in T} \lambda_t \eta_t \right) \in N(\bar{x}; \Omega), \tag{3.8}$$

$$\alpha_k \left(f_k(\bar{x}) - \max_{k \in K} f_k(\bar{x}) \right) = 0, \ k \in K, \tag{3.9}$$

$$g_t(\bar{x}) = 0, \ t \in T(\lambda). \tag{3.10}$$

Assume to the contrary that $\bar{x}$ is not a globally optimal solution of problem (P). Then there is $\hat{x} \in F$ such that

$$\phi(\bar{x}) > \phi(\hat{x}). \tag{3.11}$$

By the generalized convexity of (f, g_T) on Ω at $\bar{x}$, we deduce from (3.8) that, for such $\hat{x}$, there is $v \in N(\bar{x}; \Omega)^\circ$ such that

$$0 \leq \sum_{k \in K} \alpha_k \langle \xi_k, v \rangle + \sum_{t \in T} \lambda_t \langle \eta_t, v \rangle$$
$$\leq \sum_{k \in K} \alpha_k \left(f_k(\hat{x}) - f_k(\bar{x}) \right) + \sum_{t \in T} \lambda_t \left(g_t(\hat{x}) - g_t(\bar{x}) \right).$$

Hence,

$$\sum_{k\in K}\alpha_k f_k(\bar{x}) \le \sum_{k\in K}\alpha_k f_k(\hat{x}) + \sum_{t\in T}\lambda_t\left(g_t(\hat{x}) - g_t(\bar{x})\right). \tag{3.12}$$

In addition, $\hat{x}\in F$ which yileds $\lambda_t g_t(\hat{x}) \le 0$. So, we obtain by (3.10), (3.12) that

$$\sum_{k\in K}\alpha_k f_k(\bar{x}) \le \sum_{k\in K}\alpha_k f_k(\hat{x}).$$

On the other hand, by (3.9), it holds that

$$\sum_{k\in K}\alpha_k\phi(\bar{x}) = \sum_{k\in K}\alpha_k f_k(\bar{x}) \le \sum_{k\in K}\alpha_k\phi(\hat{x}).$$

This implies that

$$\phi(\bar{x}) \le \phi(\hat{x}). \tag{3.13}$$

Obviously, (3.13) contradicts (3.11), which completes the proof. □

4. DUALITY RELATIONS

In this section, we formulate a dual problem for problem (P), and explore weak, strong, and converse-like duality relations between problem (P) and its dual.

In connection with the semi-infinite minimax optimization problem (P), we consider a dual problem in the following form:

$$\max\left\{\tilde{\phi}(z,\alpha,\lambda) := \phi(z) + \sum_{t\in T}\lambda_t g_t(z) \mid (z,\alpha,\lambda)\in F_D\right\}, \tag{D}$$

where $\phi(z) := \max_{k\in K} f_k(z)$, and the constraint set F_D is defined by

$$F_D := \left\{(z,\alpha,\lambda)\in\Omega\times\mathbb{R}_+^l\times\mathbb{R}_+^{(T)} \mid 0\in\sum_{k\in K}\alpha_k\partial f_k(z) + \sum_{t\in T}\lambda_t\partial g_t(z) + N(z;\Omega),\right.$$

$$\left.\alpha_k\left(f_k(z) - \phi(z)\right) = 0,\ k\in K,\ \sum_{k\in K}\alpha_k = 1\right\}.$$

Definition 4.1. A point $(\bar{z},\bar{\alpha},\bar{\lambda})\in F_D$ is called a globally optimal solution of problem (D) if and only if

$$\tilde{\phi}(\bar{z},\bar{\alpha},\bar{\lambda}) \ge \tilde{\phi}(z,\alpha,\lambda),\ \forall(z,\alpha,\lambda)\in F_D.$$

The next theorem shows weak duality relations between (P) and (D).

Theorem 4.1. *Let $x\in F$, and let $(z,\alpha,\lambda)\in F_D$. If (f,g_T) is generalized convex on Ω at z, then*

$$\phi(x) \ge \tilde{\phi}(z,\alpha,\lambda),$$

where $\phi(x) := \max_{k\in K} f_k(x)$.

Proof. Since $(z, \alpha, \lambda) \in F_D$, there exist $\alpha := (\alpha_1, \ldots, \alpha_l) \in \mathbb{R}_+^l$ with $\sum_{k \in K} \alpha_k = 1$, $\lambda \in \mathbb{R}_+^{(T)}$, $\xi_k \in \partial f_k(z)$, $k \in K$, $\eta_t \in \partial g_t(z)$, $t \in T$, such that

$$-\left(\sum_{k \in K} \alpha_k \xi_k + \sum_{t \in T} \lambda_t \eta_t \right) \in N(z; \Omega), \tag{4.1}$$

$$\alpha_k (f_k(z) - \phi(z)) = 0, \ k \in K. \tag{4.2}$$

By the generalized convexity of (f, g_T) on Ω at z, there exists $\nu \in N(z; \Omega)^\circ$. Due to the definition of polar cones, it follows from (4.1) and the relation $\nu \in N(z; \Omega)^\circ$ that

$$0 \leq \sum_{k \in K} \alpha_k \langle \xi_k, \nu \rangle + \sum_{t \in T} \lambda_t \langle \eta_t, \nu \rangle$$
$$\leq \sum_{k \in K} \alpha_k (f_k(x) - f_k(z)) + \sum_{t \in T} \lambda_t (g_t(x) - g_t(z)).$$

In addition, due to $\lambda \in \mathbb{R}_+^{(T)}$, and $x \in F$, we have $\sum_{t \in T} \lambda_t g_t(x) \leq 0$. Taking now (4.2), above inequality yields

$$\phi(z) + \sum_{t \in T} \lambda_t g_t(z) = \sum_{k \in K} \alpha_k f_k(z) + \sum_{t \in T} \lambda_t g_t(z)$$
$$\leq \sum_{k \in K} \alpha_k f_k(x)$$
$$= \phi(x).$$

This completes the proof.   □

The forthcoming theorem describes strong duality relations between (P) and (D).

Theorem 4.2. Let $\bar{x} \in \Omega$ be a locally optimal solution of problem (P) such that the (CQ) is satisfied at this point. Then there exists $(\bar{\alpha}, \bar{\lambda}) \in \mathbb{R}_+^l \times \mathbb{R}_+^{(T)}$ such that $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in F_D$ and

$$\phi(\bar{x}) = \tilde{\phi}(\bar{x}, \bar{\alpha}, \bar{\lambda}),$$

where $\phi(x) := \max_{k \in K} f_k(x)$. Furthermore, if (f, g_T) is generalized convex on Ω at any $z \in \Omega$, then $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is an optimal solution of problem (D).

Proof. Applying Theorem 3.1, we find $\bar{\alpha} := (\bar{\alpha}_1, \ldots, \bar{\alpha}_l) \in \mathbb{R}_+^l$ with $\sum_{k \in K} \bar{\alpha}_k = 1$, and $\bar{\lambda} := (\bar{\lambda}_t)_{t \in T} \in \mathbb{R}_+^{(T)}$, such that

$$0 \in \sum_{k \in K} \bar{\alpha}_k \partial f_k(\bar{x}) + \sum_{t \in T} \bar{\lambda}_t \partial g_t(\bar{x}) + N(\bar{x}; \Omega),$$
$$\bar{\alpha}_k (f_k(\bar{x}) - \phi(\bar{x})) = 0, \ k \in K,$$
$$g_t(\bar{x}) = 0, \ t \in T(\lambda).$$

Hence $\sum_{t \in T} \bar{\lambda}_t g_t(\bar{x}) = 0$, we conclude that $\phi(\bar{x}) = \tilde{\phi}(\bar{x}, \bar{\alpha}, \bar{\lambda})$. Clearly, $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in F_D$. If (f, g_T) is generalized convex on Ω at any $z \in \Omega$, then by Theorem 4.1, we obtain

$$\tilde{\phi}(\bar{x}, \bar{\alpha}, \bar{\lambda}) = \phi(\bar{x}) \geq \tilde{\phi}(z, \alpha, \lambda)$$

for any $(z, \alpha, \lambda) \in F_D$. Hence, $(\bar{x}, \bar{\alpha}, \bar{\lambda})$ is an optimal solution of problem (D).   □

We close this section by proving the converse-like duality relation between (P) and (D).

Theorem 4.3. Let $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in F_D$ be such that $\phi(\bar{x}) = \tilde{\phi}(\bar{x}, \bar{\alpha}, \bar{\lambda})$ with $\phi(\bar{x}) := \max_{k \in K} f_k(\bar{x})$. If $\bar{x} \in F$ and (f, g_T) is generalized convex on Ω at $\bar{x}$, then $\bar{x}$ is a globally optimal solution of problem (P).

Proof. Since $(\bar{x}, \bar{\alpha}, \bar{\lambda}) \in F_D$, there exist $\bar{\alpha} := (\bar{\alpha}_1, \dots, \bar{\alpha}_l) \in \mathbb{R}_+^l$ with $\sum_{k \in K} \bar{\alpha}_k = 1$, $\bar{\lambda} := (\bar{\lambda}_t)_{t \in T} \in \mathbb{R}_+^{(T)}$, $\bar{\xi}_k \in \partial f_k(\bar{x})$, $k \in K$, $\bar{\eta}_t \in \partial g_t(\bar{x})$, $t \in T$, such that

$$-\left(\sum_{k \in K} \bar{\alpha}_k \bar{\xi}_k + \sum_{t \in T} \bar{\lambda} \bar{\eta}_t \right) \in N(\bar{x}; \Omega), \tag{4.3}$$

$$\bar{\alpha}_k \left(f_k(\bar{x}) - \phi(\bar{x}) \right) = 0, \ k \in K. \tag{4.4}$$

Let $\bar{x} \in F$. We have $g_t(\bar{x}) \leq 0$ for $t \in T$. Hence, $\sum_{t \in T} \bar{\lambda}_t g_t(\bar{x}) \leq 0$, which together with $\phi(\bar{x}) = \tilde{\phi}(\bar{x}, \bar{\alpha}, \bar{\lambda})$ yields $g_t(\bar{x}) = 0$ for all $t \in T(\lambda)$, i.e., $\bar{\lambda} \in \mathbb{R}_+^{(T)}$. This together with (4.3) and (4.4) confirms that $\bar{x}$ satisfies condition (3.1). To finish the proof, it remains to apply Theorem 3.2. $\square$

5. Applications to Multiobjective Optimization Problems

In this section, we establish necessary and sufficient conditions for a weak Pareto solution of (MP) by employing optimality conditions obtained by (P) in Section 3.

Consider the following semi-infinite multiobjective optimization problem:

$$\min_{\mathbb{R}_+^l} \left\{ f(x) := (f_1(x), \dots, f_l(x)) \mid x \in F \right\}, \tag{MP}$$

where F is same as the feasible set of problem (P).

Definition 5.1. A point $\bar{x} \in F$ is said to be a weak Pareto solution of problem (MP) if and only if

$$f(x) - f(\bar{x}) \notin -\mathrm{int}\,\mathbb{R}_+^l, \ \forall x \in F.$$

The following result is a Karush–Kuhn–Tucker (KKT) necessary condition for a weak Pareto solution of problem (MP).

Theorem 5.1. Let the (CQ) be satisfied at $\bar{x} \in \Omega$. If $\bar{x}$ is a weak Pareto solution of problem (MP), then there exist $\alpha := (\alpha_1, \dots, \alpha_l) \in \mathbb{R}_+^l$ with $\sum_{k \in K} \alpha_k = 1$, and $\lambda \in \mathbb{R}_+^{(T)}$, such that

$$0 \in \sum_{k \in K} \alpha_k \partial f_k(\bar{x}) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + N(\bar{x}; \Omega),$$
$$g_t(\bar{x}) = 0, \ t \in T(\lambda). \tag{5.1}$$

Proof. Let $\bar{x}$ be a weak Pareto solution of problem (MP) and let

$$\widehat{f}_k(x) := f_k(x) - f_k(\bar{x}), \ k \in K, \ x \in X.$$

By [10, Theorem 5.1], we can show that $\bar{x}$ is an optimal solution of the minimax optimization problem

$$\min_{x \in F} \max_{k \in K} \widehat{f}_k(x).$$

So, we can employ the KKT condition in Theorem 3.1. We find $\alpha \in \mathbb{R}_+^l$ with $\sum_{k \in K} \alpha_k = 1$, and $\lambda \in \mathbb{R}_+^{(T)}$, such that

$$0 \in \sum_{k \in K} \alpha_k \partial f_k(\bar{x}) + \sum_{t \in T} \lambda_t \partial g_t(\bar{x}) + N(\bar{x}; \Omega),$$
$$g_t(\bar{x}) = 0, \ t \in T(\lambda).$$

Thus, the proof is complete. □

The last theorem of this work describes sufficient optimality conditions for a weak Pareto solution of problem (MP).

Theorem 5.2. *Let $\bar{x} \in F$ satisfy condition* (5.1). *If* (f, g_T) *is generalized convex on* Ω *at* $\bar{x}$, *then* $\bar{x}$ *is a weak Pareto solution of problem* (MP).

Proof. Set

$$\widehat{f}_k(x) := f_k(x) - f_k(\bar{x}), \ k \in K, \ x \in X.$$

Let $\bar{x} \in F$ satisfy condition (5.1). It is clear that the following condition holds

$$\alpha_k \left(\widehat{f}_k(\bar{x}) - \max_{k \in K} \widehat{f}_k(\bar{x}) \right) = 0, \ k \in K.$$

Let $\widehat{f} := \left(\widehat{f}_1, \ldots, \widehat{f}_l \right)$. Since (f, g_T) is generalized convex on Ω at $\bar{x}$, it follows that $(\widehat{f}, g_T)$ is generalized convex on Ω at this point as well. Thus, we apply the sufficient criteria in Theorem 3.2 to conclude that $\bar{x}$ is a globally optimal solution of the minimax optimization problem

$$\min_{x \in F} \max_{k \in K} \widehat{f}_k(x).$$

It means that

$$\widehat{\phi}(\bar{x}) \leq \widehat{\phi}(x), \ \forall x \in F,$$

where $\widehat{\phi}(x) := \max_{k \in K} \widehat{f}_k(x)$. In other words, we obtain

$$0 \leq \max_{k \in K} \left\{ f_k(x) - f_k(\bar{x}) \right\}, \ \forall x \in F.$$

This inequality ensures that $\bar{x}$ is a weak Pareto solution of problem (MP). □

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