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A TIKHONOV-TYPE REGULARIZATION FOR BRÉZIS PSEUDOMONOTONE EQUILIBRIUM PROBLEMS IN BANACH SPACES

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Abstract. In this paper, we study the existence of solutions for nonmonotone equilibrium problems by a Tikhonov weak regularization scheme and a Galerkin-type method. In our approach, we use the notion of the pseudomonotonicity in the sense of Brézis for bifunctions initially introduced by Gwinner, and we do not assume that the considered bifunction is lower semicontinuous with respect to the second argument. We use suitable weakened coercivity conditions to obtain the convergence of generalized versions of the Tikhonov regularization procedure for nonmonotone equilibrium problems. Our results, which can be applied to rather class of nonlinear and nonmonotone problems associated to pseudomonotone operators in the sense of Brézis, are new and extend some previous results in literature.

Keywords. Equilibrium problems; Topological pseudomonotonicity; Tikhonov regularization.

1. Introduction

Let X be a Banach space, K be a nonempty subset of X and $\Phi : K \times K \to \mathbb{R}$ be a real valued bifunction such that $\Phi(x,x) = 0$ for all $x \in K$. By equilibrium problem (for short, EP), we mean the following problem: Find $\bar{x} \in K$ such that

$$\Phi(\bar{x}, y) \geq 0, \quad \text{for all } y \in K. \tag{1.1}$$

The equilibrium problem, which is also called *Ky Fan minimax inequality*, is one of the most important tools in nonlinear analysis. They appear frequently in many mathematical models arising in engineering, physics, transportation, game theory, mathematical economics and networks. The equilibrium problems formulation started implicitly in the paper by Nikaido and Isoda [1] in 1955 with an aim to characterize the Nash equilibrium.

The first existence results on equilibrium problems started in 1972 by the work of Ky Fan [2], where K is supposed a nonempty, convex and compact subset of a real topological vector space, $\Phi(\cdot, y)$ is upper semicontinuous for each $y \in K$, and $\Phi(x, \cdot)$ is quasiconvex for each $x \in K$. Later, in 1975, Mosco [3] considered a general formulation that they called implicit variational problem and which includes variational and quasi-variational inequalities, fixed points, Nash equilibrium, minimization problems. The name *equilibrium problems* appeared in the seminal paper by Blum and Oettli [4] where they highlighted the unifying aspects of this new field of investigation and gave some fundamental results and new concepts.

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In this paper, we study the existence of solutions for nonmonotone equilibrium problems by a Tikhonov weak regularization scheme and a Galerkin-type method. In our approach, we use the notion of pseudomonotonicity in the sense of Brézis for bifunctions, and without assuming that the bifunction Φ is lower semicontinuous with respect to the second argument. The concept of pseudomonone bifunctions in the sense of Brézis was initiated by Gwinner [5, 6] as an extension to bifunctions of the notion of pseudomonotone operators introduced by Brézis [7]. In the general setting of Hausdorff topological vector space, this concept was considered by Aubin [8, page 412] and Gwinner [9] with an aim to relax the continuity properties in the study of many problems related to minimax formulations in game theory as well as fixed points problems. A recent paper by Steck [10] shown the interest of the notion of pseudomonotonicity in the sense of Brézis and showed how it is strictly weaker than the notion called *Ky Fan hemicontinuity* (see [10, Definition 2.1] for operators), i.e., the weak upper semicontinuity of Φ with respect to the first argument, which has been used in many works related to the problems studied in this paper; see, for instance, [11, 12] and the references therein.

When the feasible set K is not bounded, many authors concentrate their efforts for obtaining weakened coercivity conditions which guarantee the existence of solutions for the equilibrium problem (1.1); see, e.g., [4, 11, 12, 13, 14, 15] and the references therein. In our approach, we use the weakened coercivity conditions from [11, 12] which have been shown to be suitable for obtaining the convergence of generalized versions of the Tikhonov-Browder regularization procedure for nonmonotone equilibrium problems. In a first step, we show some existence results for the equilibrium problem (1.1) under a condition on Φ weaker than the pseudomonotonicity in the sense of Brézis. By a Tikhonov-type regularization procedure, we approach the equilibrium problem (1.1) by a family of regularized nonmonotone equilibrium problems. Under the pseudomonotonicity in the sense of Brézis of bifunctions, we obtain existence of solutions for the perturbed problems and weak convergence to a solution of the problem (1.1). The results obtained are new and extend some previous results in literature, they can be applied to rather general class of nonlinear and nonmonotone problems associated to pseudomonotone operators in the sense of Brézis.

2. Preliminaries

Let X be a reflexive Banach space with X^* its dual space, and let K be a nonempty, closed and convex subset of X. We denote the norms of X by $\|\cdot\|$, and by $B_r(\mathbf{0}) := \{x \in X : \|x\| \leq r\}$, we denote the closed ball of X with center $\mathbf{0}$ and radius r. For $x \in X$ and $x^* \in X^*$, the symbol $\langle x^*, x\rangle$ stands for the value of x^* at x. For a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$, we shall use the standard notation $x_n \to x$ to denote the strong convergence to x and $x_n \rightharpoonup x$ to denote the weak convergence to x. For each subset A of X, we denote by $\mathrm{conv}(A)$ the convex hull of A and by $\mathrm{cl}(A)$ the closure of A in X. We shall use $\mathscr{F}(A)$ to denote the family of all finite subsets of A and by 2^A the family of all subsets of A.

Let M be a nonempty subset of X. A set-valued function $F : M \to 2^X$ is called a Knaster-Kuratowski-Mazurkiewicz map or simply a *KKM-map* (or that the family $\{F(x)\}_{x\in M}$ of subsets of X has the KKM-property) if $\mathrm{conv}(Z) \subset \bigcup_{x\in Z} F(x)$ for any $Z \in \mathscr{F}(M)$. This notion has its origin in the so-called KKM-principle (KKM Theorem) formulated in 1929 by Knaster-Kuratowski-Mazurkiewicz [16]. Ky Fan [17] extended the KKM theorem to topological vector spaces and gave several interesting applications in the fixed point theory, minimax theory, and game theory. The study of nonlinear analysis associated to the KKM-principle has been a rapidly developing area in mathematics which is emerging as an independent field known as the KKM theory, with numerous underlying applications; see, e.g., [18, 19] and the references therein.

Definition 2.1. [7] A single-valued mapping $T : X \to X^*$ is said to be *pseudomonotone* in the sense of Brézis if, for any sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ with $x_n \rightharpoonup x$ in X and $\limsup\langle T(x_n), x_n - x\rangle \leq 0$, we have

$$\liminf\langle T(x_n), x_n - y\rangle \geq \langle T(x), x - y\rangle, \quad \text{for all } y \in X.$$

Classical existence results on Ky Fan minimax inequality [2] require the upper semicontinuity of Φ with respect to the first argument in problem (1.1). The following definition extends to bifunctions the concept of *Ky-Fan hemicontinuity* introduced for operators by Maugeri and Raciti [20], see also [10, 21].

Definition 2.2. Let K be a nonempty closed convex subset of X. A real-valued bifunction $\Phi : K \times K \to \mathbb{R}$ is said to be *Ky-Fan hemicontinuous* if the function $x \in K \mapsto \Phi(x, y)$ is weakly upper semicontinuous for all $y \in K$.

We recall the following definition introduced by Gwinner [5, 6].

Definition 2.3. Let K be a nonempty closed convex subset of X. A real-valued bifunction $\Phi : K \times K \to \mathbb{R}$ is said to be *Brézis pseudomonotone*, (for short, B-*pseudomonotone*) if, whenever

$$\{x_n\}_{n\in\mathbb{N}} \subset K, \quad x_n \rightharpoonup x, \quad \text{and} \quad \liminf\Phi(x_n, x) \geq 0,$$

then

$$\limsup\Phi(x_n, y) \leq \Phi(x, y) \quad \text{for all} \quad y \in K.$$

Remark 2.1. (a) If $T : X \to X^*$ is pseudomonotone in the sense of Brézis, then the bifunction Φ defined by $\Phi(u, v) = \langle T(u), v - u\rangle$ is B-*pseudomonotone*.
(b) If a bifunction Φ is *Ky-Fan hemicontinuous*, then it is B-*pseudomonotone*. The converse is not true, this has been shown recently by Steck [10] in a counter example to a claim in a recent paper by Sadeqi and Paydar [21] on the equivalence of the two properties for operators.
(c) If $\Phi, \Psi : K \times K \to \mathbb{R}$ are two real-valued B-*pseudomonotone* bifunctions such that $\Phi(x, x) \leq 0$ and $\Psi(x, x) \leq 0$ for all $x \in K$, then $\Phi + \Psi$ is B-*pseudomonotone*; see [22, Proposition 2.1]. Hence, the B-pseudomonotonicity notion enjoys the stability under the sum which is not the case for the algebraic pseudomonotonicity notion, introduced by Karamardian [23] and considered by many authors to study equilibrium problems and variational inequalities, see for instance [24, 25, 26] and the references therein.

Definition 2.4. [12] A function $\varphi : X \to \mathbb{R}$ is said to be

(i) *quasiconvex* on $K \subset X$ if and only if

$$\varphi(tx + (1-t)y) \leq \max\{\varphi(x), \varphi(y)\}, \ \forall x, y \in K \ \text{ and } \ \forall t \in [0, 1];$$

(ii) *explicitly quasiconvex* on $K \subset X$ if and only if it is quasiconvex and

$$\varphi(tx + (1-t)y) < \max\{\varphi(x), \varphi(y)\}, \ \forall x, y \in K \ \text{ with } \ \varphi(x) \neq \varphi(y) \ \text{ and } \ \forall t \in (0, 1);$$

(iii) *coercive* if and only if

$$\varphi(x) \to +\infty \text{ when } \|x\| \to +\infty;$$

(iv) *weakly coercive* with respect to $K \subset X$ if and only if there exists $\gamma \in \mathbb{R}$ such that the set

$$K_\gamma := \{x \in K : \varphi(x) \leq \gamma\}$$

is nonempty and bounded.

3. Existence Results and Regularization Method

We first start by showing the following preliminary result.

Lemma 3.1. *Let X be a Banach space, K be a nonempty convex subset of X and let $\Phi : K \times K \to \mathbb{R}$ be a real-valued bifunction such that $\Phi(x,x) \geq 0$ for all $x \in K$. Suppose that $\Phi(x,\cdot)$ is quasiconvex for each $x \in K$, and $\Phi(\cdot,y)$ is upper semicontinuous on* $\operatorname{conv}(A)$ *for each $A \in \mathscr{F}(K)$ and y in K. Then, for any $A \in \mathscr{F}(K)$, there exists $x \in \operatorname{conv}(A)$ such that $\Phi(x,y) \geq 0$ for all $y \in \operatorname{conv}(A)$.*

Proof. Let $A = \{x_1, x_2, \ldots, x_n\}$ be an arbitrary finite subset of K. For each y in $\operatorname{conv}(A)$, define

$$\mathbb{T}(y) = \{x \in \operatorname{conv}(A) : \Phi(x,y) \geq 0\}.$$

Note that $\mathbb{T}(y) \neq \emptyset$ since it contains y. On the other part, $\mathbb{T}(y)$ is closed for each $y \in K$. Indeed, let $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{T}(y)$ be a sequence such that $x_n \to x$. Hence, $\{x_n\}_{n\in\mathbb{N}} \subset \operatorname{conv}(A)$ and

$$\Phi(x_n, y) \geq 0. \tag{3.1}$$

By considering the upper limit in the previous inequality, we obtain, thanks to the upper semicontinuity of $\Phi(\cdot,y)$ on $\operatorname{conv}(A)$, that $\Phi(x,y) \geq \limsup \Phi(x_n,y) \geq 0$. This implies that $x \in \mathbb{T}(y)$, and therefore $\mathbb{T}(y)$ is closed. Now let us verify that the set-valued mapping $T : \operatorname{conv}(A) \to 2^{\operatorname{conv}(A)}$ has the KKM-property. Suppose by contradiction that there exist $\{z_1, \ldots, z_m\} \subset \operatorname{conv}(A)$ and $z = \sum_{i=1}^m \lambda_i z_i \in \operatorname{conv}(\{z_1, \ldots, z_m\})$ with $\lambda_i \geq 0$ and $\sum_{i=1}^m \lambda_i = 1$ such that $\Phi(z, z_i) < 0$ for all $i \in \{1, \ldots, m\}$. As $\Phi(z,\cdot)$ is quasiconvex, it follows that $0 \leq \Phi(z,z) \leq \max_{i=1,\cdots,m} \{\Phi(z,z_i)\} < 0$, which is a contradiction. Since $\operatorname{conv}(A)$ is compact, by the Ky Fan's lemma [17] we deduce that $\cap_{y\in\operatorname{conv}(A)} \mathbb{T}(y) \neq \emptyset$, which completes the proof. □

Next, we give an existence result for problem (1.1) on bounded sets that will be needed in the sequel.

Theorem 3.1. *Let K be a nonempty, closed, bounded and convex subset of a reflexive Banach space X, and let $\Phi : K \times K \to \mathbb{R}$ be a bifunction such that $\Phi(x,x) \geq 0$ for all $x \in K$. Suppose that*

(i) For each $A \in \mathscr{F}(K)$ and y in K, $\Phi(\cdot,y)$ is upper semicontinuous on $\operatorname{conv}(A)$*;*
(ii) For each $x \in K$, $\Phi(x,\cdot)$ is quasiconvex;
(iii) For any $x,y \in K$ and $\{x_n\}_{n\in\mathbb{N}} \subset K$ such that $x_n \rightharpoonup x$, we have

$$\Phi(x_n, tx + (1-t)y) \geq 0 \textit{ for every } t \in [0,1] \implies \Phi(x,y) \geq 0.$$

Then equilibrium problem (1.1) has a solution.

Proof. For $A \in \mathscr{F}(K)$, let $\mathbb{T}_A = \{x \in K : \Phi(x,y) \geq 0$ for all $y \in \mathrm{conv}(A)\}$. From Lemma 3.1 we have that $\mathbb{T}_A \neq \emptyset$. Since K is weakly compact, we obtain that $cl^w(\mathbb{T}_A)$ is weakly compact, where the closure is considered with respect to the weak topology of X. Let us verify that $\bigcap_{A\in\mathscr{F}(K)} cl^w(\mathbb{T}_A) \neq \emptyset$. To this aim, since K is weakly compact, it suffices to verify that the family $\{cl^w(\mathbb{T}_A)\}_{A\in\mathscr{F}(K)}$ has the finite intersection property; see, e.g., [27, Theorem 2.31]. Let $A_1,\ldots,A_p \in \mathscr{F}(K)$ and set $Z = A_1 \cup \ldots \cup A_p \in \mathscr{F}(K)$. We can easily verify that $\mathbb{T}_Z \subset \cap_{i=1}^p \mathbb{T}_{A_i}$. As $\mathbb{T}_Z \neq \emptyset$, then $\bigcap_{i=1}^p \mathbb{T}_{A_i} \neq \emptyset$. Therefore, for each finite sub family $\{cl^w(\mathbb{T}_{A_i})\}_{i=1,\cdots,p}$ of $\{cl^w(\mathbb{T}_A)\}_{A\in\mathscr{F}(K)}$ we have $\bigcap_{i=1}^p cl^w(\mathbb{T}_{A_i}) \neq \emptyset$. Hence, $\bigcap_{A\in\mathscr{F}(K)} cl^w(\mathbb{T}_A) \neq \emptyset$. Let $\bar{x} \in \bigcap_{A\in\mathscr{F}(K)} cl^w(\mathbb{T}_A)$ and y be an arbitrary element in K. As $\bar{x} \in cl^w(\mathbb{T}_{\{y,\bar{x}\}})$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{T}_{\{y,\bar{x}\}}$ such that $x_n \rightharpoonup \bar{x}$. As $\{x_n\}_{n\in\mathbb{N}} \subset \mathbb{T}_{\{y,\bar{x}\}}$, we have

$$\Phi(x_n,z) \geq 0, \quad \text{for all } z \in \mathrm{conv}(\{y,\bar{x}\}). \tag{3.2}$$

Hence, by condition (iii), we get $\Phi(\bar{x},y) \geq 0$. Since y is an arbitrary element in K, it follows that $\bar{x}$ is a solution of equilibrium problem (1.1). This completes the proof. □

Remark 3.1. One can easily verify that if Φ is B-*pseudomonotone*, then condition (iii) in Theorem 3.1 is satisfied.

Now, we consider the unbounded case.

Theorem 3.2. *Let K be a nonempty, closed and convex subset of a reflexive Banach space X, and let $\Phi : K \times K \to \mathbb{R}$ be a bifunction such that $\Phi(x,x) = 0$ for all $x \in K$. Suppose that*

- *(i) For each $A \in \mathscr{F}(K)$ and y in K, $\Phi(\cdot,y)$ is upper semicontinuous on* conv(A)*;*
- *(ii) For each $x \in K$, $\Phi(x,\cdot)$ is explicitly quasiconvex;*
- *(iii) For any $x,y \in K$ and $\{x_n\}_{n\in\mathbb{N}} \subset K$ such that $x_n \rightharpoonup x$, we have*

$$\Phi(x_n, tx+(1-t)y) \geq 0 \ \text{for every } t \in [0,1] \implies \Phi(x,y) \geq 0.$$

- *(iv) There exists $n_0 \in \mathbb{N}$ such that for each $x \in K \setminus B_{n_0}(\mathbf{0})$, there exists some $y \in K$ with $\|y\| < \|x\|$ such that*

$$\Phi(x,y) \leq 0.$$

Then, equilibrium problem (1.1) has a solution.

Proof. Consider $p \in \mathbb{N}$ such that $p > n_0$. As $B_p(\mathbf{0})$ is convex and weakly compact, we find from Theorem 3.1 that there exists $x_p \in K \cap B_p(\mathbf{0})$ such that

$$\Phi(x_p,z) \geq 0, \text{ for all } z \in K \cap B_p(\mathbf{0}). \tag{3.3}$$

We have the following cases:

1- If $\|x_p\| = p$, then $\|x_p\| > n_0$. From the coercivity condition (iv), there exists $y_p \in K$ such that $\|y_p\| < \|x_p\| = p$ and

$$\Phi(x_p,y_p) \leq 0. \tag{3.4}$$

Let y be an arbitrary element in $K \setminus B_p(\mathbf{0})$. As $\|y_p\| < p$, then there exists $t \in (0,1)$ such that $y_t = ty + (1-t)y_p \in K \cap B_p(\mathbf{0})$. Let us take $z = y_t$ in (3.3). It follows that

$$\Phi(x_p,y_t) \geq 0. \tag{3.5}$$

It follows from condition (ii) and relation (3.4) that $\Phi(x_p,y) \geq 0$. Hence, x_p is a solution of equilibrium problem (1.1).

2- If $\|x_p\| < p$, then, for an arbitrary element $y \in K$, there exists $t \in (0,1)$ such that $y_t = ty + (1-t)x_p \in K \cap B_p(\mathbf{0})$. Hence, by considering $z = y_t$ in (3.3), we obtain that $\Phi(x_p, y_t) \geq 0$. Therefore, since $\Phi(x_p, \cdot)$ is quasiconvex (condition (ii)), we get

$$0 \leq \Phi(x_p, y_t) \leq \max\{\Phi(x_p, x_p), \Phi(x_p, y)\}. \tag{3.6}$$

- If $\Phi(x_p, x_p) = \Phi(x_p, y)$, then we derive from (3.6) that $\Phi(x_p, y) \geq 0$.
- If $\Phi(x_p, x_p) \neq \Phi(x_p, y)$, then from the condition (ii), the second inequality in relation (3.6) is strict, that is

$$0 \leq \Phi(x_p, y_t) < \max\{\Phi(x_p, x_p), \Phi(x_p, y)\}.$$

As $\Phi(x_p, x_p) = 0$, it follows that $\Phi(x_p, y) > 0$.

Thus, x_p is a solution of problem (1.1).

This completes the proof. □

Remark 3.2. One can verify that the coercivity assumption (iv) of Theorem 3.2 is satisfied if we suppose that there exists a nonempty weakly compact set $D \subset K$ such that, for each $x \in K \setminus D$, there exists $y \in D$ satisfying $\Phi(x, y) < 0$. Moreover, the condition (i) of Theorem 3.2 holds if we suppose that, for every fixed $y \in K$, the function $x \mapsto \Phi(x, y)$ is upper semicontinuous on the intersection of K with any finite dimensional subspace of X. Therefore, in the setting of reflexive Banach spaces, Theorem 3.2 improves Theorem 1 in [28].

Now, we modify the coercivity condition (iv) of Theorem 3.2 with the weaker one introduced in [11, 12]. For a function $\mu : X \to \mathbb{R}$ and a real number r, we define the level sets

$$\mathbb{B}_r^\mu = \{x \in X : \mu(x) \leq r\}.$$

Let us denote by $K_r^\mu = \mathbb{B}_r^\mu \cap K$. Note that if $\mu : X \to \mathbb{R}$ convex, its weak coercivity with respect to K is equivalent to the boundedeness of K_ρ^μ for each ρ; see [29, Chapter 3, Theorem 3.14].

Theorem 3.3. *Let K be a nonempty, closed and convex subset of a reflexive Banach space X, and $\Phi : K \times K \to \mathbb{R}$ be a bifunction such that $\Phi(x, x) = 0$ for all $x \in K$. Suppose that the conditions (i)-(iii) of Theorem 3.1 are satisfied and that the following coercivity condition holds*

($\mathscr{C}_1$) There exists a lower semicontinuous and convex function $\mu : X \to \mathbb{R}$ which is weakly coercive with respect to K and a real number $\tilde{r}$ such that for each $x \in K \setminus \mathbb{B}_{\tilde{r}}^\mu$ there is $z \in K$ with

$$\min\{\Phi(x,z), \mu(z) - \mu(x)\} < 0 \ \text{ and } \ \max\{\Phi(x,z), \mu(z) - \mu(x)\} \leq 0.$$

Then the equilibrium problem (1.1) has a solution.

Proof. First of all, we claim that $K_{\tilde{r}}^\mu$ is nonempty. Indeed, let $\tau := \inf_{x \in K} \mu(x)$. As $\mu(\cdot)$ is convex and weakly coercive, there exists $\gamma \in \mathbb{R}$ such that K_γ^μ is nonempty, convex and bounded, and hence weakly compact. Since $\mu(\cdot)$ is weakly lower semicontinuous, it follows that there exists $\bar{x} \in K_\gamma^\mu$ such that $\mu(\bar{x}) = \inf_{x \in K_\gamma} \mu(x)$. Suppose that $\mu(\bar{x}) > \tau$. Let $\{\lambda_n\}_{n\in\mathbb{N}} \subset (\tau, \mu(\bar{x})]$ such that $\lambda_n \to \tau$. For each $n \in \mathbb{N}$, there exists $x_n \in K_{\lambda_n}^\mu \subset K_\gamma^\mu$. Since K_γ^μ is weakly compact, it follows that, for a subsequence of $\{x_n\}_{n\in\mathbb{N}}$, also denoted by $\{x_n\}_{n\in\mathbb{N}}$, $x_n \rightharpoonup \tilde{x} \in K_\gamma^\mu$. From $\tau < \mu(x_n) \leq \lambda_n$ and the weak lower semicontinuity of $\mu(\cdot)$, we get

$$\tau \leq \mu(\tilde{x}) \leq \liminf \mu(x_n) \leq \lim \lambda_n = \tau.$$

Hence, $\mu(\tilde{x}) = \tau$, and therefore $\mu(\bar{x}) = \tau$, which is a contradiction. Consequently, there exists $\bar{x} \in K$ such that $\mu(\bar{x}) = \inf_{x \in K} \mu(x)$. It follows that K_τ^μ is nonempty, convex, closed and bounded. Thus, from Theorem 3.1 there exists $x_\tau \in K_\tau^\mu$ such that

$$\Phi(x_\tau, y) \geq 0, \text{ for all } y \in K_\tau^\mu. \tag{3.7}$$

If $K_{\tilde{r}}^\mu = \emptyset$, then $\tilde{r} < \tau$ and $x_\tau \notin \mathbb{B}_{\tilde{r}}^\mu$. Hence, $x_\tau \in K \setminus \mathbb{B}_{\tilde{r}}^\mu$. By the coercivity condition $(\mathscr{C}_1)$, we have that there exists $z \in K$ such that

$$\min\{\Phi(x_\tau, z), \mu(z) - \mu(x_\tau)\} < 0 \text{ and } \max\{\Phi(x_\tau, z), \mu(z) - \mu(x_\tau)\} \leq 0.$$

It follows that $\mu(z) = \mu(x_\tau)$ and $\Phi(x_\tau, z) < 0$, which contradicts (3.7). Thus, we have verified that $K_{\tilde{r}} \neq \emptyset$. Now, let us consider $\rho > \tilde{r}$. Then K_ρ^μ is a nonempty, closed, convex and bounded subset of X. From Theorem 3.1, we deduce that there exists $x_\rho \in K_\rho^\mu$ such that

$$\Phi(x_\rho, y) \geq 0, \text{ for all } y \in K_\rho^\mu. \tag{3.8}$$

Let $\varphi(x) := \Phi(x_\rho, x)$. Then $\varphi(x_\rho) = 0$ and $\varphi(x) \geq 0$ for all $x \in K_\rho$. We consider the following two cases:

- Case 1: $\mu(x_\rho) < \rho$. Suppose that there exists $z \in K \setminus \mathbb{B}_\rho^\mu$ such that $\varphi(z) < \varphi(x_\rho) = 0$. As $\mu(x_\rho) < \rho$ and $\mu(z) > \rho$, there exists $\alpha \in (0,1)$ such that $\alpha\mu(z) + (1-\alpha)\mu(x_\rho) < \rho$. Let $x(\alpha) = \alpha z + (1-\alpha)x_\rho \in K$. The convexity of $\mu(\cdot)$ implies that

  $$\mu(x(\alpha)) \leq \alpha\mu(z) + (1-\alpha)\mu(x_\rho) < \rho.$$

  Hence, $x(\alpha) \in K_\rho^\mu$ and therefore from (3.8) we get $\varphi(x(\alpha)) \geq 0$. The explicit quasiconvexity of $\Phi(x_\rho, \cdot)$ implies that

  $$\varphi(x(\alpha)) < \max\{\varphi(z), \varphi(x_\rho)\} = \varphi(x_\rho) = 0,$$

  which is a contradiction. Thus $\varphi(z) \geq 0$ for all $z \in K \setminus \mathbb{B}_\rho^\mu$. Therefore, taking into account of (3.8), we deduce that x_ρ is a solution of equilibrium problem (1.1).
- Case 2: $\mu(x_\rho) = \rho$. Then $x_\rho \in K \setminus \mathbb{B}_{\tilde{r}}^\mu$. From $(\mathscr{C}_1)$, there exists $z \in K$ such that

  $$\min\{\varphi(z), \mu(z) - \mu(x_\rho)\} < 0 \text{ and } \max\{\varphi(z), \mu(z) - \mu(x_\rho)\} \leq 0.$$

  It follows that $\mu(z) < \mu(x_\rho) = \rho$ and $\varphi(z) = 0$. Suppose that there exists $v \in K \setminus \mathbb{B}_\rho^\mu$ such that $\varphi(v) < \varphi(x_\rho) = 0$. As $\mu(z) < \rho$ and $\mu(v) > \rho$, then, similarly as in Case 1, there exists $\alpha \in (0,1)$ such that $x(\alpha) := \alpha v + (1-\alpha)z \in K_\rho^\mu$. By using (3.8) and the explicit quasiconvexity of $\Phi(x_\rho, \cdot)$, we get

  $$0 \leq \Phi(x_\rho, x(\alpha)) < \max\{\Phi(x_\rho, v), \Phi(x_\rho, z)\} = \max\{\varphi(v), \varphi(z)\} = \varphi(z) = 0,$$

  which is a contradiction. Hence, $\varphi(v) \geq 0$ for all $v \in K \setminus \mathbb{B}_\rho^\mu$. It follows, by taking into account of relation (3.8), that $\varphi(v) \geq 0$ for all $v \in K$. Thus, x_ρ is a solution of the equilibrium problem (1.1).

□

Remark 3.3. One can easily verify that the coercivity condition (iv) of Theorem 3.2 implies $(\mathscr{C}_1)$. To this aim, it suffices to consider the function $\mu : X \to \mathbb{R}$ defined by $\mu(x) = \|x\|$.

Now, we study the existence of solutions of the equilibrium problem (1.1) by using a Tikhonov-type regularization procedure. Given $\varepsilon > 0$ and $\Psi : K \times K \to \mathbb{R}$ a real-valued bifunction, we consider the following regularized problem: Find $x_\varepsilon \in K$ such that

$$(\mathrm{EP})_\varepsilon \qquad \Phi(x_\varepsilon, y) + \varepsilon\Psi(x_\varepsilon, y) \geq 0, \text{ for all } y \in K. \tag{3.9}$$

The existence of solutions of the problem (3.9) is given by the following theorem.

Theorem 3.4. *Let K be a nonempty, closed and convex subset of a reflexive Banach space X, and let $\Phi, \Psi : K \times K \to \mathbb{R}$ be two bifunctions such that $\Phi(x,x) = \Psi(x,x) = 0$ for all $x \in K$. Suppose that*

- *(i) For each $A \in \mathscr{F}(K)$ and y in K, $\Phi(\cdot, y)$ and $\Psi(\cdot, y)$ are upper semicontinuous on $\mathrm{conv}(A)$;*
- *(ii) For each $x \in K$, $\Phi(x, \cdot)$ and $\Psi(x, \cdot)$ are convex;*
- *(iii) Φ and Ψ are $\mathtt{B}$-pseudomonotone;*
- *(iv) There exists a lower semicontinuous and convex function $\mu : X \to \mathbb{R}$ which is weakly coercive with respect to K and a real number $\tilde{r}$ such that for each $x \in K \setminus \mathbb{B}^{\mu}_{\tilde{r}}$ there is $z \in K$ with $\mu(z) < \mu(x)$,*

$$\min\{\Phi(x,z), \Psi(x,z)\} < 0 \;\; \text{and} \;\; \max\{\Phi(x,z), \Psi(x,z)\} \leq 0.$$

Then, for each $\varepsilon > 0$, regularized equilibrium problem $(\mathrm{EP})_\varepsilon$ (3.9) has a solution $x_\varepsilon \in K_{\tilde{r}}$.

Proof. Let us set $\Phi_\varepsilon(x,y) := \Phi(x,y) + \varepsilon\Psi(x,y)$. Since Φ and Ψ are $\mathtt{B}$-*pseudomonotone*, it follows from Remark 2.1 (c) that Φ_ε is $\mathtt{B}$-*pseudomonotone*, and hence it satisfies condition (iii) of Theorem 3.1.

On the other hand, we can easily verify that conditions (i) and (ii) of Theorem 3.1 hold. Now, let us verify that Φ_ε satisfies the coercivity condition $(\mathscr{C}_1)$ of Theorem 3.3. Indeed, let $x \in K \setminus \mathbb{B}^{\mu}_{\tilde{r}}$. From condition (iv), we have that there exists $z \in K$ such $\mu(z) < \mu(x)$,

$$\min\{\Phi(x,z), \Psi(x,z)\} < 0 \;\; \text{and} \;\; \max\{\Phi(x,z), \Psi(x,z)\} \leq 0.$$

This implies that $\Phi_\varepsilon(x,z) < 0$, from which we deduce that

$$\min\{\Phi_\varepsilon(x,z), \mu(z) - \mu(x)\} < 0 \;\; \text{and} \;\; \max\{\Phi_\varepsilon(x,z), \mu(z) - \mu(x)\} \leq 0.$$

Hence, the coercivity condition $(\mathscr{C}_1)$ of Theorem 3.3 holds for Φ_ε. Therefore, from Theorem 3.3, we deduce that there exists $x_\varepsilon \in K$ such that $\Phi_\varepsilon(x_\varepsilon, y) \geq 0$, for all $y \in K$. Now, let us verify that $x_\varepsilon \in K^{\mu}_{\tilde{r}}$. On the contrary, suppose $x_\varepsilon \in K \setminus K^{\mu}_{\tilde{r}}$. From the condition (iv), we deduce that there exists $z \in K$ such that $\Phi_\varepsilon(x_\varepsilon, z) < 0$, which is a contradiction. □

Theorem 3.5. *Let K be a nonempty, closed and convex subset of a reflexive Banach space X, and let $\Phi, \Psi : K \times K \to \mathbb{R}$ be two bifunctions such that $\Phi(x,x) = \Psi(x,x) = 0$ for all $x \in K$. Let $\{\varepsilon_k\}_{k \in \mathbb{N}} \subset (0, +\infty)$ such that $\varepsilon_k \to 0$. Suppose that Φ is $\mathtt{B}$-pseudomonotone and Ψ is Ky-Fan hemicontinuous, and that conditions (i), (iii) and (iv) of Theorem 3.4 are satisfied. Then for any $\varepsilon_k > 0$, the regularized problem $(\mathrm{EP})_{\varepsilon_k}$ (3.9) has a solution $\bar{x}_{\varepsilon_k} \in K$ and the sequence $\{\bar{x}_{\varepsilon_k}\}_{k \in \mathbb{N}}$ has a subsequence weakly converging to a solution $\bar{x}$ of the equilibrium problem (1.1).*

Proof. As Ψ is *Ky-Fan hemicontinuous*, it is $\mathtt{B}$-*pseudomonotone*. Therefore, all the conditions of Theorem 3.4 are satisfied. It follows that for each $\varepsilon_k > 0$, the regularized problem $(\mathrm{EP})_{\varepsilon_k}$ (3.9) has a solution $\bar{x}_{\varepsilon_k} \in K^{\mu}_{\tilde{r}}$. Thus, the sequence $\{\bar{x}_{\varepsilon_k}\}_{k \in \mathbb{N}}$ is bounded, and hence for a subsequence we have $\bar{x}_{\varepsilon_k} \rightharpoonup \bar{x} \in K_{\bar{r}}^{\mu}$. We claim that $\bar{x}$ is a solution of equilibrium problem (1.1). Indeed, since $\bar{x}_{\varepsilon_k}$ is a solution of regularized problem (3.9), we have

$$\Phi(x_{\varepsilon_k}, y) + \varepsilon_k \Psi(x_{\varepsilon_k}, y) \geq 0, \text{ for all } y \in K. \tag{3.10}$$

By considering $y = \bar{x}$ in the previous inequality, we obtain

$$\Phi(x_{\varepsilon_k}, \bar{x}) \geq -\varepsilon_k \Psi(x_{\varepsilon_k}, \bar{x}).$$

Therefore,

$$\liminf \Phi(x_{\varepsilon_k}, \bar{x}) \geq -\limsup[\varepsilon_k \Psi(x_{\varepsilon_k}, \bar{x})].$$

But $\limsup \Psi(x_{\varepsilon_k}, \bar{x}) \leq \Psi(\bar{x}, \bar{x}) = 0$, it follows that $\liminf \Phi(x_{\varepsilon_k}, \bar{x}) \geq 0$. As Φ is B-*pseudomonotone*, we get

$$\limsup \Phi(x_{\varepsilon_k}, y) \leq \Phi(\bar{x}, y), \text{ for all } y \in K. \tag{3.11}$$

By considering the upper limit in (3.10), we obtain

$$\limsup \Phi(x_{\varepsilon_k}, y) + \limsup[\varepsilon_k \Psi(x_{\varepsilon_k}, y)] \geq 0, \text{ for all } y \in K. \tag{3.12}$$

As $\limsup \Psi(x_{\varepsilon_k}, y) \leq \Psi(\bar{x}, y) \leq C$ and $\varepsilon_k \to 0$, we deduce from (3.12) and (3.11) that

$$\Phi(\bar{x}, y) \geq 0, \text{ for all } y \in K.$$

This completes the proof. □

Example 3.1. We provide an example of a bifunction Φ, which is B-*pseudomonotone*, but not monotone, for which all the conditions of Theorem 3.4 and Theorem 3.5 are satisfied. We recall that a bifunction $\Phi : X \times X \to \mathbb{R}$ is said to be *monotone* if $\Phi(x,y) + \Phi(y,x) \leq 0$ for all $x, y \in X$.

We consider the bifunction $\Phi : X \times X \to \mathbb{R}$ defined by

$$\Phi(x,y) = \langle N(x), y - x \rangle, \text{ for all } x, y \in X, \tag{3.13}$$

where $N : X \to X^*$ is a Navier-Stokes operator. Note that in this example $K = X$.

Let us recall that an operator $N : X \to X^*$ is said to be a *Navier-Stokes operator* (see [30, 31, 32]), if

$$N(x) = A(x) + B[x],$$

where

(α) $A : X \to X^*$ is a linear, continuous, symmetric operator such that

$$\langle A(x), x \rangle \geq c\|x\|^2, \quad \text{for all } x \in X \text{ with } c > 0;$$

(β) $B[x] := B(x,x)$, $B : X \times X \to X^*$ is a bilinear continuous operator such that

(β-1) $\langle B(x,y), y \rangle = 0$ for $x, y \in X$,

(β-2) the mapping $B[\cdot] : X \to X^*$ is weakly continuous.

The operator N is pseudomonotone in the sense of Brézis (see, e.g., [32, Lemma 2.4]), and therefore, thanks to Remark 2.1(a), the bifunction Φ is B-*pseudomonotone*. For a specific example of Navier-Stokes operators in Sobolev spaces; see, e.g., [32]. When $B[\cdot]$ is not identically 0, it was proved in [33] that N is not monotone. Hence the bifunction Φ is not monotone.

Let us observe from (α) and (β-1) that

$$\Phi(x, 0) \leq -c\|x\|^2, \text{ for all } x \in X.$$

It follows that

$$\Phi(x, 0) \to -\infty, \text{ when } \|x\| \to +\infty.$$

Therefore, the set $C := \{x \in X : \Phi(x,0) > 0\}$ is bounded. Let $\psi : X \to \mathbb{R}$ be a continuous and strongly convex function, which attains its minimal value at y_0. Let us recall that a function $h : X \to \mathbb{R}$ is said to be *strongly convex* if, for each $x, y \in X$ and all $\lambda \in [0,1]$,

$$h(\lambda x + (1-\lambda)y) \leq \lambda h(x) + (1-\lambda)h(y) - \frac{1}{2}\tau\lambda(1-\lambda)\|x-y\|^2,$$

where $\tau > 0$ is a constant. We choose the bifunction $\Psi : X \times X \to \mathbb{R}$ to be defined by

$$\Psi(x,y) = \psi(y) - \psi(x),$$

and $\mu : X \to \mathbb{R}$ to be a an arbitrary strongly convex function, which attains its minimal value at 0. We can easily verify that $\mu(x) \to +\infty$ when $\|x\| \to +\infty$, which implies that μ is weakly coercive. It follows that there exists $r \in \mathbb{R}$ such that $C \subset K_r^\mu$. For $x \in K \setminus \mathbb{B}_r^\mu$ (here $K = X$), we take $z = 0$. Then, we have $\mu(z) < \mu(x)$, $\Phi(x,z) \leq 0$ and $\Psi(x,z) = \psi(z) - \psi(x) < 0$. Hence,

$$\min\{\Phi(x,z), \Psi(x,z)\} < 0 \quad \text{and} \quad \max\{\Phi(x,z), \Psi(x,z)\} \leq 0.$$

Therefore, the coercivity condition (iv) in Theorem 3.4 is satisfied. On the other hand, since ψ is continuous and strongly convex, it follows that Ψ is Ky-Fan hemicontinuous. Condition (ii) of Theorem 3.4 is obvious. Furthermore, condition (i) of Theorem 3.4 follows from the properties (α) and (β-2) of the Navier-Stokes operator N and the properties of the function ψ. Thus, all the conditions of Theorems 3.4 and 3.5 are satisfied.

In the following remark, we give a comparison with some results in literature related to the problems studied in this paper.

Remark 3.4. Theorem 3.5 improves Theorem 4 in [11], since the (weak) upper semicontinuity of $\Phi(\cdot, y)$ in [11, Theorem 4] is replaced by a strictly weaker condition which is the B-*pseudomonotonicity* property. Note that the results given in [11] are in a finite-dimensional space framework.

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