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MINIMUM ENERGY CONTROL PROBLEM FOR ONE CLASS OF SINGULARLY PERTURBED SYSTEMS WITH INPUT DELAY

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Abstract. In this paper, a singularly perturbed linear time-dependent controlled system with a point-wise delay in the input (the control variable) is considered. For this system, the problem of minimum energy control is formulated. An approximate solution to this problem is derived. An illustrative example is presented.

Keywords. Controlled system; Input delay; Singular perturbation; Minimum energy control; Approximate solution.

1. INTRODUCTION

Systems of differential equations with a small multiplier $\varepsilon > 0$ for a part of the highest order derivatives are called singularly perturbed systems, while the multiplier $\varepsilon > 0$ is called the parameter of singular perturbation. Such systems represent adequate mathematical models for various real-life processes with two-time-scale dynamics, [1]. One of important classes of singularly perturbed differential systems represents the systems with time delays. Singularly perturbed systems with delays arise in various applications (see e.g. [2, 3, 4, 5, 6, 7, 8]).

In the literature, various theoretical and applied issues of singularly perturbed controlled systems without and with delays are extensively studied (see e.g. [1, 9, 10, 11] and references therein).

In the present paper, we study the minimum energy control problem for a singularly perturbed system with a point-wise delay in the input (the control variable). The minimum energy control problem was studied extensively in the literature for systems without and with delays (see e.g. [12, 13, 14, 15, 16] and references therein). However, such a problem for a singularly perturbed system was studied much less. Thus, in [17, 18, 19], various versions of minimum energy control problem were studied for singularly perturbed systems without delays. In [20], the minimum energy control problem was considered for one class of singularly perturbed systems with point-wise and distributed delays in the state variables. Asymptotic behaviour of the solution to this problem was studied. In the present paper, we derive an approximate solution to the minimum energy control problem for a class of singularly perturbed systems with delay in

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the control variable. To the best of our knowledge, such a problem has not yet been studied in the literature.

2. Problem Statement

Consider the controlled system

$$\frac{dx(t)}{dt} = A_{11}(t)x(t) + A_{12}(t)y(t), \quad t \geq 0, \tag{2.1}$$

$$\varepsilon \frac{dy(t)}{dt} = A_{21}(t)x(t) + A_{22}(t)y(t) + B_{0y}(t)u(t) + B_{1y}(t)u(t-h), \quad t \geq 0, \tag{2.2}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$ are state variables; $u(t) \in \mathbb{R}^s$ is a control; $\varepsilon > 0$ is a small parameter; $h > 0$ is a time delay; $u(t+\eta)$, $\eta \in [-h, 0]$ is a control variable; $A_{ij}(t)$ and $B_{ky}(t)$, $(i = 1,2;\ j = 1,2;\ k = 0,1)$, $t \geq 0$, are matrix-valued functions of corresponding dimensions; $A_{ij}(t)$ and $B_{ky}(t)$, $(i = 1,2;\ j = 1,2;\ k = 0,1)$ are continuously differentiable for $t \in [0, +\infty)$.

Let $\varepsilon > 0$ be any given number and $u(t)$, $t \in [-h, +\infty)$, be any given piecewise continuous s-dimensional vector-valued function. For such ε and $u(t)$, the system (2.1)-(2.2) is a linear time-dependent nonhomogeneous system. Moreover, (2.1)-(2.2) is a singularly perturbed system, (see, e.g., [1]). In this system, the equation (2.1) is the slow mode and the state variable $x(t)$ is the slow state variable. The equation (2.2) is the fast mode and the state variable $y(t)$ is the fast state variable. Also, in (2.1)-(2.2), the control appears with time delay.

Along with the system (2.1)-(2.2), let us consider the functional

$$J(u) = \int_0^{t_c} u^T(t)u(t)dt, \tag{2.3}$$

where $t_c > h$ is a given time instant.

Let $\varphi_u(\eta)$ be an arbitrary given s-dimensional vector-valued function, piecewise continuous in the interval $[-h, 0]$. Let $x_0 \in \mathbb{R}^n$, $y_0 \in \mathbb{R}^m$, $x_c \in \mathbb{R}^n$ and $y_c \in \mathbb{R}^m$ be arbitrary given vectors. For a given $\varepsilon > 0$, consider the set $U(\varepsilon)$ of all s-dimensional controls $u(t,\varepsilon)$ piecewise continuous in the interval $[0, t_c]$, for which system (2.1)-(2.2) has a solution satisfying the initial conditions

$$u(\eta) = \varphi_u(\eta), \quad \eta \in [-h, 0); \quad x(0) = x_0, \quad y(0) = y_0, \tag{2.4}$$

and the terminal conditions

$$x(t_c) = x_c, \quad y(t_c) = y_c. \tag{2.5}$$

The minimum energy control problem for the system (2.1)-(2.2) is to find a control $u^*(t,\varepsilon) \in U(\varepsilon)$ such that

$$J\big(u^*(t,\varepsilon)\big) \leq J\big(u(t,\varepsilon)\big) \quad \forall u(t,\varepsilon) \in U(\varepsilon). \tag{2.6}$$

In what follows of the paper, we call this problem the Original Minimum Energy Problem (OMEP).

In this paper, we are going to establish ε-free sufficient conditions providing the existence of the solution to the OMEP robustly with respect to $\varepsilon > 0$, i.e., for all sufficiently small values of this parameter. Also, we are going to derive an approximate solution of the OMEP.

3. Preliminary Results

Consider the block vectors:

$$z(t)=\mathrm{col}\big(x(t),y(t)\big),\quad z_0=\mathrm{col}\big(x_0,y_0\big),\quad z_c=\mathrm{col}\big(x_c,y_c\big). \tag{3.1}$$

Also, for a given $\varepsilon>0$, consider the block matrices:

$$A(t,\varepsilon)=\begin{pmatrix} A_{11}(t) & A_{12}(t)\\ \frac{1}{\varepsilon}A_{21}(t) & \frac{1}{\varepsilon}A_{22}(t)\end{pmatrix},$$
$$B_0(t,\varepsilon)=\begin{pmatrix} 0\\ \frac{1}{\varepsilon}B_{0y}(t)\end{pmatrix},\quad B_1(t,\varepsilon)=\begin{pmatrix} 0\\ \frac{1}{\varepsilon}B_{1y}(t)\end{pmatrix}. \tag{3.2}$$

Using these block vectors and block matrices, the system (2.1)-(2.2) in the interval $[0,t_c]$ and the conditions (2.4)-(2.5) can be rewritten in the equivalent form as:

$$\frac{dz(t)}{dt}=A(t,\varepsilon)z(t)+B_0(t,\varepsilon)u(t)+B_1(t,\varepsilon)u(t-h),\quad t\in[0,t_c], \tag{3.3}$$

$$u(\eta)=\varphi_u(\eta),\quad \eta\in[-h,0);\quad z(0)=z_0, \tag{3.4}$$

$$z(t_c)=z_c. \tag{3.5}$$

The fundamental matrix solution $\Phi(t,\sigma,\varepsilon)$, $0\le\sigma\le t\le t_c$, of the homogeneous system corresponding to (3.3) is the unique solution of the initial-value problem

$$\frac{d\Phi(t)}{dt}=A(t,\varepsilon)\Phi(t),\quad t\in(\sigma,t_c];\quad \Phi(\sigma)=I_{n+m}. \tag{3.6}$$

Consider the function

$$\Psi(\sigma,\varepsilon)=\big(\Phi(t_c,\sigma,\varepsilon)\big)^T,\quad \sigma\in[0,t_c]. \tag{3.7}$$

Due to the results of [21], $\Psi(\sigma,\varepsilon)$ is the unique solution of the terminal-value problem

$$\frac{d\Psi(\sigma,\varepsilon)}{d\sigma}=-\big(A(\sigma,\varepsilon)\big)^T\Psi(\sigma,\varepsilon),\quad \sigma\in[0,t_c);\quad \Psi(t_c,\varepsilon)=I_{n+m}. \tag{3.8}$$

3.1. **Exact solution of the OMEP.** Along with the matrix-valued function $\Psi(\sigma,\varepsilon)$, we consider the following matrix-valued functions:

$$\Lambda(\sigma,\varepsilon)=\begin{cases}\Psi(\sigma,\varepsilon), & \sigma\in[0,t_c]\\ 0, & \sigma>t_c,\end{cases} \tag{3.9}$$

$$\Omega(\sigma,\varepsilon)=\big(\Lambda(\sigma,\varepsilon)\big)^T B_0(\sigma,\varepsilon)+\big(\Lambda(\sigma+h,\varepsilon)\big)^T B_1(\sigma+h,\varepsilon),\quad \sigma\in[0,t_c]. \tag{3.10}$$

Based on the matrix-valued function $\Omega(\sigma,\varepsilon)$, we construct the following matrix:

$$W(t_c,\varepsilon)=\int_0^{t_c}\Omega(\sigma,\varepsilon)\Omega^T(\sigma,\varepsilon)d\sigma. \tag{3.11}$$

Quite similarly to the results of [15, 20], we obtain the following assertion.

Proposition 3.1. *Let, for a given $\varepsilon > 0$, the matrix $W(t_c, \varepsilon)$ be invertible, i.e.,* $\det W(t_c, \varepsilon) \neq 0$. *Then, for this ε, the solution of the OMEP exists and has the form:*

$$u^*(t,\varepsilon) = \Omega^T(t,\varepsilon)W^{-1}(t_c,\varepsilon)\big(z_c - w_0(\varepsilon)\big), \quad t \in [0,t_c], \tag{3.12}$$

where the vector $w_0(\varepsilon)$ has the form

$$w_0(\varepsilon) = \Lambda^T(0,\varepsilon)z_0 + \int_{-h}^{0} \Lambda^T(\eta+h,\varepsilon)B_1(\eta+h,\varepsilon)\varphi_u(\eta)d\eta. \tag{3.13}$$

Moreover,

$$J_\varepsilon^* \stackrel{\triangle}{=} J\big(u^*(t,\varepsilon)\big) = \big(z_c - w_0(\varepsilon)\big)^T W^{-1}(t_c,\varepsilon)\big(z_c - w_0(\varepsilon)\big). \tag{3.14}$$

3.2. **Block-wise estimate of the matrix-valued function $\Lambda(\sigma,\varepsilon)$.** In what follows, we assume:

(A) All the eigenvalues $\lambda_p(t)$, $(p = 1, ..., m)$ of the matrix $A_{22}(t)$ satisfy the inequality $\mathrm{Re}\lambda_p(t) \leq -2\beta$ for all $t \in [0,t_c]$, where $\beta > 0$ is some constant.

Let the $n \times n$-matrix-valued function $\bar{\Lambda}_1(\sigma)$, $\sigma \in [0,+\infty)$ be the unique solution of the following terminal-value problem:

$$\frac{d\bar{\Lambda}_1(\sigma)}{d\sigma} = -\bar{A}^T(\sigma)\bar{\Lambda}_1(\sigma), \quad \sigma \in [0,t_c),$$
$$\bar{\Lambda}_1(t_c) = I_n; \qquad \bar{\Lambda}_1(\sigma) = 0, \quad \sigma > t_c, \tag{3.15}$$

where

$$\bar{A}(t) = A_{11}(t) - A_{12}(t)A_{22}^{-1}(t)A_{21}(t), \quad t \geq 0. \tag{3.16}$$

Also, we consider the following $m \times m$-matrix-valued function:

$$\widetilde{\Lambda}_4(\xi) = \begin{cases} 0, & \xi < 0, \\ \exp\Big(A_{22}^T(t_c)\xi\Big), & \xi \geq 0. \end{cases} \tag{3.17}$$

Due to the assumption (A), the following inequality is valid:

$$\|\widetilde{\Lambda}_4(\xi)\| \leq a\exp(-\beta\xi), \quad \xi \geq 0, \tag{3.18}$$

where $\|\cdot\|$ denotes the Euclidean norm of a matrix; $a > 0$ is some constant.

Let us represent the matrix-valued function $\Lambda(\sigma,\varepsilon)$ in the block form:

$$\Lambda(\sigma,\varepsilon) = \begin{pmatrix} \Lambda_1(\sigma,\varepsilon) & \Lambda_2(\sigma,\varepsilon) \\ \varepsilon\Lambda_3(\sigma,\varepsilon) & \varepsilon\Lambda_4(\sigma,\varepsilon) \end{pmatrix}, \tag{3.19}$$

where the matrices $\Lambda_1(\sigma,\varepsilon)$, $\Lambda_2(\sigma,\varepsilon)$, $\Lambda_3(\sigma,\varepsilon)$ and $\Lambda_4(\sigma,\varepsilon)$ are of the dimensions $n \times n$, $n \times m$, $m \times n$ and $m \times m$, respectively.

Based on the matrix-valued functions $\bar{\Lambda}_1(\sigma)$ and $\widetilde{\Lambda}_4(\xi)$, let us construct the following matrix-valued functions:

$$\widetilde{\Lambda}_2(\xi) = \begin{cases} 0, & \xi < 0, \\ \int_\xi^{+\infty} A_{21}^T(t_c)\widetilde{\Lambda}_4(\chi)d\chi, & \xi \geq 0, \end{cases} \tag{3.20}$$

$$\bar{\Lambda}_2(\sigma) = \bar{\Lambda}_1(\sigma)\widetilde{\Lambda}_2(0), \quad \sigma \geq 0, \tag{3.21}$$

$$\bar{\Lambda}_3(\sigma) = -\big(A_{22}^T(\sigma)\big)^{-1} A_{12}^T(\sigma)\bar{\Lambda}_1(\sigma), \quad \sigma \geq 0, \tag{3.22}$$

$$\widetilde{\Lambda}_3(\xi) = -\widetilde{\Lambda}_4(\xi)\bar{\Lambda}_3(t_c), \quad \xi \in (-\infty, +\infty), \tag{3.23}$$

$$\bar{\Lambda}_4(\sigma) = -\big(A_{22}^T(\sigma)\big)^{-1} A_{12}^T(\sigma)\bar{\Lambda}_2(\sigma), \quad \sigma \geq 0. \tag{3.24}$$

Using the equations (3.20), (3.23) and the inequality (3.18), we obtain

$$\|\widetilde{\Lambda}_2(\xi)\| \leq a\exp(-\beta\xi), \quad \|\widetilde{\Lambda}_3(\xi)\| \leq a\exp(-\beta\xi), \quad \xi \geq 0, \tag{3.25}$$

where $a > 0$ is some constant.

As a particular (undelayed) case of the results of [22], we obtain the following assertion.

Proposition 3.2. *Let the assumption (A) be valid. Then, there exists a positive number ε_1 such that, for all $\varepsilon \in (0, \varepsilon_1]$, the following inequalities are satisfied:*

$$\begin{aligned} \|\Lambda_1(\sigma,\varepsilon) - \bar{\Lambda}_1(\sigma)\| \leq a\varepsilon, \quad & \big\|\Lambda_2(\sigma,\varepsilon) - \bar{\Lambda}_2(\sigma) + \widetilde{\Lambda}_2\big((t_c-\sigma)/\varepsilon\big)\big\| \leq a\varepsilon, \quad \sigma \geq 0, \\ & \big\|\Lambda_3(\sigma,\varepsilon) - \bar{\Lambda}_3(\sigma) - \widetilde{\Lambda}_3\big((t_c-\sigma)/\varepsilon\big)\big\| \leq a\varepsilon, \quad \sigma \geq 0, \\ \big\|\Lambda_4(\sigma,\varepsilon) - (1/\varepsilon)\widetilde{\Lambda}_4\big((t_c-\sigma)/\varepsilon\big) - \bar{\Lambda}_4(\sigma)\big\| & \leq a\big[\varepsilon + \exp\big(-\beta(t_c-\sigma)/\varepsilon\big)\big], \quad \sigma \in [0, t_c], \end{aligned} \tag{3.26}$$

where $a > 0$ is some constant independent of ε.

Moreover,

$$\big\|\Lambda_4(\sigma,\varepsilon) - (1/\varepsilon)\widetilde{\Lambda}_4\big((t_c-\sigma)/\varepsilon\big) - \bar{\Lambda}_4(\sigma)\big\| = 0, \quad \sigma > t_c. \tag{3.27}$$

3.3. **ε-free conditions for the invertibility of the matrix $W(t_c,\varepsilon)$.** Let us introduce into the consideration the following matrix-valued functions:

$$\bar{B}_0(t) = -A_{12}(t)A_{22}^{-1}(t)B_{0y}(t), \quad \bar{B}_1(t) = -A_{12}(t)A_{22}^{-1}(t)B_{1y}(t), \ \ t \geq 0, \tag{3.28}$$

$$\bar{\Omega}(\sigma) = \bar{\Lambda}_1^T(\sigma)\bar{B}_0(\sigma) + \bar{\Lambda}_1^T(\sigma+h)\bar{B}_1(\sigma+h), \quad \sigma \in [0, t_c]. \tag{3.29}$$

Based on the matrix-valued function $\bar{\Omega}(\sigma)$, let us construct the following matrix:

$$\bar{W}(t_c) = \int_0^{t_c} \bar{\Omega}(\sigma)\bar{\Omega}^T(\sigma)d\sigma. \tag{3.30}$$

Also, we construct the matrix

$$\widetilde{W}(t_c) = \int_0^{+\infty} \widetilde{\Lambda}_4^T(\xi)\big(B_{0y}(t_c)B_{0y}^T(t_c) + B_{1y}(t_c)B_{1y}^T(t_c)\big)\widetilde{\Lambda}_4(\xi)d\xi. \tag{3.31}$$

Proposition 3.3. ([23]) *Let the assumption (A) be valid. Let the matrices $\bar{W}(t_c)$ and $\widetilde{W}(t_c)$ be invertible, i.e.,*

$$\det\bar{W}(t_c) \neq 0, \quad \det\widetilde{W}(t_c) \neq 0. \tag{3.32}$$

Then, there exists a positive number ε_2, ($\varepsilon_2 \leq \varepsilon_1$) such that, for all $\varepsilon \in (0, \varepsilon_2]$, the following inequality is satisfied:

$$\det W(t_c,\varepsilon) \neq 0. \tag{3.33}$$

3.4. **Block-wise estimate of the matrix** $W(t_c,\varepsilon)$**.** Let us partition the matrix $W(t_c,\varepsilon)$ into blocks as:

$$W(t_c,\varepsilon)=\begin{pmatrix} W_1(t_c,\varepsilon) & W_2(t_c,\varepsilon)\\ W_2^T(t_c,\varepsilon) & W_3(t_c,\varepsilon)\end{pmatrix}, \tag{3.34}$$

where the blocks $W_1(t_c,\varepsilon)$, $W_2(t_c,\varepsilon)$ and $W_3(t_c,\varepsilon)$ are of the dimensions $n\times n$, $n\times m$ and $m\times m$, respectively.

Let us introduce into the consideration the following matrix:

$$\begin{aligned}\widehat{W}(t_c)=\bar{W}(t_c)\widetilde{\Lambda}_2(0)-\bar{\Lambda}_3^T(t_c)\widetilde{W}(t_c)\\ -\bar{\Lambda}_3^T(t_c)\big(B_{0y}(t_c)B_{0y}^T(t_c)+B_{1y}(t_c)B_{1y}^T(t_c)\big)\big(A_{22}^T(t_c)\big)^{-1}\\ -\bar{\Lambda}_3^T(t_c-h)B_{0y}(t_c-h)B_{1y}^T(t_c)\big(A_{22}^T(t_c)\big)^{-1}.\end{aligned} \tag{3.35}$$

Quite similarly to the results of [23], we have the following assertion.

Proposition 3.4. *Let the assumption (A) be valid. Then, there exists a positive number* ε_3, *(*$\varepsilon_3\le\varepsilon_1$*) such that, for all* $\varepsilon\in(0,\varepsilon_3]$, *the following inequalities are satisfied:*

$$\big\|W_1(t_c,\varepsilon)-\bar{W}(t_c)\big\|\le a\varepsilon, \tag{3.36}$$

$$\big\|W_2(t_c,\varepsilon)-\widehat{W}(t_c)\big\|\le a\varepsilon, \tag{3.37}$$

$$\big\|\varepsilon W_3(t_c,\varepsilon)-\widetilde{W}(t_c)\big\|\le a\varepsilon, \tag{3.38}$$

where $a>0$ *is some constant independent of* ε*.*

3.5. **Reduced minimum energy control problem.** The equation of dynamics in this problem is obtained in the following way. First, we set formally $\varepsilon=0$ in the system (2.1)-(2.2) and replace x, y and u with x_r, y_r, u_r. This procedure yields the system

$$\frac{dx_r(t)}{dt}=A_{11}(t)x_r(t)+A_{12}(t)y_r(t),\qquad t\ge 0, \tag{3.39}$$

$$0=A_{21}(t)x_r(t)+A_{22}(t)y_r(t)+B_{0y}(t)u_r(t)+B_{1y}(t)u_r(t-h),\qquad t\ge 0, \tag{3.40}$$

where $x_r(t)\in\mathbb{R}^n$ and $y_r(t)\in\mathbb{R}^m$ are state variables; $u_r(t)\in\mathbb{R}^s$; $u_r(t+\eta)$, $\eta\in[-h,0]$ is a control variable. This system is a descriptor (differential-algebraic) system with a delay in the control.

Subject to the assumption (A),

$$\det A_{22}(t)\neq 0,\quad t\in[0,t_c]. \tag{3.41}$$

Thus, the differential-algebraic system (3.39)-(3.40) can be reduced in the interval $[0,t_c]$ to the pure differential equation with respect to $x_r(t)$ with a delay in the control

$$\frac{dx_r(t)}{dt}=\bar{A}(t)x_r(t)+\bar{B}_0(t)u_r(t)+\bar{B}_1(t)u_r(t-h),\quad t\in[0,t_c], \tag{3.42}$$

where $\bar{A}(t)$ and $\bar{B}_0(t)$, $\bar{B}_1(t)$ are given in the equations (3.16) and (3.28), respectively.

The equation (3.42) is the equation of dynamics in the reduced minimum energy control problem. The functional of this problem is obtained from the functional (2.6) by replacing there J with J_r and u with u_r. Thus, we have

$$J_r(u_r)=\int_0^{t_c}u_r^T(t)u_r(t)dt. \tag{3.43}$$

Let $\varphi_u(\eta)$ and $x_0 \in \mathbb{R}^n$, $x_c \in \mathbb{R}^n$ be the same vector-valued function and the same vectors as in (2.4) and (2.5). Consider the set U_r of all piecewise continuous s-dimensional vector-valued functions $u_r(t)$, $t \in [0,t_c]$, such that the equation (3.42) has a solution satisfying the initial and terminal conditions

$$u_r(\eta) = \varphi_u(\eta), \quad \eta \in [-h,0), \quad x_r(0) = x_0, \quad x_r(t_c) = x_c. \tag{3.44}$$

The minimum energy control problem for the equation (3.42) is to find a control $u_r^*(t) \in U_r$ such that

$$J_r\big(u_r^*(t)\big) \le J_r\big(u_r(t)\big) \quad \forall u_r(t) \in U_r. \tag{3.45}$$

In what follows of the paper, we call this problem the Reduced Minimum Energy Problem (RMEP) associated with the OMEP.

Similarly to Proposition 3.1, we have the following assertion.

Proposition 3.5. *Let the matrix $\bar{W}(t_c)$ be invertible, i.e., the first inequality in (3.32) be valid. Then, the solution of the RMEP exists and has the form:*

$$u_r^*(t) = \bar{\Omega}^T(t)\bar{W}^{-1}(t_c)(x_c - w_{r,0}), \quad t \in [0,t_c], \tag{3.46}$$

where $w_{r,0}$ is

$$w_{r,0} = \bar{\Lambda}_1^T(0)x_0 + \int_{-h}^{0} \bar{\Lambda}_1^T(\eta+h)\bar{B}_1(\eta+h)\varphi_u(\eta)d\eta. \tag{3.47}$$

Moreover,

$$J_r^* \triangleq J_r\big(u_r^*(t)\big) = (x_c - w_{r,0})^T\bar{W}^{-1}(t_c)(x_c - w_{r,0}). \tag{3.48}$$

4. Main Results

4.1. ε -free conditions for the existence of the OMEP solution.

Theorem 4.1. *Let the assumption (A) be valid. Let the matrices $\bar{W}(t_c)$ and $\widetilde{W}(t_c)$ be invertible. Then, for all $\varepsilon \in (0,\varepsilon_2]$, the OMEP has the solution given by the equations (3.12)-(3.13). The optimal value of the functional in this problem is given by the equation (3.14).*

Proof. The theorem directly follows from Propositions 3.1 and 3.3. □

4.2. Block-wise estimate of the matrix $W^{-1}(t_c,\varepsilon)$.

Denote

$$V(t_c,\varepsilon) \triangleq W^{-1}(t_c,\varepsilon) = \begin{pmatrix} V_1(t_c,\varepsilon) & V_2(t_c,\varepsilon) \\ V_2^T(t_c,\varepsilon) & V_3(t_c,\varepsilon) \end{pmatrix}, \tag{4.1}$$

where the blocks $V_1(t_c,\varepsilon)$, $V_2(t_c,\varepsilon)$ and $V_3(t_c,\varepsilon)$ are of the dimensions $n\times n$, $n\times m$ and $m\times m$, respectively.

Lemma 4.1. *Let the assumption (A) be valid. Let the matrices $\bar{W}(t_c)$ and $\widetilde{W}(t_c)$ be invertible. Then, there exists a positive number $\varepsilon_4 \le \min\{\varepsilon_2,\varepsilon_3\}$ such that, for all $\varepsilon \in (0,\varepsilon_4]$, the following inequalities are satisfied:*

$$\big\|V_1(t_c,\varepsilon) - \bar{W}^{-1}(t_c)\big\| \le a\varepsilon, \tag{4.2}$$

$$\big\|V_2(t_c,\varepsilon) + \varepsilon\bar{W}^{-1}(t_c)\widehat{W}(t_c)\widetilde{W}^{-1}(t_c)\big\| \le a\varepsilon^2, \tag{4.3}$$

$$\big\|V_3(t_c,\varepsilon) - \varepsilon\widetilde{W}^{-1}(t_c)\big\| \le a\varepsilon^2, \tag{4.4}$$

where $a > 0$ is some constant independent of ε.

Proof. Remember that we assume the matrix $\bar{W}(t_c)$ to be invertible. Therefore, by virtue of the inequality (3.36), there exists a positive number $\bar{\varepsilon}_2 \leq \min\{\varepsilon_2, \varepsilon_3\}$ such that the matrix $W_1(t_c, \varepsilon)$ is invertible for all $\varepsilon \in (0, \bar{\varepsilon}_2]$, and

$$\left\|W_1^{-1}(t_c,\varepsilon) - \bar{W}^{-1}(t_c)\right\| \leq a\varepsilon, \quad \varepsilon \in (0, \bar{\varepsilon}_2], \tag{4.5}$$

where $a > 0$ is some constant independent of ε.

Let us note that, by virtue of Proposition 3.3, the matrix $W(t_c, \varepsilon)$ is invertible for all $\varepsilon \in (0, \bar{\varepsilon}_2]$.

Applying the Frobenius formula (see e.g. [24]) to the calculation of the inverse matrix for $W(t_c, \varepsilon)$, given by its block representation (3.34), we obtain the blocks of the matrix $V(t_c, \varepsilon)$ (see (4.1)) as:

$$V_1(t_c,\varepsilon) = W_1^{-1}(t_c,\varepsilon) + W_1^{-1}(t_c,\varepsilon)W_2(t_c,\varepsilon)H(t_c,\varepsilon)W_2^T(t_c,\varepsilon)W_1^{-1}(t_c,\varepsilon), \tag{4.6}$$

$$V_2(t_c,\varepsilon) = -W_1^{-1}(t_c,\varepsilon)W_2(t_c,\varepsilon)H(t_c,\varepsilon), \tag{4.7}$$

$$V_3(t_c,\varepsilon) = H(t_c,\varepsilon), \tag{4.8}$$

where

$$H(t_c,\varepsilon) = \Big(W_3(t_c,\varepsilon) - W_2^T(t_c,\varepsilon)W_1^{-1}(t_c,\varepsilon)W_2(t_c,\varepsilon)\Big)^{-1}. \tag{4.9}$$

Using (3.37)-(3.38), (4.5) and the assumption on the invertibility of the matrix $\widetilde{W}(t_c)$, we obtain the existence of a positive number $\bar{\varepsilon}_3 \leq \bar{\varepsilon}_2$ such that, for all $\varepsilon \in (0, \bar{\varepsilon}_3]$, the matrix $H(t_c, \varepsilon)$ exists and satisfies the inequality

$$\left\|H(t_c,\varepsilon) - \varepsilon\widetilde{W}^{-1}(t_c)\right\| \leq a\varepsilon^2, \quad \varepsilon \in (0, \bar{\varepsilon}_3], \tag{4.10}$$

where $a > 0$ is some constant independent of ε.

Now, the equations (4.6)-(4.8) and (3.37)-(3.38), (4.5), (4.10) directly yield the inequalities (4.2)-(4.4) for $\varepsilon_4 = \bar{\varepsilon}_3$, which completes the proof of the lemma. □

4.3. **Estimate of the vector** $w_0(\varepsilon)$**.** Consider the matrix-valued function

$$\bar{\Lambda}(\sigma,\varepsilon) = \begin{pmatrix} \bar{\Lambda}_1(\sigma) & \bar{\Lambda}_2(\sigma) \\ \varepsilon\bar{\Lambda}_3(\sigma) & \varepsilon\bar{\Lambda}_4(\sigma) \end{pmatrix}, \quad \sigma \geq 0, \quad \varepsilon \geq 0. \tag{4.11}$$

Based on this matrix, let us construct the vector

$$\bar{w}_0(\varepsilon) = \bar{\Lambda}^T(0,0)z_0 + \int_{-h}^{0} \bar{\Lambda}^T(\eta+h,\varepsilon)B_1(\eta+h,\varepsilon)\varphi_u(\eta)d\eta, \quad \varepsilon > 0. \tag{4.12}$$

Remark 4.1. As it will be shown below, the vector $\bar{w}_0(\varepsilon)$ is independent of ε. Therefore, in what follows, we use the notation $\bar{w}_0$ for this vector.

Lemma 4.2. *Let the assumption (A) be valid. Then, there exists a positive number $\varepsilon_5 \leq \varepsilon_1$ such that, for all $\varepsilon \in (0, \varepsilon_5]$, the following inequality is satisfied:*

$$\left\|w_0(\varepsilon) - \bar{w}_0\right\| \leq a\varepsilon. \tag{4.13}$$

Proof. Using the definition of the vector $w_0(\varepsilon)$ (see equation (3.13)), as well as the block form of the vector z_0 (see the equation (3.1)) and the block representations of the matrices $B_1(t, \varepsilon)$, $\Lambda(\sigma, \varepsilon)$ (see equations (3.2), (3.19)), we can represent the vector $w_0(\varepsilon)$ in the block form as:

$$w_0(\varepsilon) = \begin{pmatrix} w_{0,1}(\varepsilon) \\ w_{0,2}(\varepsilon) \end{pmatrix}, \tag{4.14}$$

where

$$w_{0,1}(\varepsilon) = \Lambda_1^T(0,\varepsilon)x_0 + \varepsilon\Lambda_3^T(0,\varepsilon)y_0 + \int_{-h}^{0} \Lambda_3^T(\eta+h,\varepsilon)B_{1y}(\eta+h)\varphi_u(\eta)d\eta, \tag{4.15}$$

$$w_{0,2}(\varepsilon) = \Lambda_2^T(0,\varepsilon)x_0 + \varepsilon\Lambda_4^T(0,\varepsilon)y_0 + \int_{-h}^{0} \Lambda_4^T(\eta+h,\varepsilon)B_{1y}(\eta+h)\varphi_u(\eta)d\eta. \tag{4.16}$$

Similarly, using equations (4.11), (4.12), as well as (3.1) and (3.2), we can represent the vector $\bar{w}_0$ in the following block form:

$$\bar{w}_0 = \begin{pmatrix} \bar{w}_{0,1} \\ \bar{w}_{0,2} \end{pmatrix}, \tag{4.17}$$

where

$$\bar{w}_{0,1} = \bar{\Lambda}_1^T(0)x_0 + \int_{-h}^{0} \bar{\Lambda}_3^T(\eta+h)B_{1y}(\eta+h)\varphi_u(\eta)d\eta, \tag{4.18}$$

$$\bar{w}_{0,2} = \bar{\Lambda}_2^T(0)x_0 + \int_{-h}^{0} \bar{\Lambda}_4^T(\eta+h)B_{1y}(\eta+h)\varphi_u(\eta)d\eta. \tag{4.19}$$

Denote

$$\Delta w_0(\varepsilon) \stackrel{\triangle}{=} w_0(\varepsilon) - \bar{w}_0. \tag{4.20}$$

Due to the equations (4.14)-(4.16) and (4.17)-(4.19), we have

$$\Delta w_0(\varepsilon) = \begin{pmatrix} \Delta w_{0,1}(\varepsilon) \\ \Delta w_{0,2}(\varepsilon) \end{pmatrix}, \tag{4.21}$$

where

$$\begin{aligned}\Delta w_{0,1}(\varepsilon) = w_{0,1}(\varepsilon) - \bar{w}_{0,1} = \big(\Lambda_1^T(0,\varepsilon) - \bar{\Lambda}_1^T(0)\big)x_0 + \varepsilon\Lambda_3^T(0,\varepsilon)y_0 \\ + \int_{-h}^{0} \big(\Lambda_3^T(\eta+h,\varepsilon) - \bar{\Lambda}_3^T(\eta+h)\big)B_{1y}(\eta+h)\varphi_u(\eta)d\eta,\end{aligned} \tag{4.22}$$

$$\begin{aligned}\Delta w_{0,2}(\varepsilon) = w_{0,2}(\varepsilon) - \bar{w}_{0,2} = \big(\Lambda_2^T(0,\varepsilon) - \bar{\Lambda}_2^T(0)\big)x_0 + \varepsilon\Lambda_4^T(0,\varepsilon)y_0 \\ + \int_{-h}^{0} \big(\Lambda_4^T(\eta+h,\varepsilon) - \bar{\Lambda}_4^T(\eta+h)\big)B_{1y}(\eta+h)\varphi_u(\eta)d\eta.\end{aligned} \tag{4.23}$$

Now, applying Proposition 3.2 and the inequalities (3.18), (3.25) to the estimation of the vectors $\Delta w_{0,1}(\varepsilon)$ and $\Delta w_{0,2}(\varepsilon)$, we obtain the existence of a positive number $\varepsilon_5 \le \varepsilon_1$ such that:

$$\big\|\Delta w_{0,1}(\varepsilon)\big\| \le a\varepsilon, \quad \big\|\Delta w_{0,1}(\varepsilon)\big\| \le a\varepsilon, \quad \varepsilon \in (0,\varepsilon_5], \tag{4.24}$$

where $a > 0$ is some constant independent of ε.

The equations (4.20)-(4.23), along with the inequalities in (4.24) directly yield the inequality (4.13). Thus, the lemma is proven. □

4.4. **Approximate solution of the OMEP.** Let us introduce into the consideration the following matrices:

$$\Lambda_{\mathrm{appr}}(t,\varepsilon)=\begin{pmatrix}\bar{\Lambda}_1(t) & \bar{\Lambda}_2(t)+\widetilde{\Lambda}_2\big((t_c-t)/\varepsilon\big)\\ \varepsilon\big[\bar{\Lambda}_3(t)+\widetilde{\Lambda}_3\big((t_c-t)/\varepsilon\big)\big] & \widetilde{\Lambda}_4\big((t_c-t)/\varepsilon\big)+\varepsilon\bar{\Lambda}_4(t)\end{pmatrix}, \tag{4.25}$$

$$V_{\mathrm{appr}}(t_c,\varepsilon)=\begin{pmatrix}V_{\mathrm{appr},1}(t_c,\varepsilon) & V_{\mathrm{appr},2}(t_c,\varepsilon)\\ \big(V_{\mathrm{appr},2}(t_c,\varepsilon)\big)^T & V_{\mathrm{appr},3}(t_c,\varepsilon)\end{pmatrix}$$

$$=\begin{pmatrix}\bar{W}^{-1}(t_c) & -\varepsilon\bar{W}^{-1}(t_c)\widehat{W}(t_c)\widetilde{W}^{-1}(t_c)\\ -\varepsilon\big(\bar{W}^{-1}(t_c)\widehat{W}(t_c)\widetilde{W}^{-1}(t_c)\big)^T & \varepsilon\widetilde{W}^{-1}(t_c)\end{pmatrix}, \tag{4.26}$$

$$\Omega_{\mathrm{appr}}(t,\varepsilon)=\big(\Lambda_{\mathrm{appr}}(t,\varepsilon)\big)^T B_0(t,\varepsilon)+\big(\Lambda_{\mathrm{appr}}(t+h,\varepsilon)\big)^T B_1(t+h,\varepsilon). \tag{4.27}$$

Using the equations (3.2) and (4.25), we can represent the matrix $\Omega_{\mathrm{appr}}(t,\varepsilon)$ in the block form as:

$$\Omega_{\mathrm{appr}}(t,\varepsilon)=\begin{pmatrix}\Omega_{\mathrm{appr},1}(t,\varepsilon)\\ \Omega_{\mathrm{appr},2}(t,\varepsilon)\end{pmatrix}=$$

$$\begin{pmatrix}\big[\bar{\Lambda}_3(t)+\widetilde{\Lambda}_3\big((t_c-t)/\varepsilon\big)\big]^T B_{0y}(t)+\big[\bar{\Lambda}_3(t+h)+\widetilde{\Lambda}_3\big((t_c-t-h)/\varepsilon\big)\big]^T B_{1y}(t+h)\\ \Big[\frac{1}{\varepsilon}\widetilde{\Lambda}_4\big((t_c-t)/\varepsilon\big)+\bar{\Lambda}_4(t)\Big]^T B_{0y}(t)+\Big[\frac{1}{\varepsilon}\widetilde{\Lambda}_4\big((t_c-t-h)/\varepsilon\big)+\bar{\Lambda}_4(t+h)\Big]^T B_{1y}(t+h)\end{pmatrix}. \tag{4.28}$$

Based on the matrices $V_{\mathrm{appr}}(t_c,\varepsilon)$, $\Omega_{\mathrm{appr}}(t,\varepsilon)$ and on the vector $\bar{w}_0$, let us construct the following control function:

$$u_{\mathrm{appr}}(t,\varepsilon)=\big(\Omega_{\mathrm{appr}}(t,\varepsilon)\big)^T V_{\mathrm{appr}}(t_c,\varepsilon)\big(z_c-\bar{w}_0\big),\quad t\in[0,t_c]. \tag{4.29}$$

Denote

$$\varepsilon^*\stackrel{\triangle}{=}\min\{\varepsilon_4,\varepsilon_5\}. \tag{4.30}$$

Theorem 4.2. *Let the assumption (A) be valid. Let the matrices $\bar{W}(t_c)$ and $\widetilde{W}(t_c)$ be invertible. Then, for all $\varepsilon\in(0,\varepsilon^*]$, the following inequality is satisfied:*

$$\big\|u^*(t,\varepsilon)-u_{\mathrm{appr}}(t,\varepsilon)\big\|\le a\varepsilon,\quad t\in[0,t_c], \tag{4.31}$$

where $a>0$ is some constant independent of ε

Proof. Using the block representations of the matrices $B_0(t,\varepsilon)$, $B_1(t,\varepsilon)$ and $\Lambda(t,\varepsilon)$ (see the equations (3.2) and (3.19)), we can represent the matrix $\Omega(t,\varepsilon)$ in the following block form:

$$\Omega(t,\varepsilon)=\begin{pmatrix}\Omega_1(t,\varepsilon)\\ \Omega_2(t,\varepsilon)\end{pmatrix}=$$
$$\begin{pmatrix}\big(\Lambda_3(t,\varepsilon)\big)^T B_{0y}(t)+\big(\Lambda_3(t+h,\varepsilon)\big)^T B_{1y}(t+h)\\ \\ \big(\Lambda_4(t,\varepsilon)\big)^T B_{0y}(t)+\big(\Lambda_4(t+h,\varepsilon)\big)^T B_{1y}(t+h)\end{pmatrix}. \tag{4.32}$$

Further, using this block representation, as well as the block representations of the matrix $W^{-1}(t_c,\varepsilon)$ and the vectors z_c, $w_0(\varepsilon)$ (see equations (4.1) and (3.1), (4.14)), we can rewrite the expression (3.12) for the control $u^*(t,\varepsilon)$ as:

$$u^*(t,\varepsilon)=\big(\Omega_1^T(t,\varepsilon)V_1(t_c,\varepsilon)+\Omega_2^T(t,\varepsilon)V_2^T(t_c,\varepsilon)\big)\big(x_c-w_{0,1}(\varepsilon)\big)$$
$$+\big(\Omega_1^T(t,\varepsilon)V_2(t_c,\varepsilon)+\Omega_2^T(t,\varepsilon)V_3(t_c,\varepsilon)\big)\big(y_c-w_{0,2}(\varepsilon)\big),\quad t\in[0,t_c]. \tag{4.33}$$

Similarly, using t (4.17), (4.26) and (4.28), we can rewrite the expression (4.29) for the control function $u_{\mathrm{appr}}(t,\varepsilon)$ in the form

$$u_{\mathrm{appr}}(t,\varepsilon)=\big(\Omega_{\mathrm{appr},1}^T(t,\varepsilon)V_{\mathrm{appr},1}(t_c,\varepsilon)+\Omega_{\mathrm{appr},2}^T(t,\varepsilon)V_{\mathrm{appr},2}^T(t_c,\varepsilon)\big)(x_c-\bar{w}_{0,1})$$
$$+\big(\Omega_{\mathrm{appr},1}^T(t,\varepsilon)V_{\mathrm{appr},2}(t_c,\varepsilon)+\Omega_{\mathrm{appr},2}^T(t,\varepsilon)V_{\mathrm{appr},3}(t_c,\varepsilon)\big)(y_c-\bar{w}_{0,2}),\quad t\in[0,t_c]. \tag{4.34}$$

Using (4.14), (4.17), (4.20) and (4.21), we can rewrite equation (4.33) as follows:

$$u^*(t,\varepsilon)=\big(\Omega_1^T(t,\varepsilon)V_1(t_c,\varepsilon)+\Omega_2^T(t,\varepsilon)V_2^T(t_c,\varepsilon)\big)(x_c-\bar{w}_{0,1})$$
$$+\big(\Omega_1^T(t,\varepsilon)V_2(t_c,\varepsilon)+\Omega_2^T(t,\varepsilon)V_3(t_c,\varepsilon)\big)\big(y_c-\bar{w}_{0,2})-\Theta(t,t_c,\varepsilon),\quad t\in[0,t_c], \tag{4.35}$$

where

$$\Theta(t,t_c,\varepsilon)=\big(\Omega_1^T(t,\varepsilon)V_1(t_c,\varepsilon)+\Omega_2^T(t,\varepsilon)V_2^T(t_c,\varepsilon)\big)\Delta w_{0,1}(\varepsilon)$$
$$+\big(\Omega_1^T(t,\varepsilon)V_2(t_c,\varepsilon)+\Omega_2^T(t,\varepsilon)V_3(t_c,\varepsilon)\big)\Delta w_{0,2}(\varepsilon),\quad t\in[0,t_c]. \tag{4.36}$$

Estimating the vector-valued function $\Theta(t,t_c,\varepsilon)$ by application of Proposition 3.2 and Lemmas 4.1, 4.2, we directly obtain the inequality

$$\big\|\Theta(t,t_c,\varepsilon)\big\|\le a\varepsilon,\quad t\in[0,t_c],\quad \varepsilon\in\big(0,\min\{\varepsilon_4,\varepsilon_5\}\big], \tag{4.37}$$

where $a>0$ is some constant independent of ε.

Subtracting equation (4.34) from equation (4.35), we have

$$\begin{aligned} u^*(t,\varepsilon)-u_{\text{appr}}(t,\varepsilon) &= \Big[\big(\Omega_1^T(t,\varepsilon)V_1(t_c,\varepsilon)-\Omega_{\text{appr},1}^T(t,\varepsilon)V_{\text{appr},1}(t_c,\varepsilon)\big) \\ &+\big(\Omega_2^T(t,\varepsilon)V_2^T(t_c,\varepsilon)-\Omega_{\text{appr},2}^T(t,\varepsilon)V_{\text{appr},2}^T(t_c,\varepsilon)\big)\Big](x_c-\bar{w}_{0,1}) \\ &+\Big[\big(\Omega_1^T(t,\varepsilon)V_2(t_c,\varepsilon)-\Omega_{\text{appr},1}^T(t,\varepsilon)V_{\text{appr},2}(t_c,\varepsilon)\big) \\ &+\big(\Omega_2^T(t,\varepsilon)V_3(t_c,\varepsilon)-\Omega_{\text{appr},2}^T(t,\varepsilon)V_{\text{appr},3}(t_c,\varepsilon)\big)\Big](y_c-\bar{w}_{0,2}) \\ &-\Theta(t,t_c,\varepsilon), \quad t\in[0,t_c], \end{aligned} \tag{4.38}$$

which yields

$$\begin{aligned} \big\|u^*(t,\varepsilon)-u_{\text{appr}}(t,\varepsilon)\big\| &\le \Big[\big\|\Omega_1^T(t,\varepsilon)V_1(t_c,\varepsilon)-\Omega_{\text{appr},1}^T(t,\varepsilon)V_{\text{appr},1}(t_c,\varepsilon)\big\| \\ &+\big\|\Omega_2^T(t,\varepsilon)V_2^T(t_c,\varepsilon)-\Omega_{\text{appr},2}^T(t,\varepsilon)V_{\text{appr},2}^T(t_c,\varepsilon)\big\|\Big]\big\|x_c-\bar{w}_{0,1}\big\| \\ &+\Big[\big\|\Omega_1^T(t,\varepsilon)V_2(t_c,\varepsilon)-\Omega_{\text{appr},1}^T(t,\varepsilon)V_{\text{appr},2}(t_c,\varepsilon)\big\| \\ &+\big\|\Omega_2^T(t,\varepsilon)V_3(t_c,\varepsilon)-\Omega_{\text{appr},2}^T(t,\varepsilon)V_{\text{appr},3}(t_c,\varepsilon)\big\|\Big]\big\|y_c-\bar{w}_{0,2}\big\| \\ &+\big\|\Theta(t,t_c,\varepsilon)\big\|, \quad t\in[0,t_c]. \end{aligned} \tag{4.39}$$

Let us estimate the terms in the square brackets in the right-hand side of this inequality. Applying Proposition 3.2 and Lemma 4.1 to this estimation, we obtain the following inequalities for all $t\in[0,t_c]$ and $\varepsilon\in(0,\varepsilon_4]$:

$$\begin{aligned} \big\|\Omega_1^T(t,\varepsilon)V_1(t_c,\varepsilon)-\Omega_{\text{appr},1}^T(t,\varepsilon)V_{\text{appr},1}(t_c,\varepsilon)\big\| &\le a\varepsilon, \\ \big\|\Omega_2^T(t,\varepsilon)V_2^T(t_c,\varepsilon)-\Omega_{\text{appr},2}^T(t,\varepsilon)V_{\text{appr},2}^T(t_c,\varepsilon)\big\| &\le a\varepsilon\big[\varepsilon+\exp\big(-\beta(t_c-t)/\varepsilon\big)\big], \\ \big\|\Omega_1^T(t,\varepsilon)V_2(t_c,\varepsilon)-\Omega_{\text{appr},1}^T(t,\varepsilon)V_{\text{appr},2}(t_c,\varepsilon)\big\| &\le a\varepsilon^2, \\ \big\|\Omega_2^T(t,\varepsilon)V_3(t_c,\varepsilon)-\Omega_{\text{appr},2}^T(t,\varepsilon)V_{\text{appr},3}(t_c,\varepsilon)\big\| &\le a\varepsilon\big[\varepsilon+\exp\big(-\beta(t_c-t)/\varepsilon\big)\big], \end{aligned} \tag{4.40}$$

where $a>0$ is some constant independent of ε.

Now, inequality (4.39), along with inequalities (4.37) and (4.40) immediately yields inequality (4.31), which completes the proof of the theorem. □

Let, for any given $\varepsilon\in(0,\varepsilon^*]$, $z_{\text{appr}}(t,\varepsilon)$, $t\in[0,t_c]$ be the solution of the system (3.3) with $u(t)=u_{\text{appr}}(t,\varepsilon)$ and subject to the initial conditions (3.4). Thus,

$$\begin{aligned} z_{\text{appr}}(t,\varepsilon) &= \Phi(t,0,\varepsilon)z_0 \\ +\int_0^t \Phi(t,\sigma,\varepsilon)\big[B_0(\sigma,\varepsilon)u_{\text{appr}}(\sigma,\varepsilon)&+B_1(\sigma,\varepsilon)u_{\text{appr}}(\sigma-h,\varepsilon)\big]d\sigma, \quad t\in[0,t_c]. \end{aligned} \tag{4.41}$$

Substituting $t = t_c$ into this expression for $z_{\mathrm{appr}}(t,\varepsilon)$, and using equations (3.7) and (3.9), we obtain

$$z_{\mathrm{appr}}(t_c,\varepsilon) = \Lambda^T(0,\varepsilon)z_0 + \int_0^{t_c} \Lambda^T(\sigma,\varepsilon)\big[B_0(\sigma,\varepsilon)u_{\mathrm{appr}}(\sigma,\varepsilon) + B_1(\sigma,\varepsilon)u_{\mathrm{appr}}(\sigma - h,\varepsilon)\big]d\sigma. \tag{4.42}$$

Corollary 4.1. *Let the assumption (A) be valid. Let the matrices $\bar{W}(t_c)$ and $\widetilde{W}(t_c)$ be invertible. Then, for all $\varepsilon \in (0,\varepsilon^*]$, the following inequality is satisfied:*

$$\big\|z_c - z_{\mathrm{appr}}(t_c,\varepsilon)\big\| \le a\varepsilon, \tag{4.43}$$

where $a > 0$ is some constant independent of ε.

Proof. Since the control $u^*(t,\varepsilon)$, $t \in [0,t_c]$ is the solution of the OMEP, then the vector z_c can be represented as:

$$z_c = \Lambda^T(0,\varepsilon)z_0 + \int_0^{t_c} \Lambda^T(\sigma,\varepsilon)\big[B_0(\sigma,\varepsilon)u^*(\sigma,\varepsilon) + B_1(\sigma,\varepsilon)u^*(\sigma - h,\varepsilon)\big]d\sigma. \tag{4.44}$$

Subtracting equation (4.42) from equation (4.44), we obtain

$$\begin{aligned} z_c - z_{\mathrm{appr}}(t_c,\varepsilon) = \int_0^{t_c} \Big[&\Lambda^T(\sigma,\varepsilon)B_0(\sigma,\varepsilon)\big(u^*(\sigma,\varepsilon) - u_{\mathrm{appr}}(\sigma,\varepsilon)\big) \\ &+\Lambda^T(\sigma,\varepsilon)B_1(\sigma,\varepsilon)\big(u^*(\sigma - h,\varepsilon) - u_{\mathrm{appr}}(\sigma - h,\varepsilon)\big)\Big]d\sigma, \end{aligned} \tag{4.45}$$

which yields

$$\begin{aligned} \big\|z_c - z_{\mathrm{appr}}(t_c,\varepsilon)\big\| \le \int_0^{t_c} \Big[&\big\|\Lambda^T(\sigma,\varepsilon)B_0(\sigma,\varepsilon)\big\|\big\|u^*(\sigma,\varepsilon) - u_{\mathrm{appr}}(\sigma,\varepsilon)\big\| \\ &+\big\|\Lambda^T(\sigma,\varepsilon)B_1(\sigma,\varepsilon)\big\|\big\|u^*(\sigma - h,\varepsilon) - u_{\mathrm{appr}}(\sigma - h,\varepsilon)\big\|\Big]d\sigma. \end{aligned} \tag{4.46}$$

Using the block representations of the matrices $B_0(\sigma,\varepsilon)$, $B_1(\sigma,\varepsilon)$ and $\Lambda(\sigma,\varepsilon)$ (see equations (3.2) and (3.19)), we obtain, for $\sigma \in [0,t_c]$,

$$\Lambda^T(\sigma,\varepsilon)B_0(\sigma,\varepsilon) = \begin{pmatrix} \Lambda_3^T(\sigma,\varepsilon)B_{0y}(\sigma) \\ \Lambda_4^T(\sigma,\varepsilon)B_{0y}(\sigma) \end{pmatrix}, \ \Lambda^T(\sigma,\varepsilon)B_1(\sigma,\varepsilon) = \begin{pmatrix} \Lambda_3^T(\sigma,\varepsilon)B_{1y}(\sigma) \\ \Lambda_4^T(\sigma,\varepsilon)B_{1y}(\sigma) \end{pmatrix}. \tag{4.47}$$

Now, by substitution of (4.47) into the right-hand side of (4.46), and using Proposition 3.2, Theorem 4.2 and inequality (3.18), one directly converts inequality (4.46) to inequality (4.43). Thus, the corollary is proven. □

Let, for any given $\varepsilon \in (0,\varepsilon^*]$, $J_{\mathrm{appr}}(\varepsilon)$ be the value of the functional (2.3) calculated for $u = u_{\mathrm{appr}}(t,\varepsilon)$, i.e.,

$$J_{\mathrm{appr}}(\varepsilon) = \int_0^{t_c} \big(u_{\mathrm{appr}}(t,\varepsilon)\big)^T u_{\mathrm{appr}}(t,\varepsilon)dt. \tag{4.48}$$

Corollary 4.2. *Let the assumption (A) be valid. Let the matrices $\bar{W}(t_c)$ and $\widetilde{W}(t_c)$ be invertible. Then, the following inequality is satisfied:*

$$\left|J\big(u^*(t,\varepsilon)\big) - J_{\text{appr}}(\varepsilon)\right| \le a\varepsilon, \quad \varepsilon \in (0,\varepsilon^*], \tag{4.49}$$

where $J(u)$ is the functional in the OMEP (see the equation (2.3)); $u^*(t,\varepsilon)$ is the solution of the OMEP; $a>0$ is some constant independent of ε.

Proof. First of all, let us observe the following. Due to equations (4.26), (4.28) and (4.34), and inequalities (3.18) and (3.25), the control function $u_{\text{appr}}(t,\varepsilon)$ is bounded for $(t,\varepsilon) \in [0,t_c] \times (0,\varepsilon^*]$. Therefore, due to inequality (4.31), the control function $u^*(t,\varepsilon)$ also is bounded for $(t,\varepsilon) \in [0,t_c] \times (0,\varepsilon^*]$. Now, the statement of the corollary directly follows from this observation, as well as equations (2.3), (4.48) and Theorem 4.2. □

Theorem 4.3. *Let the assumption (A) be valid. Let the matrices $\bar{W}(t_c)$ and $\widetilde{W}(t_c)$ be invertible. Then, the following inequality is satisfied:*

$$\left|J_\varepsilon^* - J_r^*\right| \le a\varepsilon, \quad \varepsilon \in (0,\varepsilon^*], \tag{4.50}$$

where J_ε^* is the optimal value of the functional in the OMEP (see the equation (3.14)); J_r^* is the optimal value of the functional in the RMEP (see the equation (3.48); $a>0$ is some constant independent of ε.

Proof. First of all, let us note the following. Using equations (3.22), (3.28), (3.47) and (4.18), we immediately obtain that

$$w_{r,0} = \bar{w}_{0,1}. \tag{4.51}$$

Substitution of the block representation for the matrix $W^{-1}(t_c,\varepsilon)$ (see the equation (4.1)) and the block representations for the vectors z_c, $w_0(\varepsilon)$ (see equations (3.1) and (4.14)) into the equation (3.14) yields after a routine algebra the following expression for J_ε^*:

$$J_\varepsilon^* = \big(x_c - w_{0,1}(\varepsilon)\big)^T V_1(t_c,\varepsilon)\big(x_c - w_{0,1}(\varepsilon)\big) + 2\big(x_c - w_{0,1}(\varepsilon)\big)^T V_2(t_c,\varepsilon)\big(y_c - w_{0,2}(\varepsilon)\big) \\ + \big(y_c - w_{0,2}(\varepsilon)\big)^T V_3(t_c,\varepsilon)\big(y_c - w_{0,2}(\varepsilon)\big). \tag{4.52}$$

Let us estimate each of the addends in the right-hand side of this equation. Let us start with the first one. Using equations (3.48), (4.14), (4.17), (4.20), (4.21) and (4.51), we obtain

$$\big(x_c - w_{0,1}(\varepsilon)\big)^T V_1(t_c,\varepsilon)\big(x_c - w_{0,1}(\varepsilon)\big) = \\ \big(x_c - \bar{w}_{0,1} - \Delta w_{0,1}(\varepsilon)\big)^T \left[\bar{W}^{-1}(t_c) + \big(V_1(t_c,\varepsilon) - \bar{W}^{-1}(t_c)\big)\right]\big(x_c - \bar{w}_{0,1} - \Delta w_{0,1}(\varepsilon)\big) = \\ (x_c - \bar{w}_{0,1})^T \bar{W}^{-1}(t_c)(x_c - \bar{w}_{0,1}) + \Gamma(t_c,\varepsilon) = J_r^* + \Gamma(t_c,\varepsilon), \tag{4.53}$$

where

$$\Gamma(t_c,\varepsilon) = -2(x_c - \bar{w}_{0,1})^T \bar{W}^{-1}(t_c)\Delta w_{0,1}(\varepsilon) \\ + (x_c - \bar{w}_{0,1})^T \big(V_1(t_c,\varepsilon) - \bar{W}^{-1}(t_c)\big)\big](x_c - \bar{w}_{0,1}) \\ -2(x_c - \bar{w}_{0,1})^T \big(V_1(t_c,\varepsilon) - \bar{W}^{-1}(t_c)\big)\Delta w_{0,1}(\varepsilon) \\ + \big(\Delta w_{0,1}(\varepsilon)\big)^T \bar{W}^{-1}(t_c)\Delta w_{0,1}(\varepsilon) + \big(\Delta w_{0,1}(\varepsilon)\big)^T \big(V_1(t_c,\varepsilon) - \bar{W}^{-1}(t_c)\big)\Delta w_{0,1}(\varepsilon). \tag{4.54}$$

By virtue of inequalities (4.2) and (4.24), we have the estimate of $\Gamma(t_c,\varepsilon)$:

$$\big|\Gamma(t_c,\varepsilon)\big| \le a\varepsilon, \quad \varepsilon\in(0,\varepsilon^*], \tag{4.55}$$

where $a>0$ is some constant independent of ε.

This inequality, along with equation (4.53), yields

$$\big|\big(x_c - w_{0,1}(\varepsilon)\big)^T V_1(t_c,\varepsilon)\big(x_c - w_{0,1}(\varepsilon)\big) - J_r^*\big| \le a\varepsilon, \quad \varepsilon\in(0,\varepsilon^*], \tag{4.56}$$

where $a>0$ is the same constant as in (4.55).

Proceed to the estimation of the second and third addends in the right-hand side of equation (4.52). Due to Lemma 4.2, the vectors $w_{0,1}(\varepsilon)$ and $w_{0,2}(\varepsilon)$ are bounded for all $\varepsilon\in(0,\varepsilon^*]$. Therefore, by virtue of Lemma 4.1, we have

$$\begin{aligned}
&\big|\big(x_c - w_{0,1}(\varepsilon)\big)^T V_2(t_c,\varepsilon)\big(y_c - w_{0,2}(\varepsilon)\big)\big| \le a\varepsilon, \quad \varepsilon\in(0,\varepsilon^*],\\
&\big|\big(y_c - w_{0,2}(\varepsilon)\big)^T V_3(t_c,\varepsilon)\big(y_c - w_{0,2}(\varepsilon)\big)\big| \le a\varepsilon, \quad \varepsilon\in(0,\varepsilon^*],
\end{aligned} \tag{4.57}$$

where $a>0$ is some constant independent of ε.

Now, equation (4.52), along with inequalities (4.56) and (4.57), directly yields inequality (4.50). Thus, the theorem is proven. □

Corollary 4.3. *Let the assumption (A) be valid. Let the matrices $\bar{W}(t_c)$ and $\widetilde{W}(t_c)$ be invertible. Then, the following inequality is satisfied:*

$$\big|J_{appr}(\varepsilon) - J_r^*\big| \le a\varepsilon, \quad \varepsilon\in(0,\varepsilon^*], \tag{4.58}$$

where $a>0$ is some constant independent of ε.

Proof. The statement of the corollary directly follows from Corollary 4.2, Theorem 4.3 and equation (3.14). □

Remark 4.2. It is important to note the following. First, although the control $u_{\rm appr}(t,\varepsilon)$ depends on ε, the matrices constituting this control can be obtained independently of ε. This means that it is simpler to derive the control $u_{\rm appr}(t,\varepsilon)$ than the control $u^*(t,\varepsilon)$. Second, the control $u_{\rm appr}(t,\varepsilon)$ provides the approximate fulfilment of the terminal condition (3.5) with the accuracy of order of ε for all sufficiently small positive values of this parameter. Third, the value of the functional (2.3), calculated for $u=u_{\rm appr}(t,\varepsilon)$ differs from the optimal value of this functional by the value of order of ε for all sufficiently small $\varepsilon>0$. Based on these observations, we can conclude that $u_{\rm appr}(t,\varepsilon)$ is the approximate solution of the OMEP, and this solution is simpler than the exact solution $u^*(t,\varepsilon)$.

5. Example

Consider the following particular case of the OMEP:

$$\frac{dx(t)}{dt} = -x(t)+y(t), \quad t\ge 0, \tag{5.1}$$

$$\varepsilon\frac{dy(t)}{dt} = x(t)-y(t)+u(t)-u(t-1), \quad t\ge 0, \tag{5.2}$$

$$u(\eta)=0, \quad \eta\in[-1,0); \quad x(0)=1, \quad y(0)=2, \tag{5.3}$$

$$x(2) = 2, \quad y(2) = 1, \tag{5.4}$$

$$J(u) = \int_0^2 u^2(t)dt. \tag{5.5}$$

It is seen that this problem satisfies assumption (A).

Let us derive the approximate solution $u_{\text{appr}}(t,\varepsilon)$ of problem (5.1)-(5.5). We start with obtaining the functions $\bar{\Lambda}_i(\sigma)$, $(i = 1,...,4)$, and $\widetilde{\Lambda}_j(\xi)$, $(j = 2,3,4)$. Using equations (3.15)-(3.16), (3.17), (3.20)-(3.24), we have

$$\bar{\Lambda}_1(\sigma) = \begin{cases} 1, & \sigma \in [0,2], \\ 0, & \sigma > 2, \end{cases} \qquad \widetilde{\Lambda}_4(\xi) = \begin{cases} 0, & \xi < 0, \\ \exp(-\xi), & \xi \geq 0, \end{cases}$$
$$\widetilde{\Lambda}_2(\xi) = \begin{cases} 0, & \xi < 0, \\ \exp(-\xi), & \xi \geq 0, \end{cases} \qquad \bar{\Lambda}_2(\sigma) = \begin{cases} 1, & \sigma \in [0,2], \\ 0, & \sigma > 2, \end{cases}$$
$$\bar{\Lambda}_3(\sigma) = \begin{cases} 1, & \sigma \in [0,2], \\ 0, & \sigma > 2, \end{cases} \qquad \widetilde{\Lambda}_3(\xi) = \begin{cases} 0, & \xi < 0, \\ -\exp(-\xi), & \xi \geq 0, \end{cases}$$
$$\bar{\Lambda}_4(\sigma) = \begin{cases} 1, & \sigma \in [0,2], \\ 0, & \sigma > 2. \end{cases} \tag{5.6}$$

Further, using equations (3.28)-(3.31) and equation (5.6), we obtain the function $\bar{\Omega}(\sigma)$ and the values $\bar{W}(t_c)$, $\widetilde{W}(t_c)$ as:

$$\bar{\Omega}(\sigma) = \begin{cases} 0, & \sigma \in [0,1], \\ 1, & \sigma \in (1,2], \end{cases} \tag{5.7}$$

$$\bar{W}(t_c) = 1, \qquad \widetilde{W}(t_c) = 1. \tag{5.8}$$

Thus both values, $\bar{W}(t_c)$ and $\widetilde{W}(t_c)$, differ from zero.

Using equation (3.35), as well as equations (5.6) and (5.8), we also obtain

$$\widehat{W}(t_c) = 1. \tag{5.9}$$

Now, let us calculate the values $\bar{w}_{0,1}$, $\bar{w}_{0,2}$, $V_{\text{appr,k}}(t_c,\varepsilon)$, $(k = 1,2,3)$, and functions $\Omega_{\text{appr},1}(t,\varepsilon)$, $\Omega_{\text{appr},2}(t,\varepsilon)$. Due to equations (4.18)-(4.19) and equations (5.3) and (5.6), we obtain

$$\bar{w}_{0,1} = 1, \qquad \bar{w}_{0,2} = 1. \tag{5.10}$$

Using equations (4.26),(4.28), as well as equations (5.6), (5.8) and (5.9), we have

$$V_{\text{appr},1}(t_c,\varepsilon) = 1, \quad V_{\text{appr},2}(t_c,\varepsilon) = -\varepsilon, \quad V_{\text{appr},3}(t_c,\varepsilon) = \varepsilon, \tag{5.11}$$

$$\Omega_{\text{appr},1}(t,\varepsilon) = \begin{cases} \exp\big((t-1)/\varepsilon\big) - \exp\big((t-2)/\varepsilon\big), & t \in [0,1], \\ 1 - \exp\big((t-2)/\varepsilon\big), & t \in (1,2], \end{cases}$$
$$\Omega_{\text{appr},2}(t,\varepsilon) = \begin{cases} (1/\varepsilon)\big[\exp\big((t-2)/\varepsilon\big) - \exp\big((t-1)/\varepsilon\big)\big], & t \in [0,1], \\ (1/\varepsilon)\exp\big((t-2)/\varepsilon\big) + 1, & t \in (1,2]. \end{cases} \tag{5.12}$$

Now, the substitution of $V_{\mathrm{appr,k}}(t_c,\varepsilon)$, $(k=1,2,3)$, $\Omega_{\mathrm{appr},1}(t,\varepsilon)$, $\Omega_{\mathrm{appr},2}(t,\varepsilon)$, $\bar{w}_{0,1}$, $\bar{w}_{0,2}$, as well as the terminal values of the state variables (see equation (5.4)), into equation (4.34) yields the expression for the approximate solution of problem (5.1)-(5.5)

$$u_{\mathrm{appr}}(t,\varepsilon)=\begin{cases}2\big[\exp\big(-(1-t)/\varepsilon\big)-\exp\big(-(2-t)/\varepsilon\big)\big], & t\in[0,1],\\ 1-\varepsilon-2\exp\big(-(2-t)/\varepsilon\big), & t\in(1,2].\end{cases} \tag{5.13}$$

Let $\mathrm{col}\big(x_{\mathrm{appr}}(t,\varepsilon),y_{\mathrm{appr}}(t,\varepsilon)\big)$, $t\in[0,2]$, be the solution of the initial-value problem (5.1)-(5.3) for $u(t)=u_{\mathrm{appr}}(t,\varepsilon)$. Calculating this solution at $t=t_c=2$, we obtain

$$x_{\mathrm{appr}}(2,\varepsilon)=2+\frac{\varepsilon(2-3\varepsilon-4\varepsilon^2-2\varepsilon^3)}{(1+\varepsilon)(2+\varepsilon)}+\frac{2\varepsilon}{2+\varepsilon}\exp\left(-\frac{1}{\varepsilon}\right)$$
$$-\frac{\varepsilon^2(3+\varepsilon)}{(1+\varepsilon)^2(2+\varepsilon)}\exp\left(-1-\frac{1}{\varepsilon}\right)-\frac{\varepsilon}{1+\varepsilon}\exp\left(-2-\frac{2}{\varepsilon}\right)-\varepsilon\alpha(\varepsilon), \tag{5.14}$$

$$y_{\mathrm{appr}}(2,\varepsilon)=1-\frac{2\varepsilon(3+3\varepsilon+\varepsilon^2)}{(1+\varepsilon)^2(2+\varepsilon)}+\frac{2(1+\varepsilon)}{2+\varepsilon}\exp\left(-\frac{1}{\varepsilon}\right)$$
$$+\frac{\varepsilon(3+\varepsilon^2)}{(1+\varepsilon)^2(2+\varepsilon)}\exp\left(-1-\frac{1}{\varepsilon}\right)+\frac{1}{1+\varepsilon}\exp\left(-2-\frac{2}{\varepsilon}\right)+\alpha(\varepsilon), \tag{5.15}$$

where

$$\alpha(\varepsilon)=\frac{2}{(1+\varepsilon)(2+\varepsilon)}\exp\left(-1-\frac{2}{\varepsilon}\right)\left[1-\exp\left(-\frac{1}{\varepsilon}\right)\right.$$
$$\left.-\exp\left(-1-\frac{1}{\varepsilon}\right)+\exp\left(-1-\frac{2}{\varepsilon}\right)\right]. \tag{5.16}$$

Using these equations and the terminal conditions (5.4), we obtain the following inequalities:

$$\big|x_{\mathrm{appr}}(2,\varepsilon)-x(2)\big|\le 1.00006\varepsilon,\qquad \big|y_{\mathrm{appr}}(2,\varepsilon)-y(2)\big|\le 3.31055\varepsilon,\qquad \varepsilon\in(0,0.1], \tag{5.17}$$

which are consistent with Corollary 4.1.

Now, let us calculate J_r^* and $J_{\mathrm{appr}}(\varepsilon)$. Using equations (3.48) and (4.48), and equations (5.8), (5.10) and (5.13), we obtain

$$J_r^*=1, \tag{5.18}$$

$$J_{\mathrm{appr}}(\varepsilon)=1-2\varepsilon+5\varepsilon^2-4\varepsilon^2\exp\left(-\frac{1}{\varepsilon}\right)$$
$$-2\varepsilon\exp\left(-\frac{2}{\varepsilon}\right)+4\varepsilon\exp\left(-\frac{3}{\varepsilon}\right)-2\varepsilon\exp\left(-\frac{4}{\varepsilon}\right). \tag{5.19}$$

Equations (5.18) and (5.19) directly yield

$$\big|J_{\mathrm{appr}}(\varepsilon)-J_r^*\big|\le 2.50002\varepsilon,\qquad \varepsilon\in(0,0.1], \tag{5.20}$$

which is consistent with Corollary 4.3.

6. Conclusions

In this paper, the minimum energy control problem for a singularly perturbed linear time-dependent system with point-wise delay in the input (the control variable) was considered. For this problem sufficient conditions of its solvability, which are independent of the parameter of singular perturbation $\varepsilon > 0$, were established. These conditions, being independent of ε, are valid for all sufficiently small values of this parameter. Along with these conditions, an approximate solution (a suboptimal control) to the considered minimum energy control problem was derived. This solution is obtained in a simpler way than the exact solution (the optimal control) to the problem. The approximate solution provides the fulfilment of a given terminal condition in the minimum energy control problem with the accuracy of order of ε for all sufficiently small values of this parameter. Moreover, the value of the functional in the considered problem, corresponding to the suboptimal control, differs from the optimal one by the value of order of ε for all sufficiently small $\varepsilon > 0$.

By setting formally $\varepsilon = 0$ in the original minimum energy control problem, its reduced version was obtained. The reduced minimum energy control problem is of a lower dimension than the original one and it is ε-free. It was shown that the optimal value of the functional in the original problem differs from such a value in the reduced problem by the value of order of ε for all sufficiently small $\varepsilon > 0$. Also, it was shown that the similar property is valid for the value of the functional in the original problem, corresponding to the suboptimal control.

The illustrating example, presented in the paper, directly shows the fulfilment of the above mentioned properties of the suboptimal control in the original minimum energy problem.

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