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CONVERGENCE OF RELAXED INERTIAL METHODS FOR EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we first introduce a relaxed inertial algorithm for solving a pseudomonotone equilibrium problem with a Lipschitz-type condition in a Hilbert space. The algorithm is constructed around the proximal-like mapping and the inertial technique. The weak convergence of the algorithm is proved under some mild conditions. We also present a modified version of the first algorithm which can be implemented more easily without the prior knowledge of the Lipschitz-type constant of bifunction. Finally, several experiments are performed to illustrate the numerical behavior of the new algorithms, and also to show their computational efficiency over others.

Keywords. Extragradient method; Equilibrium problem; Monotone bifunction; Lipschitz-type condition.

1. INTRODUCTION

The equilibrium problem (EP) [1, 2], which is well known as the Ky Fan inequality, were early studied in [3]. This problem unifies in a simple form various problems, which arise in economics, optimization and operators research. More precisely, in this general formulation, the equilibrium problem includes an important class of variational inequalities as well as the classes of complementarity problems, convex optimisation, saddle point problems, fixed point problems and famous Nash equilibria problem. In recent years, the equilibrium problem has been intensively and widely investigated both in theoretically and algorithmically. Many numerical methods have been proposed for approximating solutions of this problem; see, e.g., [4]-[16] and the references therein.

One of the most popular methods for solving the equilibrium problem is the proximal point method (PPM). This method was first introduced by Martinet [17] for monotone variational inequality problems. Later, it was extended by Rockafellar [18] to monotone operators. Moudafi [14] and Konnov [19] further extended the PPM to the equilibrium problem with monotone and weakly monotone bifunctions, respectively. The PPM is constructed around the resolvent of a bifunction. Based on this method, many works have been devoted to presenting numerical

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approaches for finding an approximation solution of the equilibrium problem in various ways with different types of conditions.

In this paper, we are first interested in a method of inertial-type. This method originates from heavy ball method (an implicit discretization) of the second-order dynamical systems in time [20, 21, 22]. The main feature of which is that the construction of the next iterate is based on at least the previous two iterates. Recently, inertial-type algorithms have been applied to numerous kinds of problems; see, e.g., [23, 24, 25, 26, 27, 28, 29, 30, 31, 32]. In this direction, by using the resolvent of a bifunction, Moudafi [33] proposed a new inertial method, so-called the second-order differential proximal method, which combines the (relaxed) original PPM with inertial effect. Under certain suitable conditions, Moudafi established the weak convergence of the algorithm. Recently, Chbani and Riahi [34] developed the relaxed inertial proximal point methods in incorporating with the Mann iteration or the Halpern iteration and obtained associated convergence results. Several other recent methods in this direction can be found in [35, 36].

We next focus on the method which is based on the auxiliary problem principle, namely, the proximal-like algorithm [4]. This method requires at each iteration to compute two-proximal mappings which is equivalent to solve two strongly convex optimization programs. The proximal-like algorithm [4] is also called the extragradient method [15] due to the early obtained results in [37] on saddle point problems. The extragradient method was investigated and further extended the convergence in [15] under the hypotheses of the pseudomonotonicity and the Lipschitz-type condition of bifunctions. In recent years, the extragradient method has received a lot of attention due to its importance in numerical computations. We remark here that the extragradient methods seem to be easier to solve numerically by optimization tools than the PPM [5, 6, 7, 8, 16]. A reason to explain this remark may be due to the fact that it is not easy to compute the resolvent of a bifunction.

In this paper, motivated and inspired by the results in [33, 34, 35, 36, 38], we first introduce a new algorithm with inertial form for solving a pseudomonotone equilibrium problem with a Lipschitz-type condition in a Hilbert space. The algorithm uses a relaxed version of the extragradient method to incorporate with inertial terms. The chosen stepsizes in the first algorithm depend on the Lipschitz-type constants of bifunctions. The theorem of convergence is established under certain mild conditions. In the inverse case when the information of the Lipschitz-type constants of bifunctions is unknown, we propose the second algorithm which can be performed more easily. The stepsizes in the second algorithm are updated at each iteration by a cheap computation based on the previous iterates. In order to show the efficiency of the new algorithms, several numerical experiments in comparison with others are also implemented.

The remainder of this paper is organized as follows: Section 2 recalls some definitions and preliminary results used in the paper. Sections 3 and 4 deal with the description of the algorithms and the analysis of their convergence. Finally, in Section 5, we perform several experiments to show the numerical behavior of the new algorithms and also to compare them with others.

2. PRELIMINARIES

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let f be a bifunction from $H \times H$ to the set of real numbers $\Re$ such that $f(x,x) = 0$ for all $x \in C$. The equilibrium problem (EP) for the bifunction f on C is to find $x^* \in C$ such that

$$f(x^*, y) \geq 0, \ \forall y \in C. \tag{EP}$$

Let us denote by $EP(f,C)$ the solution set of problem (EP). It was well known that solution methods are often relative to some concepts of the monotonicity of a bifunction. Recall that a bifunction $f : H \times H \to \Re$ is said to be:

(i) *strongly monotone* on C if there exists a constant $\gamma > 0$ such that

$$f(x,y) + f(y,x) \leq -\gamma ||x-y||^2, \ \forall x,y \in C;$$

(ii) *monotone* on C if

$$f(x,y) + f(y,x) \leq 0, \ \forall x,y \in C;$$

(iii) *pseudomonotone* on C if

$$f(x,y) \geq 0 \Longrightarrow f(y,x) \leq 0, \ \forall x,y \in C;$$

(iv) *strongly pseudomonotone* on C if there exists a constant $\gamma > 0$ such that

$$f(x,y) \geq 0 \Longrightarrow f(y,x) \leq -\gamma ||x-y||^2, \ \forall x,y \in C.$$

From the aforementioned definitions, it is easy to see that

$$\text{(i)} \Longrightarrow \text{(ii)} \Longrightarrow \text{(iii) and (i)} \Longrightarrow \text{(iv)} \Longrightarrow \text{(iii)}.$$

A bifunction f is said to satisfy *Lipschitz-type condition* on C if there exist two positive constants c_1 and c_2 such that

$$f(x,y) + f(y,z) \geq f(x,z) - c_1 ||x-y||^2 - c_2 ||y-z||^2, \ \forall x,y,z \in C.$$

If $A : C \to H$ is a L-Lipschitz continuous operator, the bifunction $f(x,y) = \langle Ax, y-x \rangle$ satisfies the Lipschitz-type condition with $c_1 = c_2 = L/2$. Dealing with the analysis of the convergence of the proposed algorithms, we consider the following conditions imposed on a bifunction $f : H \times H \to \Re$:

(A1) f is pseudomonotone on C and $f(x,x) = 0$ for all $x \in C$;

(A2) f satisfies Lipschitz-type condition on H with some constants c_1, c_2;

(A3) $f(x,.)$ is convex and lower semicontinuos on C for every fixed $x \in H$;

(A4) $\limsup_{n\to\infty} f(x_n, y) \leq f(x,y)$ for each sequence $\{x_n\} \subset C$ converging weakly to x;

(A5) $EP(f,C)$ is nonempty;

It is easy to show that under assumptions (A1) and (A3), the solution set $EP(f,C)$ is closed and convex. Recall that the proximal mapping of a proper, convex and lower semicontinuous function $g : C \to \Re$ with a parameter $\lambda > 0$ is defined by

$$\text{prox}_{\lambda g}(x) = \arg\min \left\{ \lambda g(y) + \frac{1}{2} ||x-y||^2 : y \in C \right\}, \ x \in H.$$

The following is a property of the proximal mapping.

Lemma 2.1. [39] *For each $x \in H$ and $\lambda > 0$,*

$$\lambda \left\{ g(y) - g(\text{prox}_{\lambda g}(x)) \right\} \geq \left\langle x - \text{prox}_{\lambda g}(x), y - \text{prox}_{\lambda g}(x) \right\rangle, \forall\, y \in C.$$

We need the following technical lemmas.

Lemma 2.2. [39] *For all $x, y \in H$ and $\alpha \in \Re$, the following equality holds,*

$$||\alpha x + (1-\alpha)y||^2 = \alpha||x||^2 + (1-\alpha)||y||^2 - \alpha(1-\alpha)||x-y||^2.$$

Lemma 2.3. [21] *Let $\{\Phi_n\}$, $\{\Delta_n\}$ and $\{\theta_n\}$ be sequences in $[0, +\infty)$ such that*

$$\Phi_{n+1} \leq \Phi_n + \theta_n(\Phi_n - \Phi_{n-1}) + \Delta_n, \ \forall n \geq 1, \quad \sum_{n=1}^{+\infty} \Delta_n < +\infty,$$

and there exists a real number θ with $0 \leq \theta_n \leq \theta < 1$ for all $n \geq 0$. Then the following assertions hold:

(i) $\sum_{n=1}^{+\infty} [\Phi_n - \Phi_{n-1}]_+ < +\infty$, *where* $[t]_+ := \max\{t, 0\}$*;*

(ii) *There exists $\Phi^* \in [0, +\infty)$ such that* $\lim_{n\to+\infty} \Phi_n = \Phi^*$.

Lemma 2.4. [39, Lemma 2.39] *Let C be a nonempty set of H and $\{x_n\}$ be a sequence in H such that the following two conditions hold:*

(i) *for every $x \in C$,* $\lim_{n\to\infty} \|x_n - x\|$ *exists;*

(ii) *every sequentially weak cluster point of $\{x_n\}$ is in C.*

Then $\{x_n\}$ converges weakly to a point in C.

3. Relaxed Inertial Extragradient Method

In this section, we present a new relaxed inertial algorithm for solving problem (EP) in a Hilbert space. The algorithm can be considered as a combination between a relaxed version of the extragradient method and the inertial technique. Precisely, the algorithm is described as follows:

Algorithm 1. [Relaxed Inertial Extragradient Method for EPs]

Initialization: *Choose x_0, $x_1 \in H$ and three control parameter sequences $\{\lambda_n\} \subset (0, +\infty)$, $\{\alpha_n\} \subset (0, +\infty)$ and $\{\theta_n\} \subset [0, +\infty)$.*

Iterative steps: *Assume that x_{n-1}, $x_n \in H$ are known. Calculate x_{n+1} as follows:*

Step 1. *Compute $w_n = x_n + \theta_n(x_n - x_{n-1})$ and*

$$y_n = \text{prox}_{\lambda_n f(w_n,.)}(w_n).$$

Step 2. *Compute*

$$x_{n+1} = (1-\alpha_n)w_n + \alpha_n \text{prox}_{\lambda_n f(y_n,.)}(w_n).$$

*Set $n := n+1$ and go back **Step 1**.*

Stopping criterion: *If $y_n = w_n$ then stop and y_n is a solution of problem (EP).*

Remark 3.1. In the case where problem (EP) is a variational inequality problem, i.e., $f(x,y) = \langle Ax, y-x\rangle$, where $A : H \to H$ is an operator, then Algorithm 1 becomes the following algorithm

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda_n A w_n), \\ x_{n+1} = (1-\alpha_n)w_n + \alpha_n P_C(w_n - \lambda_n A y_n). \end{cases} \tag{3.1}$$

If $\theta_n = 0$, then $w_n = x_n$ and (3.1) reduces to the algorithm

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_C(x_n - \lambda_n A y_n), \end{cases}$$

which is a relaxed version of the extragradient method in [37].

In order to establish the convergence of Algorithm 1, we assume that conditions (A1)-(A5) hold, and that three parameter sequences $\{\lambda_n\}$, $\{\alpha_n\}$, $\{\theta_n\}$ satisfy the following conditions:

(B1) $0 < \lambda_* \le \lambda_n \le \lambda^* < \min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$;

(B2) $0 < \alpha_* \le \alpha_n \le \alpha^* \le \frac{1}{2} + \frac{1}{4}\min\{1-2c_1\lambda^*, 1-2c_2\lambda^*\}$;

(B3) $0 \le \theta_n \le \theta < \frac{\varepsilon}{2\varepsilon+1}$ and $\{\theta_n\}$ is non-decreasing, where

$$\varepsilon := \frac{0.5\min\{1-2c_1\lambda^*, 1-2c_2\lambda^*\} + 1 - \alpha^*}{\alpha^*}.$$

Remark 3.2. From (B2), we have $2\alpha^* \le 1 + 0.5\min\{1-2c_1\lambda^*, 1-2c_2\lambda^*\}$. Thus

$$0.5\min\{1-2c_1\lambda^*, 1-2c_2\lambda^*\} + 1 - \alpha^* \ge \alpha^*.$$

This implies $\varepsilon \ge 1$.

We begin with the following lemma.

Lemma 3.1. *Under assumptions* (A1)-(A3), (A5) *and* (B1)-(B3), *the following estimate holds for all $n \ge 0$ and $x^* \in EP(f,C)$,*

$$||x_{n+1} - x^*||^2 \le ||w_n - x^*||^2 - \frac{\kappa_n + 1 - \alpha_n}{\alpha_n}||x_{n+1} - w_n||^2,$$

where $\kappa_n = 0.5\min\{1-2\lambda_n c_1, 1-2\lambda_n c_2\}$.

Proof. Set $z_n = \text{prox}_{\lambda_n f(y_n,.)}(w_n)$. Thus, from Lemma 2.1, we obtain

$$\lambda_n(f(y_n,y) - f(y_n,z_n)) \ge \langle w_n - z_n, y - z_n\rangle, \ \forall y \in C. \tag{3.2}$$

Similarly, by the definition of y_n and Lemma 2.1, we get

$$\lambda_n(f(w_n,y) - f(w_n,y_n)) \ge \langle w_n - y_n, y - y_n\rangle, \ \forall y \in C. \tag{3.3}$$

Substituting $y = z_n$ into relation (3.3), we obtain

$$\lambda_n(f(w_n,z_n) - f(w_n,y_n)) \ge \langle w_n - y_n, z_n - y_n\rangle. \tag{3.4}$$

Now, since $x^* \in EP(f,C)$, we have $f(x^*,y_n) \ge 0$. Thus, from hypothesis (A1), we obtain $f(y_n,x^*) \le 0$. Next, substituting $y = x^* \in EP(f,C) \subset C$ into relation (3.2) and using $f(y_n,x^*) \le 0$, we see that

$$-\lambda_n f(y_n,z_n) \ge \langle w_n - z_n, x^* - z_n\rangle.$$

Thus $\langle w_n - z_n, z_n - x^* \rangle \geq \lambda_n f(y_n, z_n)$, which together with the Lipschitz-type condition of f, implies that

$$\langle w_n - z_n, z_n - x^* \rangle \geq \lambda_n \left\{ f(w_n, z_n) - f(w_n, y_n) - c_1 ||w_n - y_n||^2 - c_2 ||y_n - z_n||^2 \right\}.$$

Combining the last inequality with relation (3.4), we obtain

$$\langle w_n - z_n, z_n - x^* \rangle \geq \langle w_n - y_n, z_n - y_n \rangle - c_1 \lambda_n ||w_n - y_n||^2 - c_2 \lambda_n ||y_n - z_n||^2. \tag{3.5}$$

We have the following facts:

$$2 \langle w_n - z_n, z_n - x^* \rangle = ||w_n - x^*||^2 - ||w_n - z_n||^2 - ||z_n - x^*||^2,$$
$$2 \langle w_n - y_n, z_n - y_n \rangle = ||w_n - y_n||^2 + ||z_n - y_n||^2 - ||w_n - z_n||^2.$$

Multiplying both two sides of inequality (3.5) by 2, after that, using the two last equalities, we obtain

$$||z_n - x^*||^2 \leq ||w_n - x^*||^2 - (1 - 2\lambda_n c_1) ||y_n - w_n||^2 - (1 - 2\lambda_n c_2) ||z_n - y_n||^2. \tag{3.6}$$

Thus, from the definition of κ_n, we get

$$\begin{aligned} ||z_n - x^*||^2 &\leq ||w_n - x^*||^2 - 2\kappa_n \left(||y_n - w_n||^2 + ||z_n - y_n||^2 \right) \\ &\leq ||w_n - x^*||^2 - \kappa_n \left(||y_n - w_n|| + ||z_n - y_n|| \right)^2 \\ &\leq ||w_n - x^*||^2 - \kappa_n ||z_n - w_n||^2. \end{aligned} \tag{3.7}$$

Note that, from the definitions of x_{n+1} and z_n, one has $x_{n+1} = (1 - \alpha_n) w_n + \alpha_n z_n$. Moreover, two hypotheses (B1) and (B2) imply that $\alpha_n \in (0,1)$. Thus, it follows from Lemma 2.2 and relation (3.7) that

$$\begin{aligned} ||x_{n+1} - x^*||^2 &= ||(1 - \alpha_n)(w_n - x^*) + \alpha_n (z_n - x^*)||^2 \\ &= (1 - \alpha_n) ||w_n - x^*||^2 + \alpha_n ||z_n - x^*||^2 - \alpha_n (1 - \alpha_n) ||z_n - w_n||^2 \\ &\leq ||w_n - x^*||^2 - \alpha_n (\kappa_n + 1 - \alpha_n) ||z_n - w_n||^2, \end{aligned}$$

which, together with the fact $||z_n - w_n|| = \frac{1}{\alpha_n} ||x_{n+1} - w_n||$, implies that

$$||x_{n+1} - x^*||^2 \leq ||w_n - x^*||^2 - \frac{\kappa_n + 1 - \alpha_n}{\alpha_n} ||x_{n+1} - w_n||^2. \tag{3.8}$$

This completes the proof of Lemma 3.1. □

Now, we set

$$\varphi_n := ||x_n - x^*||^2 - \theta_n ||x_{n-1} - x^*||^2 + \theta_n (1 + \varepsilon) ||x_n - x_{n-1}||^2,$$

where ε is defined in hypothesis (B3). Let $K := \varepsilon - (2\varepsilon + 1)\theta$. From (B3), we see that $K > 0$. We have the following lemma.

Lemma 3.2. *Under assumptions as in Lemma 3.1, the following estimate holds, for all $n \geq 1$,*

$$\varphi_{n+1} - \varphi_n \leq -K ||x_{n+1} - x_n||^2.$$

Proof. It follows from (B1)-(B2) and the definition of κ_n that

$$\begin{aligned} \frac{\kappa_n + 1 - \alpha_n}{\alpha_n} &= \frac{0.5 \min \{1 - 2\lambda_n c_1, 1 - 2\lambda_n c_2\} + 1 - \alpha_n}{\alpha_n} \\ &\geq \frac{0.5 \min \{1 - 2\lambda^* c_1, 1 - 2\lambda^* c_2\} + 1 - \alpha^*}{\alpha^*} = \varepsilon. \end{aligned}$$

Thus, from Lemma 3.1, we obtain

$$||x_{n+1}-x^*||^2 \leq ||w_n-x^*||^2 - \varepsilon||x_{n+1}-w_n||^2. \tag{3.9}$$

By the definition of w_n and Lemma 2.2, we have

$$\begin{aligned} ||w_n-x^*||^2 &= ||x_n+\theta_n(x_n-x_{n-1})-x^*||^2 \\ &= ||(1+\theta_n)(x_n-x^*)+(-\theta_n)(x_{n-1}-x^*)||^2 \\ &= (1+\theta_n)||x_n-x^*||^2-\theta_n||x_{n-1}-x^*||^2+\theta_n(1+\theta_n)||x_n-x_{n-1}||^2. \end{aligned} \tag{3.10}$$

Moreover, it also follows from the definition of w_n and the Cauchy-Schwarz inequality that

$$\begin{aligned} ||x_{n+1}-w_n||^2 &= ||x_{n+1}-x_n-\theta_n(x_n-x_{n-1})||^2 \\ &= ||x_{n+1}-x_n||^2+\theta_n^2||x_n-x_{n-1}||^2-2\theta_n\langle x_{n+1}-x_n,x_n-x_{n-1}\rangle \\ &\geq ||x_{n+1}-x_n||^2+\theta_n^2||x_n-x_{n-1}||^2-2\theta_n||x_{n+1}-x_n||||x_n-x_{n-1}|| \\ &\geq ||x_{n+1}-x_n||^2+\theta_n^2||x_n-x_{n-1}||^2-\theta_n\left[||x_{n+1}-x_n||^2+||x_n-x_{n-1}||^2\right] \\ &= (1-\theta_n)||x_{n+1}-x_n||^2+(\theta_n^2-\theta_n)||x_n-x_{n-1}||^2. \end{aligned} \tag{3.11}$$

From relations (3.9) - (3.11), we have

$$\begin{aligned} ||x_{n+1}-x^*||^2 &\leq (1+\theta_n)||x_n-x^*||^2-\theta_n||x_{n-1}-x^*||^2-\varepsilon(1-\theta_n)||x_{n+1}-x_n||^2 \\ &\quad -\varepsilon(\theta_n^2-\theta_n)||x_n-x_{n-1}||^2+\theta_n(1+\theta_n)||x_n-x_{n-1}||^2 \\ &\leq (1+\theta_n)||x_n-x^*||^2-\theta_n||x_{n-1}-x^*||^2-\varepsilon(1-\theta_n)||x_{n+1}-x_n||^2 \\ &\quad +\theta_n(1+\varepsilon-\theta_n(\varepsilon-1))||x_n-x_{n-1}||^2 \\ &\leq (1+\theta_n)||x_n-x^*||^2-\theta_n||x_{n-1}-x^*||^2-\varepsilon(1-\theta_n)||x_{n+1}-x_n||^2 \\ &\quad +\theta_n(1+\varepsilon)||x_n-x_{n-1}||^2, \end{aligned} \tag{3.12}$$

in which the last inequality follows from the fact that $\varepsilon \geq 1$ and $\theta_n \geq 0$. Now, it follows from the definition of φ_n, the non-decreasing property of $\{\theta_n\}$, and relation (3.12) that

$$\begin{aligned} \varphi_{n+1}-\varphi_n &= ||x_{n+1}-x^*||^2-(1+\theta_{n+1})||x_n-x^*||^2+\theta_{n+1}(1+\varepsilon)||x_{n+1}-x_n||^2 \\ &\quad +\theta_n||x_{n-1}-x^*||^2-\theta_n(1+\varepsilon)||x_n-x_{n-1}||^2 \\ &\leq ||x_{n+1}-x^*||^2-(1+\theta_n)||x_n-x^*||^2+\theta_n||x_{n-1}-x^*||^2 \\ &\quad -\theta_n(1+\varepsilon)||x_n-x_{n-1}||^2+\theta_{n+1}(1+\varepsilon)||x_{n+1}-x_n||^2 \\ &\leq -\varepsilon(1-\theta_n)||x_{n+1}-x_n||^2+\theta_{n+1}(1+\varepsilon)||x_{n+1}-x_n||^2 \\ &= -\left[\varepsilon(1-\theta_n)-\theta_{n+1}(1+\varepsilon)\right]||x_{n+1}-x_n||^2 \\ &\leq -\left[\varepsilon(1-\theta)-\theta(1+\varepsilon)\right]||x_{n+1}-x_n||^2 \text{ (due to } 0\leq\theta_n,\theta_{n+1}\leq\theta) \\ &= -\left[\varepsilon-(2\varepsilon+1)\theta\right]||x_{n+1}-x_n||^2 = -K||x_{n+1}-x_n||^2. \end{aligned}$$

Lemma 3.2 is proved. □

Lemma 3.3. *Under assumptions as in Lemma 3.1, the following assertions hold:*

(i) $\sum_{n=1}^{\infty}||x_{n+1}-x_n||^2 < +\infty$;

(ii) $\lim_{n\to\infty}||x_n-y_n|| = \lim_{n\to\infty}||z_n-w_n|| = \lim_{n\to\infty}||y_n-w_n|| = \lim_{n\to\infty}||y_n-z_n|| = 0$;

(iii) $\lim_{n\to\infty}||x_n-x^*||^2 \in \Re$ *for each* $x^* \in EP(f,C)$ *and* $\{x_n\}$, $\{y_n\}$, $\{z_n\}$, $\{w_n\}$ *are bounded.*

Proof. (i) It follows from Lemma 3.2 that φ_n is a non-increasing. Thus, from the definition of φ_n, we have

$$||x_n - x^*||^2 \le \theta_n||x_{n-1} - x^*||^2 + \varphi_n \le \theta_n||x_{n-1} - x^*||^2 + \varphi_1.$$

Thus, from hypothesis (B3), we get

$$||x_n - x^*||^2 \le \theta||x_{n-1} - x^*||^2 + \varphi_1.$$

This implies that

$$||x_n - x^*||^2 \le \theta^n||x_0 - x^*||^2 + \varphi_1\left(\theta^{n-1} + \theta^{n-2} + \ldots + \theta + 1\right) \le \theta^n||x_0 - x^*||^2 + \frac{\varphi_1}{1-\theta}. \quad (3.13)$$

Moreover,

$$\varphi_{n+1} = ||x_{n+1} - x^*||^2 - \theta_{n+1}||x_n - x^*||^2 + \theta_{n+1}(1+\varepsilon)||x_{n+1} - x_n||^2 \ge -\theta_{n+1}||x_n - x^*||^2.$$

This together with (3.13) implies that

$$-\varphi_{n+1} \le \theta_{n+1}||x_n - x^*||^2 \le \theta||x_n - x^*||^2 \le \theta^{n+1}||x_0 - x^*||^2 + \frac{\theta\varphi_1}{1-\theta}. \quad (3.14)$$

It follows from Lemma 3.2 that

$$K||x_{n+1} - x_n||^2 \le \varphi_n - \varphi_{n+1}$$

for all $n \ge 1$. Let a fixed $N \ge 1$. Applying the last inequality for $n = 1, 2, \ldots, N$ and summing up those inequalities, we obtain

$$K\sum_{n=1}^{N}||x_{n+1} - x_n||^2 \le \varphi_1 - \varphi_{N+1}.$$

Combining this with relation (3.14), we obtain

$$K\sum_{n=1}^{N}||x_{n+1} - x_n||^2 \le \varphi_1 + \theta^{N+1}||x_0 - x^*||^2 + \frac{\theta\varphi_1}{1-\theta} \le \theta^{N+1}||x_0 - x^*||^2 + \frac{\varphi_1}{1-\theta}.$$

From (B3), we obtain $\theta \le \frac{\varepsilon}{2\varepsilon+1} < 1$. Thus, passing to the limit in the last inequality as $N \to \infty$ and noting that $K > 0$, we get

$$\sum_{n=1}^{\infty}||x_{n+1} - x_n||^2 \le \frac{\varphi_1}{1-\theta} < \infty. \quad (3.15)$$

(ii) It follows from relation (3.15) that $||x_{n+1} - x_n|| \to 0$. Hence

$$||x_{n+1} - w_n|| = ||x_{n+1} - x_n + \theta_n||x_n - x_{n-1}|| \le ||x_{n+1} - x_n|| + \theta_n||x_n - x_{n-1}|| \to 0.$$

It is obvious that $||x_n - w_n|| \le ||x_n - x_{n+1}|| + ||x_{n+1} - w_n|| \to 0$. Moreover, from the defition of x_{n+1}, we also have

$$||z_n - w_n|| = \frac{1}{\alpha_n}||x_{n+1} - w_n|| \to 0$$

because $\alpha_n \ge \alpha_* > 0$. It follows from (3.12) that

$$||x_{n+1} - x^*||^2 \le (1+\theta_n)||x_n - x^*||^2 - \theta_n||x_{n-1} - x^*||^2 + \theta_n(1+\varepsilon)||x_n - x_{n-1}||^2.$$

Thus, by setting $\Phi_n = ||x_n - x^*||^2$ and $\Delta_n = \theta_n(1+\varepsilon)||x_n - x_{n-1}||^2$, we obtain

$$\Phi_{n+1} \le \Phi_n + \theta_n(\Phi_n - \Phi_{n-1}) + \Delta_n \ \forall n \ge 1. \quad (3.16)$$

It follows from $0 \le \theta_n \le \theta \le \frac{\varepsilon}{2\varepsilon+1} < 1$, the definition of Δ_n and relation (3.15) that $\sum_{n=1}^{+\infty} \Delta_n < +\infty$. Thus, Lemma 2.3 and relation (3.16) ensure that the limit of $\{\Phi_n\}$ exists, i.e., $\lim_{n\to\infty} ||x_n - x^*||^2 = \Phi^* \in \Re$. We also have $\lim_{n\to\infty} ||w_n - x^*||^2 = \lim_{n\to\infty} ||z_n - x^*||^2 = \Phi^*$ because of $||x_n - w_n|| \to 0$ and $||z_n - w_n|| \to 0$. Thus, it follows from relation (3.6) that

$$(1-2\lambda_n c_1)||y_n - w_n||^2 + (1-2\lambda_n c_2)||z_n - y_n||^2 \le ||w_n - x^*||^2 - ||z_n - x^*||^2 \to 0. \tag{3.17}$$

This together with hypothesis (B1) implies that

$$||y_n - w_n|| \to 0, \quad ||z_n - y_n|| \to 0. \tag{3.18}$$

Thus, we also have $||x_n - y_n|| \to 0$ due to $||x_n - w_n|| \to 0$.

(iii) Since $\lim_{n\to\infty} ||x_n - x^*||^2 = \Phi^*$, we easily see that $\{x_n\}$ is bounded. Therefore $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ are also bounded. □

Lemma 3.4. *Under assumptions* (A1)-(A5) *and* (B1)-(B3)*, every weakly cluster point of* $\{x_n\}$ *belongs to* $EP(f,C)$.

Proof. From relation (3.2), we have

$$\lambda_n f(y_n, y) \ge \lambda_n f(y_n, z_n)) + \langle w_n - z_n, y - z_n \rangle, \ \forall y \in C. \tag{3.19}$$

It follows from the Lipschitz-type condition of f that

$$f(y_n, z_n) \ge f(w_n, z_n) - f(w_n, y_n) - c_1||w_n - y_n||^2 - c_2||y_n - z_n||^2.$$

Multiplying both two sides of this inequality by $\lambda_n > 0$, after that, combining the obtained inequality with relation (3.4), we get

$$\lambda_n f(y_n, z_n) \ge \langle w_n - y_n, z_n - y_n \rangle - c_1\lambda_n||w_n - y_n||^2 - c_2\lambda_n||y_n - z_n||^2,$$

which, together with relation (3.19), implies that

$$\begin{aligned} \lambda_n f(y_n, y) \ \ge \ & \langle w_n - y_n, z_n - y_n \rangle + \langle w_n - z_n, y - z_n \rangle \\ & -c_1\lambda_n||w_n - y_n||^2 - c_2\lambda_n||y_n - z_n||^2 \end{aligned} \tag{3.20}$$

for all $y \in C$ and $n \ge 0$. Thus, passing to the limit in (3.20) as $n \to \infty$, and using Lemma 3.3(ii-iii) and (B1), we obtain

$$\lim_{n\to\infty} f(y_n, y) \ge 0, \ \forall y \in C. \tag{3.21}$$

Note that, from Lemma 3.3(iii), $\{x_n\}$ is bounded. Now, assume that p is some weakly cluster point of $\{x_n\}$, i.e., there exists a subsequence $\{x_m\}$ of $\{x_n\}$ converging weakly to p. From $||x_m - y_m|| \to 0$, we also obtain $y_m \rightharpoonup p$. Since C is closed and convex in H, we have that C is weakly closed. Thus, from $\{y_m\} \subset C$, we obtain $p \in C$. It follows from (A5) and relation (3.21) that

$$f(p, y) \ge \lim_{m\to\infty} \sup f(y_m, y) \ge 0, \ \forall y \in C.$$

This means that $p \in EP(f,C)$. The proof of Lemma 3.4 is complete. □

Finally, we obtain the following main result.

Theorem 3.1. *Under assumptions* (A1)-(A5) *and* (B1)-(B3)*, the sequences* $\{x_n\}$*,* $\{y_n\}$ *and* $\{w_n\}$ *generated by Algorithm 1 converge weakly to some solution of problem (EP).*

Proof. From Lemmas 2.4, 3.3 (iii) and 3.4, we obtain that $\{x_n\}$ converges weakly to some solution $x^\dagger$ of problem (EP). From Lemma 3.3 (ii), we have that $\{y_n\}$ and $\{w_n\}$ also converge weakly to $x^\dagger$. This completes the proof of Theorem 3.1. □

4. Modified Relaxed Inertial Extragradient Method

In view of hypothesis (B1), we see that the stepsize λ_n depends on the two Lipschitz-type constants c_1, c_2 of bifunction f. This means that the convergence of Algorithm 1 is only ensured when these constants are known. Actually, the Lipschitz-type constants are often unknown or difficult to approximate in nonlinear problems. In that case, a method of linesearch type can be used to replace. A linesearch involves an inner loop with some finite stopping criterion and requires many extra-computations at each (outer) iteration. This can be expensive and time-consuming. In this section, we introduce a modified version of Algorithm 1 where can be implemented more easily. For the presentation of Algorithm 2 below, we use the notation $[t]_+ = \max\{0,t\}$ and adopt the convention $\frac{a}{0} = +\infty$ for $a \geq 0$.

Algorithm 2. [Modified Relaxed Inertial Extragradient Method for EPs]

Initialization: *Choose $x_0,\ x_1 \in H$. Take two constants $\lambda_0 > 0,\ \mu \in (0,1)$ and two control parameter sequences, $\{\alpha_n\} \subset (0,+\infty)$, $\{\theta_n\} \subset [0,+\infty)$.*

Iterative steps: *Assume that $x_{n-1},\ x_n \in H$ and λ_n are known. Calculate x_{n+1} and λ_{n+1} as follows:*

Step 1. *Compute $w_n = x_n + \theta_n(x_n - x_{n-1})$ and*

$$y_n = \mathrm{prox}_{\lambda_n f(w_n,.)}(w_n).$$

Step 2. *Compute*

$$x_{n+1} = (1-\alpha_n)w_n + \alpha_n \mathrm{prox}_{\lambda_n f(y_n,.)}(w_n).$$

Step 3. *Update*

$$\lambda_{n+1} = \min\left\{\lambda_n, \frac{\mu(||w_n - y_n||^2 + ||z_n - y_n||^2)}{2\,[f(w_n,z_n) - f(w_n,y_n) - f(y_n,z_n)]_+}\right\},$$

where $z_n = \mathrm{prox}_{\lambda_n f(y_n,.)}(w_n)$. Set $n := n+1$ and go back ***Step 1****.*

Stopping criterion: *If $y_n = w_n$ then stop and y_n is a solution of problem (EP).*

The main difference between Algorithm 2 and Algorithm 1 is at Step 3. The stepsize λ_{n+1} is updated at each iteration based on the previous iterates, by a simple computation, and without a linesearch. Although the convergence of Algorithm 2 requires the Lipschitz-type condition of bifunction f, the Lipschitz-type constants of f is not necessary to be known. In order to establish the convergence of Algorithm 2, we consider the following assumptions imposed on the two sequences $\{\alpha_n\}$, $\{\theta_n\}$:

(B4) $0 < \alpha_* \leq \alpha_n \leq \alpha^* < \frac{3-\mu}{4}$;

(B5) $0 \leq \theta_n \leq \theta < \frac{\bar{\varepsilon}}{2\bar{\varepsilon}+1}$ and $\{\theta_n\}$ is non-decreasing, where $\bar{\varepsilon} \in [1, \frac{3-\mu-2\alpha^*}{2\alpha^*})$.

Remark 4.1. Since $0 < \alpha^* < \frac{3-\mu}{4}$, we immediately obtain $\frac{3-\mu-2\alpha^*}{2\alpha^*} > 1$.

Theorem 4.1. *Under assumptions* (A1)-(A5) *and* (B4)-(B5)*, the sequences* $\{x_n\}$*,* $\{y_n\}$ *and* $\{w_n\}$ *generated by Algorithm 2 converge weakly to some solution of problem (EP).*

Proof. We first state that $\lambda_n > 0$ for all $n \geq 0$ and

$$f(w_n,z_n) - f(w_n,y_n) - f(y_n,z_n) \leq \frac{\mu(||w_n - y_n||^2 + ||z_n - y_n||^2)}{2\lambda_{n+1}}, \ \forall n \geq 1. \tag{4.1}$$

Indeed, we see that $\lambda_0 > 0$. Assume $\lambda_n > 0$ for some $n \geq 0$. Now, if $f(w_n,z_n) - f(w_n,y_n) - f(y_n,z_n) \leq 0$, then from the definition of λ_{n+1}, we obtain that $\lambda_{n+1} = \lambda_n > 0$, and inequality (4.1) is obviously true. In the inverse case, if $f(w_n,z_n) - f(w_n,y_n) - f(y_n,z_n) > 0$, then $||w_n - y_n||^2 + ||z_n - y_n||^2 > 0$ and

$$[f(w_n,z_n) - f(w_n,y_n) - f(y_n,z_n)]_+ = f(w_n,z_n) - f(w_n,y_n) - f(y_n,z_n) > 0.$$

Thus, it follows from the definition of λ_{n+1} that $\lambda_{n+1} > 0$ and

$$\lambda_{n+1} \leq \frac{\mu(||w_n - y_n||^2 + ||z_n - y_n||^2)}{2\left(f(w_n,z_n) - f(w_n,y_n) - f(y_n,z_n)\right)},$$

which follows inequality (4.1). Moreover, using the result in [9, Theorem 1] and hypothesis (A2), we obtain

$$\lim \lambda_n = \lambda > 0. \tag{4.2}$$

Now, we obtain from Lemma 2.1 and the definition of z_n that

$$\lambda_n(f(y_n,x^*) - f(y_n,z_n)) \geq \langle w_n - z_n, x^* - z_n \rangle, \ \forall x^* \in EP(f,C). \tag{4.3}$$

Similarly, it follows from the definition of y_n that

$$\lambda_n(f(w_n,z_n) - f(w_n,y_n)) \geq \langle w_n - y_n, z_n - y_n \rangle. \tag{4.4}$$

Combining relation (4.3) and (4.4) and noting $f(y_n,x^*) \leq 0$, we derive

$$2\langle w_n - z_n, x^* - z_n \rangle + 2\langle w_n - y_n, z_n - y_n \rangle \leq 2\lambda_n\left(f(w_n,z_n) - f(w_n,y_n) - f(y_n,z_n)\right),$$

which together with relation (4.1) implies that

$$2\langle w_n - z_n, x^* - z_n \rangle + 2\langle w_n - y_n, z_n - y_n \rangle \leq \frac{\mu\lambda_n}{\lambda_{n+1}}(||w_n - y_n||^2 + ||z_n - y_n||^2). \tag{4.5}$$

Applying the equality $2\langle a,b \rangle = ||a||^2 + ||b||^2 - ||a-b||^2$ to relation (4.5), we come to the following inequality

$$||z_n - x^*||^2 \leq ||w_n - x^*||^2 - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)\left(||w_n - y_n||^2 + ||z_n - y_n||^2\right).$$

Thus, from the fact $(a^2 + b^2) \geq \frac{1}{2}(a+b)^2$ and the triangle inequality, we get

$$\begin{aligned}
||z_n - x^*||^2 &\leq ||w_n - x^*||^2 - \frac{1}{2}\left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)(||w_n - y_n|| + ||z_n - y_n||)^2 \\
&\leq ||w_n - x^*||^2 - \frac{1}{2}\left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)||z_n - w_n||^2 \\
&= ||w_n - x^*||^2 - \bar{\kappa}_n||z_n - w_n||^2,
\end{aligned} \tag{4.6}$$

where $\bar{\kappa}_n = \frac{1}{2}\left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right)$. Thus, as relation (3.8), we also obtain

$$||x_{n+1} - x^*||^2 \le ||w_n - x^*||^2 - \frac{\bar{\kappa}_n + 1 - \alpha_n}{\alpha_n}||x_{n+1} - w_n||^2. \tag{4.7}$$

Since $\lambda_n \to \lambda > 0$, we have $\bar{\kappa}_n \to \frac{1}{2}(1-\mu)$. Thus, it follows from $0 < \alpha_n \le \alpha^*$ that

$$\frac{\bar{\kappa}_n + 1 - \alpha_n}{\alpha_n} \ge \frac{\bar{\kappa}_n + 1 - \alpha^*}{\alpha^*} \to \frac{\frac{1}{2}(1-\mu) + 1 - \alpha^*}{\alpha^*} = \frac{3 - \mu - 2\alpha^*}{2\alpha^*} > \bar{\varepsilon}.$$

Thus, there exists $n_0 \ge 1$ such that

$$\frac{\bar{\kappa}_n + 1 - \alpha_n}{\alpha_n} > \bar{\varepsilon},\ \forall n \ge n_0. \tag{4.8}$$

Combining relations (4.7) and (4.8), we obtain

$$||x_{n+1} - x^*||^2 \le ||w_n - x^*||^2 - \bar{\varepsilon}||x_{n+1} - w_n||^2,\ \forall n \ge n_0. \tag{4.9}$$

Set $\bar{\varphi}_n := ||x_n - x^*||^2 - \theta_n||x_{n-1} - x^*||^2 + \theta_n(1+\bar{\varepsilon})||x_n - x_{n-1}||^2$ and $\bar{K} := \bar{\varepsilon} - (2\bar{\varepsilon}+1)\theta > 0$. Using Lemma 3.2, we also obtain

$$\varphi_{n+1} - \varphi_n \le -\bar{K}||x_{n+1} - x_n||^2,\ \forall n \ge n_0.$$

The rest of the proof is similar to Theorem 3.1. Theorem 4.1 is proved. □

5. Computational Experiments

In this section, we consider some examples to illustrate the affect of the inertial term to the numerical behavior of Algorithm 1. We show the behavior of this algorithm because, in the mentioned examples below, the Lipschitz-type constants of the bifunction are known. Note that if $\theta_n = 0$, then Algorithm 1 becomes the original algorithm, which is without inertial effect. All the programs are written in Matlab 7.0 and computed on a PC Desktop Intel(R) Core(TM) i5-3210M CPU @ 2.50GHz 2.50 GHz, RAM 2.00 GB.

We choose $\lambda_n = 0.5\min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$, $\lambda^* = 0.99\min\left\{\frac{1}{2c_1}, \frac{1}{2c_2}\right\}$ and

$$\alpha_n = \alpha^* := \frac{1}{2} + \frac{1}{4}\min\{1 - 2c_1\lambda^*, 1 - 2c_2\lambda^*\}.$$

Hence $\varepsilon = 1$. Six parameters of θ_n are here chosen to check as

$$\theta_n \in \{0,\ 0.05,\ 0.1,\ 0.2,\ 0.25,\ 0.3\}.$$

The starting points are $x_0 = x_1 = (1,1,\dots,1)^T \in \Re^m$. The feasible set is a box defined by

$$C = \{x \in \Re^m : -5 \le x_i \le 5,\ i = 1,2,\dots,m\}.$$

From Algorithm 1, we see that if $y_n = w_n$ then y_n is a solution of the problem. Thus, we use the sequence $D_n = ||y_n - w_n||^2$, $n = 0,1,2,\dots$ to study the convergence of the proposed algorithm. The data is generated randomly such that all the conditions of the problem are satisfied.

Example 5.1. Consider a linear bifunction $f : \Re^m \times \Re^m \to \Re$ defined by $f(x,y) = \langle Px + Qy + q, y - x\rangle$, where $q \in \Re^m$ and P, Q are two $m \times m$ matrices such that Q is symmetric positive semidefinite and $Q - P$ is symmetric negative semidefinite [7]. The bifunction f is pseudomonotone and satisfies the Lipschitz-type condition with $c_1 = c_2 = ||P - Q||/2$. The numerical behavior of D_n is described in Figure 1 and Figure 2 for respectively $m = 20$ and $m = 50$.

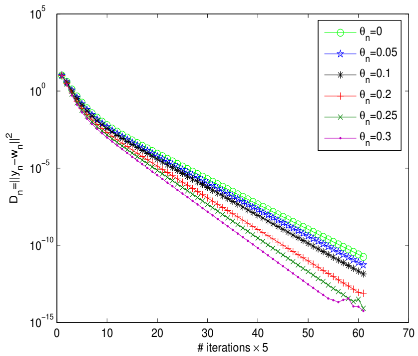

FIGURE 1. Example 1 for $m = 20$.

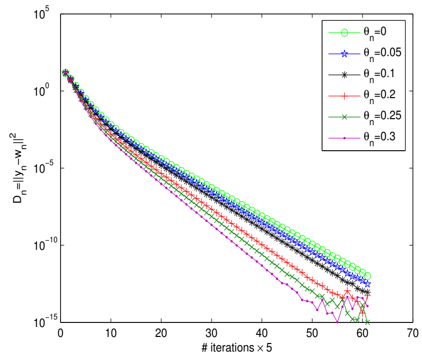

FIGURE 2. Example 1 for $m = 50$.

Example 2. Consider a nonlinear bifunction $f : \Re^m \times \Re^m \to \Re$ of the form $f(x,y) = \langle F(x), y-x\rangle$, where $F(x) = Ax + P(x)$ and A is a $m \times m$ symmetric semidefinite matrix and $P(x)$ is the proximal mapping of the function $g(x) = \frac{1}{4}||x||^4$, i.e.,

$$P(x) = \arg\min\left\{\frac{||y||^4}{4} + \frac{1}{2}||y-x||^2 : y \in \Re^m\right\}.$$

In this case, f is pseudomonotone and satisfies the Lipschitz-type condition with $c_1 = c_2 = \frac{1}{2}(||A|| + 1)$. The numerical results are showed in Figure 3 and Figure 4.

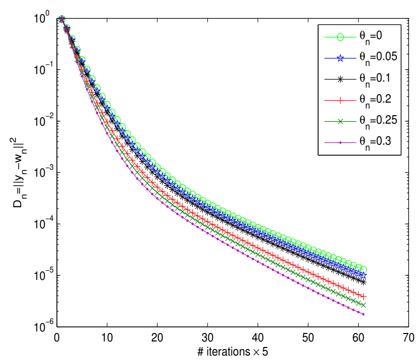

FIGURE 3. Example 2 for $m = 20$.

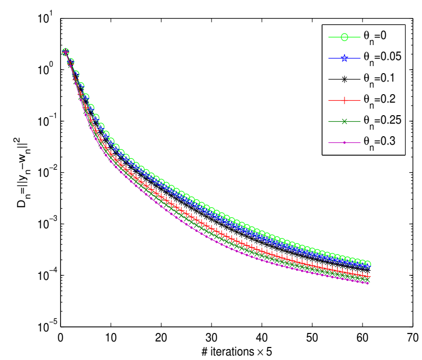

FIGURE 4. Example 2 for $m = 50$.

From the aforementioned numerical results, we see that the proposed algorithm with inertial effect ($\theta_n \neq 0$) seems better than the classical extragradient algorithm ($\theta_n = 0$). Moreover, it is also seen that the larger θ_n is, the better the convergence of the new algorithm is.

6. CONCLUSIONS

The paper proposed two relaxed inertial extragradient methods for solving the equilibrium problem in Hilbert spaces. The algorithms are constructed around the proximal-like mapping of a bifunction and the inertial method. The algorithms can be implemented with or without knowing previously the Lipschitz-type constants of bifunctions. Theorems of weak convergence were established under some mild conditions. Several of numerical results confirmed that the proposed algorithm with inertial effect seems to be better than the original method.

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