J. Appl. Numer. Optim. 3 (2021), No. 1, pp. 211-214
Available online at http://jano.biemdas.com
https://doi.org/10.23952/jano.3.2021.1.12

OPTIMAL CONTROL IN THE TRAFFIC FLOW: MULTI-LANE SIGNALIZED INTERSECTION

ILYA IOSLOVICH

Faculty of Civil and Environmental Engineering, Technion, Haifa 32000, Israel

Abstract. The continuous traffic flow model is modified and extended to find optimal control analytical solutions for oversaturated multi-phase and multi-lane signalized intersection. The considered objectives are minimal weighted total delay and maximal weighted throughput.

Keywords. Signalized intersection; Optimal control; Traffic flow.

1. Introduction

Recently, the optimal control for the signalized intersection was investigated in much researches; see, e.g., [1], [2], [3], and [4]. In this paper, we modify and extend the traffic flow model to find the optimal control analytical solution for oversaturated multi-phase and multi-lane signalized intersection. Both minimal delay and maximal throughput objectives are considered.

2. Preliminaries

We consider a continuous model for multi-lane and muli-phase signalized intersection. The control is associated with green split for each phase u_j of m signal phases, where each phase has the minimal green split value $\underline{u}_j$. Values a_i correspond to the demand rate in the lane i. Values q_i represent the queue for the lane i. For each of n lanes, the throughway velocity d_i is known. The set K_i contains all the signal phase indices j that allow the movement for lane i. The set L_j contains all the lanes indices i that are actively moved during the green phase j. The optimization criterion is the weighted with weights w_i total delay during time interval T. It is assumed that the intersection is oversaturated for all lanes for the time interval $[0,T]$.

E-mail address: agrilyaster@gmail.com.
Received September 12, 2020; September 21, 2020.

The model has the following form:

$$\begin{aligned} \frac{dq_i}{dt} &= a_i - d_i(\sum_{j\in K_i} u_j), \\ J &= \int_0^T (\sum_{i=1}^n w_i q_i)dt \to \min \\ u_j &\in U;\, U = \{\sum_{j=1}^m u_j = 1,\, \underline{u}_j \leq u_j\}, \end{aligned} \tag{2.1}$$

It follows that

$$u_j \leq \bar{u}_j = 1 - \sum_{k=1}^m \underline{u}_k + \underline{u}_j. \tag{2.2}$$

3. Optimization

Utilizing Pontryagin maximum principle (PMP) according to [5], we consider the vector p of costate variables $p_i(t)$ and form the Hamiltonian $H(p,q,u)$ as follows

$$H = \sum_{i=1}^n p_i(a_i - d_i \sum_{j\in K_i} u_j) - \sum_{i=1}^n w_i q_i. \tag{3.1}$$

The costate values must satisfy the differential equations

$$\frac{dp_i}{dt} = -\frac{\partial H}{dq_i} = w_i. \tag{3.2}$$

The final values of the queues $q_i(T)$ are free, which means that the transversality conditions must hold

$$p_i(T) = 0,\; i = 1, 2, ..., n. \tag{3.3}$$

The control values u_i have to be found from the maximization of the Hamiltonian over control variables with respect to their constraints. Thus, we have

$$u_j = arg \max_{u_j \in U} H(p,q,u). \tag{3.4}$$

As far as the lower bounds of u_j are given, we just need to find

$$\tilde{u}_j = u_j - \underline{u}_j.$$

Note that the coefficient for u_j in H is equal to

$$S_j = -\sum_{i\in L_j} p_i d_i. \tag{3.5}$$

From equation (3.2) and transversality condition (3.3), we obtain that

$$p_i(0) = -w_i T, \tag{3.6}$$

and

$$p_i(t) = -w_i T + w_i t. \tag{3.7}$$

Substituting (3.7) into S_j, we get

$$S_j(t) = (T-t)\sum_{i\in L_j} w_i d_i. \tag{3.8}$$

Now, the maximization of the Hamiltonian is reduced to the rather simple linear program for the variables $\tilde{u}_j$

$$\begin{aligned} \sum_{j=1}^{m} S_j\tilde{u}_j &\rightarrow \max, \\ \sum_{j=1}^{m} \tilde{u}_j &= 1-\sum_{j=1}^{m} \underline{u}_j, \;\; \tilde{u}_j \geq 0. \end{aligned} \tag{3.9}$$

It follows that the optimal control is obtained from

$$\begin{aligned} j^{opt} &= arg\max_j S_j, \\ \tilde{u}_{j^{opt}} &= \bar{\tilde{u}} = 1-\sum_{j=1}^{m} \underline{u}_j, \\ \tilde{u}_{j\neq j^{opt}} &= 0, \\ u_{j^{opt}} &= \tilde{u}_{j^{opt}} + \underline{u}_{j^{opt}} \\ u_{j\neq j^{opt}} &= \underline{u}_j. \end{aligned} \tag{3.10}$$

The correspondent dual problem for Lagrange multiplier λ has the form

$$\begin{aligned} I &= \lambda(1-\sum_{j=1}^{m} \underline{u}_j) \rightarrow \min \\ \lambda &\geq S_j, \, \lambda \geq 0. \end{aligned} \tag{3.11}$$

According to the KKT conditions, we have

$$I - J \geq 0. \tag{3.12}$$

This is a so-called "dual gap". The solution of the dual problem is

$$\lambda = S_{j^{opt}}. \tag{3.13}$$

Thus we see that

$$I - J = 0,$$

and the solution of the linear problem is optimal.

It is clear that the value of the index j^{opt} is not changed with time because the coefficient $(T-t)$ is always positive. This concludes the analytical solution of the optimal control problem.

4. Maximal Throughput Solution

In the case of the maximal weighted throughput the objective has form

$$J = \sum_{i=1}^{n} w_i q_i(T) \rightarrow \min, \tag{4.1}$$

i.e., here it is minimization of the function of the final states $F(q(T))$. In this case, the Hamiltonian does not contain terms with state variables and thus the costate equations are

$$\frac{dp_i}{dt} = \frac{\partial H}{\partial q_i} = 0,\ i = 1,2,...,n. \tag{4.2}$$

Accordingly, the costates p_i are constant. From the transversality conditions, we have

$$p_i(T) = -grad(F(q(T)) = -w_i. \tag{4.3}$$

It follows that $p_i(t) = w_i$, and that the coefficients of controls u_j in the Hamiltonian are

$$S_j = \sum_{i \in L_j} w_i d_i, \tag{4.4}$$

i.e., they are the same as for the case of weighted minimal delay - just without common positive coefficient $(T - t)$. This means that the optimal solution is the same.

5. Conclusion

The optimal control for the multi-lane signalized intersection was fully investigated and the analytical solution was presented. The solution is somehow similar to the known solution of the Continuous Knapsack Problem (CKP) [6].

References

[1] D. Gazis, Optimum control of a system of oversaturated intersections, Oper. Res. 12 (1964), 815–831.
[2] J. Haddad, B.D. Schutter, D. Mahalel, et al., Optimal steady-state control for isolated traffic intersections, IEEE Trans. Auto. Control 55 (2010), 2612–2617.
[3] I. Ioslovich, J. Haddad, P.O. Gutman, et al., Optimal traffic control synthesis for an isolated intersection, Control Engineering Practice 19 (2011), 900–911.
[4] I. Ioslovich, J. Haddad, P.O. Gutman, et al., Design of optimal traffic flow control at intersection with regard for queue length constraints, Automation and Remote Control 72 (2011), 1833–1840.
[5] L. Pontryagin, V. Boltyanskii, R. Gamkrelidze, et al., The mathematical theory of optimal processes, Wiley-Interscience, NY, 1962.
[6] I. Ioslovich, Robust reduction of a class of large-scale linear programs, SIAM J Optim. 12 (2001), 262–282.