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A WEAK CONVERGENCE THEOREM FOR RELATIVELY NONEXPANSIVE MAPPINGS AND MAXIMAL MONOTONE OPERATORS IN A BANACH SPACE

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Abstract. In this paper, using the idea of Mann's iteration, we prove a weak convergence theorem for finding a common element of the fixed point sets of two relatively nonexpansive mappings and the zero point set of a maximal monotone operator in a Banach space. We apply this result to get well-known and new weak convergence theorems which are connected with relatively nonexpansive mappings and maximal monotone operators in Hilbert spaces and in Banach spaces.

Keywords. Relatively nonexpansive mapping; Generalized projection; Maximal monotone operator; Generalized resolvent; Fixed point.

1. Introduction

Let E be a Banach space and let C be a nonempty, closed and convex subset of E. Let T be a mapping of C into E. We denote the set of fixed points of T by $F(T)$. A mapping $T : C \to E$ is called demiclosed if for a sequence $\{x_n\}$ in C such that $\{x_n\}$ converges weakly to a point p and $x_n - Tx_n \to 0$, $p = Tp$ holds.

Assume that E is a smooth Banach space and C is a nonempty, closed and convex subset of E. A mapping $T : C \to E$ is called relatively nonexpansive [1] if $F(T) \neq \emptyset$, it is demiclosed and

$$\phi(z, Tx) \leq \phi(z, x), \quad \forall x \in C,\ z \in F(T),$$

where $\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$ and J is the duality mapping of E.

In 1953, Mann [2] introduced the following iteration process. Let $T : C \to C$ be a nonexpansive mapping, that is, $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. For an initial guess $x_1 \in C$, an iteration process $\{x_n\}$ is defined recursively by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \quad \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$. Later, Reich [3] discussed Mann's iteration process in a uniformly convex Banach space with a Fréchet differentiable norm and obtained that the sequence $\{x_n\}$ converges weakly to a fixed point of T under some conditions. On the other

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hand, Matsushita and Takahashi [4] proved a weak convergence theorem under Mann's iteration process for relatively nonexpansive mappings in a smooth and uniformly convex Banach space.

In this paper, using the idea of Mann's iteration, we prove a weak convergence theorem for finding a common element of the fixed point sets of two relatively nonexpansive mappings and the zero point set of a maximal monotone operator in a Banach space. We apply this result to get well-known and new weak convergence theorems which are connected with relatively nonexpansive mappings and maximal monotone operators in Hilbert spaces and in Banach spaces

2. Preliminaries

We denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty, closed and convex subset of a Hilbert space H. The nearest point projection of H onto C is denoted by P_C, that is, $\|x-P_Cx\| \le \|x-y\|$ for all $x\in H$ and $y\in C$. Such P_C is called the metric projection of H onto C. We know that the metric projection P_C is firmly nonexpansive, i.e.,

$$\|P_Cx-P_Cy\|^2 \le \langle P_Cx-P_Cy, x-y\rangle \tag{2.1}$$

for all $x,y\in H$. Furthermore, $\langle x-P_Cx, y-P_Cx\rangle \le 0$ holds for all $x\in H$ and $y\in C$; see [5].

Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^*\in E^*$ at $x\in E$ by $\langle x,y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x\in E$ by $x_n\to x$ and the weak convergence by $x_n\rightharpoonup x$. The modulus δ_E of convexity of E is defined by

$$\delta_E(\varepsilon)=\inf\left\{1-\frac{\|x+y\|}{2} : \|x\|\le 1, \|y\|\le 1, \|x-y\|\ge\varepsilon\right\}$$

for every ε with $0\le\varepsilon\le 2$. A Banach space E is said to be uniformly convex if $\delta_E(\varepsilon)>0$ for every $\varepsilon>0$. A uniformly convex Banach space is strictly convex and reflexive. The duality mapping J_E from E into 2^{E^*} is defined by

$$J_Ex=\{x^*\in E^* : \langle x,x^*\rangle=\|x\|^2=\|x^*\|^2\}$$

for every $x\in E$. We also denote J_E by J simply. Let $U=\{x\in E : \|x\|=1\}$. The norm of E is said to be Gâteaux differentiable if for each $x,y\in U$, the limit

$$\lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \tag{2.2}$$

exists. In this case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. The norm of E is said to be Fréchet differentiable if for each $x\in U$, the limit (2.2) is attained uniformly for $y\in U$. The norm of E is said to be uniformly smooth if the limit (2.2) is attained uniformly for $x,y\in U$. If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [6, 7]. We also know the following result.

Lemma 2.1 ([6]). *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x-y, Jx-Jy\rangle\ge 0$ for all $x,y\in E$. Furthermore, if E is strictly convex and $\langle x-y, Jx-Jy\rangle=0$, then $x=y$.*

Let E be a smooth Banach space. The function $\phi : E \times E \to (-\infty, \infty)$ is defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{2.3}$$

for $x, y \in E$, where J is the duality mapping of E; see [8, 9]. We have from the definition of ϕ that

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle \tag{2.4}$$

for all $x, y, z \in E$. From $(\|x\| - \|y\|)^2 \leq \phi(x,y)$ for all $x, y \in E$, we can see that $\phi(x,y) \geq 0$. Furthermore, we can obtain the following equality:

$$2\langle x - y, Jz - Jw\rangle = \phi(x,w) + \phi(y,z) - \phi(x,z) - \phi(y,w) \tag{2.5}$$

for $x, y, z, w \in E$. If E is additionally assumed to be strictly convex, then from Lemma 2.1 we have

$$\phi(x,y) = 0 \Longleftrightarrow x = y. \tag{2.6}$$

Let E be a smooth, strictly convex and reflexive Banach space. Let $\phi_* : E^* \times E^* \to (-\infty, \infty)$ be the function defined by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^*\rangle + \|y^*\|^2$$

for all $x^*, y^* \in E^*$, where J is the duality mapping of E. It is easy to see that

$$\phi_*(Jy, Jx) = \phi(x,y) \tag{2.7}$$

for all $x, y \in E$. The following lemma which was by Kamimura and Takahashi [9] is well-known.

Lemma 2.2 ([9]). *Let E be a smooth and uniformly convex Banach space. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in space E such that either sequence $\{x_n\}$ or sequence $\{y_n\}$ is bounded. If* $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, *then* $\lim_{n\to\infty} \|x_n - y_n\| = 0$.

The following lemmas are in Xu [10] and Kamimura and Takahashi [9].

Lemma 2.3 ([10]). *Let E be a uniformly convex Banach space and let* $r > 0$. *Then there exists a strictly increasing, continuous and convex function* $g : [0, \infty) \to [0, \infty)$ *such that* $g(0) = 0$ *and*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)g(\|x - y\|)$$

for all $x, y \in B_r$ and λ with $0 \leq \lambda \leq 1$, *where* $B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 2.4 ([9]). *Let E be a smooth and uniformly convex Banach space and let* $r > 0$. *Then there exists a strictly increasing, continuous and convex function* $g : [0, 2r] \to \mathbb{R}$ *such that* $g(0) = 0$ *and*

$$g(\|x - y\|) \leq \phi(x,y)$$

for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. For any $x \in E$, we also know that there exists a unique element $z \in C$ such that

$$\phi(z,x) = \min_{y\in C} \phi(y,x).$$

The mapping $\Pi_C : E \to C$ defined by $z = \Pi_C x$ is called the generalized projection of E onto C. We know the following result.

Lemma 2.5 ([8, 9]). *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = \Pi_C x$;
(2) $\langle z - y, Jx - Jz \rangle \geq 0, \quad \forall y \in C$.

Let E be a Banach space and let B be a mapping of of E into 2^{E^*}. A multi-valued mapping B on E is said to be monotone if $\langle x - y, u^* - v^* \rangle \geq 0$ for all $u^* \in Bx$, and $v^* \in By$. A monotone operator B on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to Browder [11]; see also [7, Theorem 3.5.4].

Theorem 2.1 ([11]). *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let B be a monotone operator of E into 2^{E^*}. Then B is maximal if and only if, for any $r > 0$,*

$$R(J + rB) = E^*,$$

where $R(J + rB)$ is the range of $J + rB$.

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let B be a maximal monotone operator of E into 2^{E^*}. The set of null points of a maximal monotone operator B is defined by $B^{-1}0 = \{z \in E : 0 \in Bz\}$. We know that $B^{-1}0$ is closed and convex; see [7].

For all $x \in E$ and $r > 0$, we also consider the following equation

$$Jx \in Jx_r + rBx_r.$$

This equation has a unique solution x_r; see [12]. We define Q_r by $x_r = Q_r x$. Such a Q_r is called the generalized resolvent of B. For $r > 0$, the Yosida approximation $B_r : E \to E^*$ is defined by

$$B_r x = \frac{Jx - JQ_r x}{r}, \quad \forall x \in E.$$

When the Banach space is a Hilbert space, we have that the generalized resolvent Q_r is called the resolvent of B simply. We know the following result.

Lemma 2.6 ([12]). *Let E be a uniformly convex and smooth Banach space and let $B \subset E \times E^*$ be a maximal monotone operator. Let $r > 0$ and let Q_r and B_r be the generalized resolvent and the Yosida approximation of B, respectively. Then, the following hold:*

(1) $\phi(u, Q_r x) + \phi(Q_r x, x) \leq \phi(u, x), \quad \forall x \in E, u \in B^{-1}0$;
(2) $(Q_r x, B_r x) \in B, \quad \forall x \in E$;
(3) $F(Q_r) = B^{-1}0$.

3. Weak convergence theorem

In this section, we prove a weak convergence theorem of Mann's type iteration for two relatively nonexpansive mappings and maximal monotone operators in a Banach space. The following lemma was proved by Matsushita and Takahashi [1].

Lemma 3.1 ([1]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let $T: C \to E$ be a mapping satisfying the following;*

$$\phi(z, Tx) \le \phi(z, x), \quad \forall x \in C,\ z \in F(T).$$

Then $F(T)$ is closed and convex.

The following is our main result.

Theorem 3.1. *Let E be a uniformly convex and uniformly smooth Banach space which the duality mapping J_E is weakly sequentially continuous. Let C be a nonempty, closed and convex subset of E such that J_EC is closed and convex. Let $A \subset E \times E^*$ be a maximal monotone operator satisfying $D(A) \subset C$ and let Q_μ be a generalized resolvent of A, i.e., $Q_\mu = (J_E + \mu A)^{-1} J_E$ for all $\mu > 0$. Let T and U be relatively nonexpansive mappings of C into itself. Suppose that*

$$\Omega = F(T) \cap F(U) \cap A^{-1}0 \ne \emptyset.$$

For any $x_1 = x \in C$, define $\{x_n\}$ as follows:

$$\begin{cases} y_n = J_E^{-1}\big((1-r_n)J_E x_n + r_n J_E U Q_{\mu_n} x_n\big), \\ x_{n+1} = J_E^{-1}\big((1-\beta_n)J_E x_n + \beta_n J_E T y_n\big), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\mu_n\} \subset (0,\infty)$, $\{\beta_n\} \subset (0,1)$, $a, b, \delta, \gamma \in \mathbb{R}$ and $\{r_n\} \subset (0,1)$ satisfy the following:

$$0 < a \le \beta_n \le b < 1,\ 0 < \delta \le r_n \le \gamma < 1 \ \text{ and } \ 0 < c \le \mu_n, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges weakly to an elementt $z_0 \in \Omega$, where $z_0 = \lim_{n\to\infty} \Pi_\Omega x_n$.

Proof. Since T and U are relatively nonexpansive, we have that $F(T)$ and $F(U)$ are closed and convex. Since A is a maximal monotone operator, we have that $A^{-1}0$ is closed and convex. It follows that $\Omega = F(T) \cap F(U) \cap A^{-1}0$ is closed and convex. Let $z \in \Omega$. Then $z = Q_{\mu_n} z$, $z = Tz$ and $z = Uz$. Put

$$y_n = J_E^{-1}\big((1-r_n)J_E x_n + r_n J_E U Q_{\mu_n} x_n\big)$$

and $z_n = Q_{\mu_n} x_n$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned} \phi(z, y_n) &= \phi\big(z, J_E^{-1}((1-r_n)J_E x_n + r_n J_E U z_n)\big) \\ &= \|z\|^2 - 2\langle z, (1-r_n)J_E x_n + r_n J_E U z_n\rangle \\ &\quad + \|(1-r_n)J_E x_n + r_n J_E U z_n\|^2 \\ &\le \|z\|^2 - 2(1-r_n)\langle z, J_E x_n\rangle - 2r_n\langle z, J_E U z_n\rangle \\ &\quad + (1-r_n)\|x_n\|^2 + r_n\|U z_n\|^2 \\ &= (1-r_n)\phi(z, x_n) + r_n\phi(z, U z_n) \\ &\le (1-r_n)\phi(z, x_n) + r_n\phi(z, z_n) \\ &\le (1-r_n)\phi(z, x_n) + r_n\phi(z, x_n) \\ &= \phi(z, x_n). \end{aligned} \tag{3.1}$$

Similarly, we also have

$$\begin{aligned}\phi(z,x_{n+1}) &= \phi\big(z,J_E^{-1}((1-\beta_n)J_Ex_n+\beta_nJ_ETy_n)\big)\\ &\le (1-\beta_n)\phi(z,x_n)+\beta_n\phi(z,Ty_n)\\ &\le (1-\beta_n)\phi(z,x_n)+\beta_n\phi(z,y_n)\\ &\le (1-\beta_n)\phi(z,x_n)+\beta_n\phi(z,x_n)\\ &= \phi(z,x_n).\end{aligned} \tag{3.2}$$

Then $\lim_{n\to\infty}\phi(z,x_n)$ exists. Thus, $\{x_n\}$, $\{Uz_n\}$, $\{y_n\}$ and $\{Ty_n\}$ are bounded. Putting

$$r=\max\Big\{\sup_{n\in\mathbb{N}}\|J_Ex_n\|,\ \sup_{n\in\mathbb{N}}\|J_EUz_n\|,\ \sup_{n\in\mathbb{N}}\|J_ETy_n\|\Big\},$$

we have from Lemma 2.3 that there exists a strictly increasing, continuous and convex function $g:[0,\infty)\to[0,\infty)$ such that $g(0)=0$ and

$$\|\lambda x+(1-\lambda)y\|^2\le\lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)g(\|x-y\|)$$

for all $x,y\in B_r$ and λ with $0\le\lambda\le1$, where $B_r=\{z\in E^*:\|z\|\le r\}$. Using this, we have that, for $n\in\mathbb{N}$ and $z\in\Omega$,

$$\begin{aligned}\phi(z,y_n) &= \phi\big(z,J_E^{-1}((1-r_n)J_Ex_n+r_nJ_EUz_n)\big)\\ &= \|z\|^2-2\langle z,(1-r_n)J_Ex_n+r_nJ_EUz_n\rangle+\|(1-r_n)J_Ex_n+r_nJ_EUz_n\|^2\\ &\le \|z\|^2-2\langle z,(1-r_n)J_Ex_n+r_nJ_EUz_n\rangle\\ &\quad+(1-r_n)\|x_n\|^2+r_n\|Uz_n\|^2-r_n(1-r_n)g(\|J_Ex_n-J_EUz_n\|)\\ &= (1-r_n)\phi(z,x_n)+r_n\phi(z,Uz_n)-r_n(1-r_n)g(\|J_Ex_n-J_EUz_n\|)\\ &\le (1-r_n)\phi(z,x_n)+r_n\phi(z,z_n)-r_n(1-r_n)g(\|J_Ex_n-J_EUz_n\|)\\ &\le \phi(z,x_n)-r_n(1-r_n)g(\|J_Ex_n-J_EUz_n\|).\end{aligned}$$

Similarly, we have that

$$\begin{aligned}\phi(z,x_{n+1}) &= \phi(z,J_E^{-1}((1-\beta_n)J_Ex_n+\beta_nJ_ETy_n))\\ &= \|z\|^2-2\langle z,(1-\beta_n)J_Ex_n+\beta_nJ_ETy_n\rangle+\|(1-\beta_n)J_Ex_n+\beta_nJ_ETy_n\|^2\\ &\le \|z\|^2-2\langle z,(1-\beta_n)J_Ex_n+\beta_nJ_ETy_n\rangle\\ &\quad+(1-\beta_n)\|x_n\|^2+\beta_n\|Ty_n\|^2-\beta_n(1-\beta_n)g(\|J_Ex_n-J_ETy_n\|)\\ &= (1-\beta_n)\phi(z,x_n)+\beta_n\phi(z,Ty_n)-\beta_n(1-\beta_n)g(\|J_Ex_n-J_ETy_n\|)\\ &\le (1-\beta_n)\phi(z,x_n)+\beta_n\phi(z,y_n)-\beta_n(1-\beta_n)g(\|J_Ex_n-J_ETy_n\|)\\ &\le (1-\beta_n)\phi(z,x_n)+\beta_n\big(\phi(z,x_n)-r_n(1-r_n)g(\|J_Ex_n-J_EUz_n\|)\big)\\ &\quad-\beta_n(1-\beta_n)g(\|J_Ex_n-J_ETy_n\|)\\ &= \phi(z,x_n)-\beta_nr_n(1-r_n)g(\|J_Ex_n-J_EUz_n\|)-\beta_n(1-\beta_n)g(\|J_Ex_n-J_ETy_n\|).\end{aligned}$$

Therefore, we have that

$$\beta_n(1-\beta_n)g(\|J_Ex_n-J_ETy_n\|).\le\phi(z,x_n)-\phi(z,x_{n+1})$$

and

$$\beta_nr_n(1-r_n)g(\|J_Ex_n-J_EUz_n\|)\le\phi(z,x_n)-\phi(z,x_{n+1}).$$

We have from $0 < a \leq \beta_n \leq b < 1$ and $0 < \delta \leq r_n \leq \gamma < 1$ that

$$\lim_{n\to\infty} g(\|J_E x_n - J_E T y_n\|) = \lim_{n\to\infty} g(\|J_E x_n - J_E U z_n\|) = 0. \tag{3.3}$$

From the properties of g, we have that

$$\lim_{n\to\infty} \|J_E x_n - J_E T y_n\| = \lim_{n\to\infty} \|J_E x_n - J_E U z_n\| = 0. \tag{3.4}$$

From the definition of y_n, we also have that

$$\|J_E x_n - J_E y_n\| \leq r_n \|J_E x_n - J_E U z_n\|.$$

Since $\lim_{n\to\infty} \|J_E x_n - J_E U z_n\| = 0$, we have $\|J_E x_n - J_E y_n\| \to 0$ and hence $\|J_E y_n - J_E T y_n\| \to 0$. Since E^* is uniformly smooth, we have that

$$\|y_n - T y_n\| \to 0 \text{ and } \|x_n - U z_n\| \to 0 \tag{3.5}$$

as $n \to \infty$. Using $z_n = Q_{\mu_n} x_n$ and Lemma 2.6, we have that, for $z \in \Omega$,

$$\phi(z_n, x_n) = \phi(Q_{\mu_n} x_n, x_n) \leq \phi(z, x_n) - \phi(z, Q_{\mu_n} x_n) = \phi(z, x_n) - \phi(z, z_n).$$

It follows from (3.1) that

$$\begin{aligned}
\phi(z_n, x_n) &\leq \phi(z, x_n) - \phi(z, z_n) \\
&\leq \phi(z, x_n) - \frac{1}{r_n}\big(\phi(z, y_n) - (1 - r_n)\phi(z, x_n)\big) \\
&= \frac{1}{r_n}\big(\phi(z, x_n) - \phi(z, y_n)\big) \\
&= \frac{1}{r_n}\big(\|x_n\|^2 - \|y_n\|^2 - 2\langle z, Jx_n - Jy_n\rangle\big) \\
&\leq \frac{1}{r_n}\big(|\|x_n\|^2 - \|y_n\|^2| + 2|\langle z, Jx_n - Jy_n\rangle|\big) \\
&\leq \frac{1}{r_n}\big(|\|x_n\| - \|y_n\||(\|x_n\| + \|y_n\|) + 2\|z\|\|Jx_n - Jy_n\|\big) \\
&\leq \frac{1}{r_n}\big(\|x_n - y_n\|(\|x_n\| + \|y_n\|) + 2\|z\|\|Jx_n - Jy_n\|\big).
\end{aligned}$$

This implies that $\lim_{n\to\infty} \phi(z_n, x_n) = 0$. Since E is uniformly convex and smooth, we have from Lemma 2.2 that

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.6}$$

Since

$$\|z_n - U z_n\| \leq \|z_n - x_n\| + \|x_n - U z_n\|,$$

we obtain that

$$\lim_{n\to\infty} \|z_n - U z_n\| = 0. \tag{3.7}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup w$. From $\lim_{n\to\infty} \|x_n - y_n\| = 0$ and $\lim_{n\to\infty} \|x_n - z_n\| = 0$, we have $y_{n_i} \rightharpoonup w$ and $z_{n_i} \rightharpoonup w$. Using $\lim_{n\to\infty} \|z_n - U z_n\| = 0$ and the fact that U is relatively nonexpansive, we have that $w = Uw$ and hence $w \in F(U)$. Since T is relatively nonexpansive, we have from $y_{n_i} \rightharpoonup w$ and $\|y_n - T y_n\| \to 0$ that $w \in F(T)$. This implies $w \in F(T) \cap F(U)$.

Next, we show $w \in A^{-1}0$. Since J_E is uniformly norm-to-norm continuous on bounded sets, we conclude from (3.6)

$$\lim_{n\to\infty} \|J_E x_n - J_E z_n\| = 0.$$

From $\mu_n \geq c$, we have

$$\lim_{n\to\infty} \frac{1}{\mu_n} \|J_E x_n - J_E z_n\| = 0.$$

Therefore,

$$\lim_{n\to\infty} \|B_{\mu_n} x_n\| = \lim_{n\to\infty} \frac{1}{\mu_n} \|J_E x_n - J_E z_n\| = 0.$$

For $(p, p^*) \in A$, from the monotonicity of A, we have $\langle p - z_n, p^* - B_{\mu_n} x_n \rangle \geq 0$ for all $n \geq 0$. Replacing n by n_i and letting $i \to \infty$, we get $\langle p - w, p^* \rangle \geq 0$. From the maximallity of A, we have $w \in A^{-1}0$. Therefore, $w \in \Omega$.

We next show that if $x_{n_i} \rightharpoonup u$ and $x_{n_j} \rightharpoonup v$, then $u = v$. In fact, we have that $u, v \in \Omega$. Put $a = \lim_{n\to\infty}(\phi(u, x_n) - \phi(v, x_n))$. Since

$$\phi(u, x_n) - \phi(v, x_n) = 2\langle v - u, J_E x_n \rangle + \|u\|^2 - \|v\|^2$$

and the duality mapping J_E of E is weakly sequentially continuous, we have $a = 2\langle v - u, J_E u \rangle + \|u\|^2 - \|v\|^2$ and $a = 2\langle v - u, J_E v \rangle + \|u\|^2 - \|v\|^2$. From these equalities, we obtain $2\langle v - u, Ju - Jv \rangle = 0$ and hence $\langle u - v, Ju - Jv \rangle = 0$. From Lemma 2.1, it follows that $u = v$. Therefore, $\{x_n\}$ converges weakly to an element $z_0 \in \Omega$.

Put $P = \Pi_\Omega$. We have from Lemma 2.5 and (3.2) that

$$\begin{aligned} \phi(Px_{n+1}, x_{n+1}) &\leq \phi(Px_{n+1}, x_{n+1}) + \phi(Px_n, Px_{n+1}) \\ &\leq \phi(Px_n, x_{n+1}) \\ &\leq \phi(Px_n, x_n) \end{aligned}$$

for all $n \in \mathbb{N}$. Hence $\lim_{n\to\infty} \phi(Px_n, x_n)$ exists. It follows from Lemma 2.5 that, for $k \in \mathbb{N}$,

$$\begin{aligned} \phi(Px_n, x_{n+k}) &= \phi(Px_n, Px_{n+k}) + \phi(Px_{n+k}, x_{n+k}) \\ &\quad + 2\langle Px_n - Px_{n+k}, J_E Px_{n+k} - J_E x_{n+k} \rangle \\ &\geq \phi(Px_n, Px_{n+k}) + \phi(Px_{n+k}, x_{n+k}) \end{aligned}$$

and hence

$$\begin{aligned} \phi(Px_n, Px_{n+k}) &\leq \phi(Px_n, x_{n+k}) - \phi(Px_{n+k}, x_{n+k}) \\ &\leq \phi(Px_n, x_n) - \phi(Px_{n+k}, x_{n+k}). \end{aligned}$$

We also have from Lemma 2.5 that, for $p \in \Omega$,

$$\phi(p, Px_n) \leq \phi(p, Px_n) + \phi(Px_n, x_n) \leq \phi(p, x_n) \leq \phi(p, x)$$

and hence $\{Px_n\}$ is bounded. Using Lemma 2.4, we have that, for $m, n \in \mathbb{N}$ with $m > n$,

$$g'(\|Px_n - Px_m\|) \leq \phi(Px_n, Px_m) \leq \phi(Px_n, x_n) - \phi(Px_m, x_m),$$

where g' is a strictly increasing, continuous and convex function such that $g'(0) = 0$. The the properties of g' yieeld that $\{Px_n\}$ ia a Cauchy sequence. Since E is complete, $\{Px_n\}$ converges strongly to a point $u \in \Omega$. Furthermore, we have from Lemma 2.5 that

$$\langle Px_n - z_0, J_E x_n - J_E Px_n \rangle \geq 0.$$

Since $x_n \rightharpoonup z_0$ and the duality mapping J_E on E is weakly sequentially continuous, we have that

$$\langle u - z_0, J_E z_0 - J_E u\rangle \geq 0$$

and hence $\phi(u,z_0) + \phi((z_0,u) \leq 0$. This implies that $\phi(u,z_0) = \phi(z_0,u) = 0$ and hence $u = z_0$. Therefore, $z_0 = \lim_{n\to\infty} Px_n = \lim_{n\to\infty} \Pi_\Omega x_n$. This completes the proof. □

4. Applications

In this section, using Theorem 3.1, we get well-known and new weak convergence theorems which are connected with relatively nonexpansive mappings and maximal monotone operators in Hilbert spaces and in Banach spaces. We first prove a weak convergence theorem for finding a zero point of a maximal monotone operator in a Banach space.

Theorem 4.1. *Let E be a uniformly convex and uniformly smooth Banach space which the duality mapping J_E is weakly sequentially continuous. Let $A \subset E \times E^*$ be a maximal monotone operator and let Q_μ be a generalized resolvent of A, i.e., $Q_\mu = (J_E + \mu A)^{-1} J_E$ for all $\mu > 0$. Suppose that $A^{-1}0 \neq \emptyset$. For any $x_1 = x \in E$, define $\{x_n\}$ as follows:*

$$x_{n+1} = J_E^{-1}\big((1-r_n)J_E x_n + r_n J_E Q_{\mu_n} x_n\big),$$

for all $n \in \mathbb{N}$, where $\{\mu_n\} \subset (0,\infty)$, $\delta, \gamma \in \mathbb{R}$ and $\{r_n\} \subset (0,1)$ satisfy the following:

$$0 < \delta \leq r_n \leq \gamma < 1 \;\; and \;\; 0 < c \leq \mu_n, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges weakly to an element $z_0 \in A^{-1}0$, where $z_0 = \lim_{n\to\infty} \Pi_{A^{-1}0} x_n$.

Proof. Putting $C = E$ and $T = U = I$ in Theorem 3.1, we obtain the desired result from Theorem 3.1. □

Let E be a Banach space and let $f : E \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function. Define the subdifferential of f as follows:

$$\partial f(x) = \{x^* \in E^* : f(y) \geq \langle y - x, x^*\rangle + f(x), \; \forall y \in E\}$$

for all $x \in E$. Then we know that ∂f is a maximal monotone operator; see [13] for more details. Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E. We have that there exists the generalized projection Π_C of E onto C. We also have that, for the indicator function i_C, that is,

$$i_C x = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C, \end{cases}$$

the subdifferential $\partial i_C \subset E \times E^*$ is a maximal monotone operator and the generalized resolvent $Q_r = \Pi_C$ of ∂i_C for every $r > 0$. In fact, for any $x \in E$ and $r > 0$, we have that

$$\begin{aligned} z = Q_r x &\Leftrightarrow J_E z + r\partial i_C z \ni J_E x \\ &\Leftrightarrow J_E x - J_E z \in r\partial i_C z \\ &\Leftrightarrow i_C y \geq \left\langle y - z, \frac{J_E x - J_E z}{r} \right\rangle + i_C z,\ \forall y \in E \\ &\Leftrightarrow 0 \geq \langle y - z, J_E x - J_E z \rangle,\ \forall y \in C \\ &\Leftrightarrow z = \arg\min_{y \in C} \phi(y, x) \\ &\Leftrightarrow z = \Pi_C x. \end{aligned} \tag{4.1}$$

Using (4.1) and Theorem 3.1, we get the following weak convergence theorem for two relatively nonexpansive mappings in a Banach space.

Theorem 4.2. *Let E be a uniformly convex and uniformly smooth Banach space which the duality mapping J_E is weakly sequentially continuous. Let C be a nonempty, closed and convex subset of E such that $J_E C$ is closed and convex. Let T and U be relatively nonexpansive mappings of C into itself such that*

$$\Omega = F(T) \cap F(U) \neq \emptyset.$$

For any $x_1 = x \in C$, define $\{x_n\}$ as follows:

$$\begin{cases} y_n = J_E^{-1}\big((1 - r_n)J_E x_n + r_n J_E U x_n\big), \\ x_{n+1} = J_E^{-1}\big((1 - \beta_n)J_E x_n + \beta_n J_E T y_n\big), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\beta_n\} \subset (0,1)$, $a, b, \delta, \gamma \in \mathbb{R}$ and $\{r_n\} \subset (0,1)$ satisfy the following:

$$0 < a \leq \beta_n \leq b < 1 \text{ and } 0 < \delta \leq r_n \leq \gamma < 1, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges weakly to an element $z_0 \in \Omega$, where $z_0 = \lim_{n\to\infty} \Pi_\Omega x_n$.

Proof. Putting $A = \partial i_C$ in Theorem 3.1, we obtain that $Q_{\mu_n} = \Pi_C$ for all $\mu_n > 0$. Therefore, we obtain the desired result from Theorem 3.1. □

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called generalized hybrid [15] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1-\alpha)\|x - Uy\|^2 \leq \beta\|Ux - y\|^2 + (1-\beta)\|x - y\|^2$$

for all $x, y \in C$. Such a mapping U is called (α, β)-generalized hybrid. Notice that the class of (α, β)-generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is nonspreading [12, 16] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Ux - Uy\|^2 \leq \|Ux - y\|^2 + \|Uy - x\|^2, \quad \forall x, y \in C.$$

It is also hybrid [17] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Ux - Uy\|^2 \leq \|x - y\|^2 + \|Ux - y\|^2 + \|Uy - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [14]. We know the following result obtained by Kocourek, Takahashi and Yao [15]; see also [18].

Lemma 4.1 ([15, 18]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $U : C \to H$ be generalized hybrid. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

The following are two weak convergence theorems for finding a common element of the fixed point sets of two nonlinear operators and the zero point set of a maximal monotone operator in a Hilbert space.

Theorem 4.3. *Let H bea Hilbert space and let C be a nonempty, closed and convex subset of H. Let $A \subset H \times H$ be a maximal monotone operator satisfying $D(A) \subset C$ and let J_μ be the resolvent of A, i.e., $J_\mu = (I + \mu A)^{-1}$ for all $\mu > 0$. Let $T : C \to C$ be a nonspreading mapping and let $U : C \to C$ be a hybrid mapping. Suppose that $\Omega = F(T) \cap F(U) \cap A^{-1}0 \neq \emptyset$. For any $x_1 = x \in C$, define $\{x_n\}$ as follows:*

$$\begin{cases} y_n = (1 - r_n)x_n + r_n U J_{\mu_n} x_n, \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n T y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\mu_n\} \subset (0, \infty)$, $\{\beta_n\} \subset (0, 1)$, $a, b, \delta, \gamma \in \mathbb{R}$ and $\{r_n\} \subset (0, 1)$ satisfy the following:

$$0 < a \leq \beta_n \leq b < 1, \ 0 < \delta \leq r_n \leq \gamma < 1 \ \text{ and } \ 0 < c \leq \mu_n, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges weakly to an element $z_0 \in \Omega$, where $z_0 = \lim_{n\to\infty} P_\Omega x_n$ anf P_Ω is the metric projection of H onto Ω.

Proof. Since T is nonspreading of C into C, it satisfies the following:

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

Putting $y = p$ for $p \in F(T)$, we have that

$$2\|Tx - p\|^2 \leq \|Tx - p\|^2 + \|p - x\|^2, \quad \forall x \in C$$

and hence

$$\|Tx - p\|^2 \leq \|p - x\|^2, \quad \forall x \in C.$$

This implies that T is quasi-nonexpansive. Furthermore, we have from Lemma 4.1 that T is demiclosed.

Similarly, since U is a hybrid mapping of C into C such that $F(U) \neq \emptyset$, it satisfies the following:

$$3\|Ux - Uy\|^2 \leq \|x - y\|^2 + \|Ux - y\|^2 + \|Uy - x\|^2, \quad \forall x, y \in C.$$

Putting $y = p$ for $p \in F(U)$, we have that

$$3\|Ux - p\|^2 \leq \|x - p\|^2 + \|Ux - p\|^2 + \|p - x\|^2, \quad \forall x \in C$$

and hence

$$\|Ux - p\|^2 \leq \|p - x\|^2, \quad \forall x \in C.$$

This implies that U is quasi-nonexpansive. Furthermore, we have from Lemma 4.1 that U is demiclosed. Therefore, we have the desired result from Theorem 3.1. □

Theorem 4.4. *Let H bea Hilbert space and let C be a nonempty, closed and convex subset of H. Let $A \subset H \times H$ be a maximal monotone operator satisfying $D(A) \subset C$ and let J_μ be the resolvent of A, i.e., $J_\mu = (I + \mu A)^{-1}$ for all $\mu > 0$. Let $T : C \to C$ be a nonspreading mapping and let $U : C \to C$ be a generalized hybrid mapping. Suppose that*

$$\Omega = F(T) \cap F(U) \cap A^{-1}0 \neq \emptyset.$$

For any $x_1 = x \in C$, define $\{x_n\}$ as follows:

$$\begin{cases} y_n = (1-r_n)x_n + r_n U J_{\mu_n} x_n, \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T y_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\mu_n\} \subset (0,\infty)$, $\{\beta_n\} \subset (0,1)$, $a,b,\delta,\gamma \in \mathbb{R}$ and $\{r_n\} \subset (0,1)$ satisfy the following:

$$0 < a \leq \beta_n \leq b < 1,\ 0 < \delta \leq r_n \leq \gamma < 1 \ \text{ and } \ 0 < c \leq \mu_n, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges weakly to an element $z_0 \in \Omega$, where $z_0 = \lim_{n\to\infty} P_\Omega x_n$ anf P_Ω is the metric projection of H onto Ω.

Proof. Since T is a nonexpansive mapping of C into C with $F(T) \neq \emptyset$, we have that T is quasi-nonexpansive. Furthermore, we have from Lemma 4.1 that T is demiclosed. Since U is a generalized hybrid mapping of C into C such that $F(U) \neq \emptyset$, U is quasi-nonexpansive. Furthermore, from Lemma 4.1, U is demiclosed. Therefore, we have the desired result from Theorem 3.1. □

The following is a weak convergence theorems for finding a common point of three sets in a Banach space.

Theorem 4.5. *Let E be a uniformly convex and uniformly smooth Banach space which the duality mapping J_E of E is weakly suquentially continuous. Let C, D and F be nonempty, closed and convex subsets of E. Let Π_C, Π_D and Π_F be the generalized projections of E onto C, D and F, respectively. Suppose that $C \cap D \cap F \neq \emptyset$. For any $x_1 = x \in E$, define*

$$\begin{cases} y_n = J_E^{-1}\big((1-r_n)J_E x_n - + r_n J_E \Pi_D \Pi_F x_n)\big), \\ x_{n+1} = J_E^{-1}((1-\beta_n)J_E x_n + \beta_n J_E \Pi_C y_n), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\beta_n\} \subset (0,1)$ and $\{r_n\} \subset (0,1)$ satisfy the following:

$$0 < a \leq \beta_n \leq b < 1 \ \text{ and } \ 0 < \delta \leq r_n \leq \gamma < 1, \quad \forall n \in \mathbb{N}$$

for some $a,b,\delta,\gamma \in \mathbb{R}$. Then $\{x_n\}$ converges weakly to $z_0 \in C \cap D \cap F$. where $z_0 = \lim_{n\to\infty} \Pi_{C\cap D\cap F} x_n$.

Proof. Take $A = \partial i_F$ in Theorem 3.1. Then we have that $Q_{\mu_n} = \Pi_F$ for all $n \in \mathbb{N}$. Furthermore, since Π_C is the genralized projection of E onto C, we have from Lemma 2.5 that

$$\phi(z, \Pi_C x) \leq \phi(z,x), \quad \forall x \in E,\ z \in C.$$

We show that Π_C is demiclosed. In fact, assume that $x_n \rightharpoonup p$ and $x_n - \Pi_C x_n \to 0$. It is clear that $\Pi_C x_n \rightharpoonup p$. Since E is uniformly smooth, we have that $\|J_E x_n - J_E \Pi_C x_n\| \to 0$. Since Π_C is the generalized projection of E onto C, we have that

$$\langle \Pi_C x_n - \Pi_C p, J_E x_n - J_E \Pi_C x_n - (J_E p - J_E \Pi_C p)\rangle \geq 0.$$

Therefore, $\langle p - \Pi_C p, -(J_E p - J_E \Pi_C p)\rangle \geq 0$ and hence $\phi(p, \Pi_C p) + \phi(\Pi_C p, p) \leq 0$. This implies that $p = \Pi_C p$ and hence Π_C is demiclosed. Similarly,

$$\phi(z, \Pi_D x) \leq \phi(z,x), \quad \forall x \in E,\ z \in D$$

and $\Pi_D$is demiclosed. Therefore, we have the desired result from Theorem 3.1. □

The following is a weak convergence theorem for finding a common element of zero point sets of three maximal monotone operators of a Banach space.

Theorem 4.6. *Let E be a uniformly convex and uniformly smooth Banach space which the duality mapping J_E of E is weakly suquentially continuous. Let A, B and G be maximal monotone operators of E into E^*. Let Q_r^A be the generalized resolvent of A for $r > 0$, Q_μ^B be the generalized resolvent of B for $\mu > 0$ and let Q_λ^G be the generalized resolvent of G for $\lambda > 0$. Suppose that*

$$\Omega = A^{-1}0 \cap B^{-1}0 \cap G^{-1}0 \neq \emptyset.$$

For any $x_1 = x \in E$, define

$$\begin{cases} y_n = J_E^{-1}\big((1-r_n)J_E x_n + r_n Q_\lambda^G Q_r^A x_n)\big), \\ x_{n+1} = J_E^{-1}((1-\beta_n)J_E x_n + \beta_n J_E Q_\mu^B y_n), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\beta_n\} \subset (0,1)$ and $\{r_n\} \subset (0,1)$ satisfy the following:

$$0 < a \leq \beta_n \leq b < 1 \ \ and \ \ 0 < \delta \leq r_n \leq \gamma < 1, \quad \forall n \in \mathbb{N}$$

for some $a, b, \delta, \gamma \in \mathbb{R}$. Then the sequence $\{x_n\}$ converges weakly to a point $z_0 \in \Omega$. where $z_0 = \lim_{n\to\infty} \Pi_\Omega x_n$.

Proof. Take $\mu_n = r$ for $r > 0$ in Theorem 3.1. Then we have that $Q_{\mu_n}^A = Q_r^A$ for all $n \in \mathbb{N}$. Furthermore, since Q_μ^B is the generalized resolvent of B, we have from Lemma 2.6 that

$$\phi(z, Q_\mu^B x) \leq \phi(z, x), \quad \forall x \in E, \ z \in B^{-1}0.$$

Next, we show that Q_μ^B is demiclosed. In fact, assume that $x_n \rightharpoonup p$ and $x_n - Q_\mu^B x_n \to 0$. It is clear that $Q_\mu^B x_n \rightharpoonup p$ as $n \to \infty$. Since E is unifrmly smooth, we have that $\|J_E x_n - J_E Q_\mu^B x_n\| \to 0$. Since Q_μ^B is the generalized resolvent of B, we have from [19] that

$$\langle Q_\mu^B x_n - Q_\mu^B p, J_E x_n - J_E Q_\mu^B x_n - (J_E p - J_E Q_\mu^B p)\rangle \geq 0.$$

Therefore, $\langle p - Q_\mu^B p, -(J_F p - J_E Q_\mu^B p)\rangle \geq 0$ and hence $\phi(p, Q_\mu^B p) + \phi(Q_\mu^B p, p) \leq 0$. This implies that $p = Q_\mu^B p$ and hence Q_μ^B is demiclosed.

Similarly,

$$\phi(z, Q_\lambda^G x) \leq \phi(z, x), \quad \forall x \in E, \ z \in G^{-1}0$$

and Q_λ^G is demiclosed. Therefore, we have from Theorem 3.1 the desired result immediately. □

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