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L^∞-STABILITY FOR A CLASS OF PARAMETRIC OPTIMAL CONTROL PROBLEMS WITH MIXED POINTWISE CONSTRAINTS

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Abstract. This paper studies the local stability of a parametric optimal control problem with mixed pointwise constraints. We show that if the unperturbed problem satisfies the strictly second-order optimality conditions, then the solution map is upper Hölder continuous in L^∞-norm of control variable.

Keywords. Locally Hölder upper continuity; Optimality condition; Solution stability; Second-order sufficient optimality condition.

1. INTRODUCTION

In this paper, we study the following parametric optimal control problem. For each fixed parameter $w \in L^\infty([0,1],\mathbb{R}^k)$, one determines a control vector $\hat{u}_w \in L^2([0,1],\mathbb{R}^m)$ and a corresponding trajectory $\hat{x}_w \in C([0,1],\mathbb{R}^n)$ which solve

$$I(x,u,w) = \int_0^1 \big(\varphi_1(t,x(t),w(t)) + \varphi_2(t,x(t),w(t))u(t) + \varphi_3(w(t))|u(t)|^2\big)dt \to \min \quad (1.1)$$

s.t.

$$x(t) = x_0 + \int_0^t \big(f_1(s,x(s),w(s)) + f_2(s,w(s))u(s)\big)ds, \quad (1.2)$$

$$g(t,x(t),w(t)) + \varepsilon u(t) \le 0 \quad \text{a.e. } t \in [0,1], \quad (1.3)$$

$$u(t) \le 0 \quad \text{a.e. } t \in [0,1], \quad (1.4)$$

where $\varepsilon > 0$, $\varphi_1 : [0,1]\times\mathbb{R}^n\times\mathbb{R}^k \to \mathbb{R}$, $\varphi_2 : [0,1]\times\mathbb{R}^n\times\mathbb{R}^k \to \mathbb{R}^m$, $f_1 : [0,1]\times\mathbb{R}^n\times\mathbb{R}^k \to \mathbb{R}^n$, $f_2 : [0,1]\times\mathbb{R}^k \to \mathbb{R}^{n\times m}$ and $g : [0,1]\times\mathbb{R}^n\times\mathbb{R}^k \to \mathbb{R}^m$ are Carathéodory vector functions and $\varphi_3 : \mathbb{R}^k \to \mathbb{R}$ is a continuous function.

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Throughout this paper, we denote by $(P(w))$ the problem (1.1)-(1.4), by $\Phi(w)$ the feasible set of $(P(w))$ and by $S(w)$ the optimal solution set of (1.1)-(1.4) corresponding to parameter w. Note that the constraints (1.3) and (1.4) aim at regularizing the pure state constraint $g(t,x(t),w(t)) \leq 0$. Besides, it is failed to simplify $(P(w))$ by changing variable $v(t) = g(t,x(t),w(t)) + \varepsilon u(t)$.

Fixing a parameter $\bar{w}$, we call $(P(\bar{w}))$ the unperturbed problem and assume that $(\bar{x},\bar{u}) \in S(\bar{w})$. Our main concern is to estimate the error $\|\hat{x}_w - \bar{x}\| + \|\hat{u}_w - \bar{u}\|$ whenever $(\hat{x}_w, \hat{u}_w) \in S(w)$. This problem has been studied by several authors in the last decade. For the papers which are related to the present work, we refer the readers to [7, 8, 11, 20, 21, 22, 23, 24] and the references therein.

The solution stability of optimal control problems has some important applications in parameter estimation problems and in numerical methods (see, for instance, [11, 19]). It is known that when $(P(w))$ is a convex problem and $J(\cdot,\cdot,w)$ is strongly convex, then the solution map of $S(w)$ is single-valued. In this case, under certain conditions, the solution map is continuous in parameters (see, for instance, [8]). Also, in [20, 21, 22, 23, 24], Malanowski showed that if weak second-order optimality conditions and standard constraints qualifications are satisfied at the reference point, then the solution map is a Lipschitz continuous function of parameters. However, when these conditions are not satisfied, the solution map may not be singleton. In this situation, we need to use the tools of set-valued and variational analysis to deal with the problem. Such treatments have been developed recently in [1, 2, 13, 16, 17]. Particularly, in [1, 2] Alt showed that under certain conditions, the solution map is locally upper Lipschitz continuous in L^2-norm of control variables.

Motivated by results in [1] and [2], in this paper, we show that if the unperturbed problem satisfies strictly second-order optimality conditions, then the solution map $S(w)$ is locally Hölder continuous not only in L^2-norm but also in $L^\infty-$norm. Note that the norm $\|\cdot\|_\infty$ is shaper than norms $\|\cdot\|_p$ and so it is often used for error estimate in numerical methods of optimal control. However, the estimation of $\|\hat{u}_w - \bar{u}\|_\infty$ depends on the number of Lagrange multipliers which equals to the number of constraints. The more constraints the problem has the more difficult we estimate $\|\hat{u}_w - \bar{u}\|_\infty$. In our problem, the number of constraints equals to $2m$. Therefore, the estimation of $\|\hat{u}_w - \bar{u}\|_\infty$ is difficult to obtain. To tackle the problem, we first transform it to an abstract parametric optimal control problem, which satisfies the Robinson constrain qualification condition. We then establish the regularity of optimal solutions by techniques of metric projections. Based on this fact, we prove Hölder continuity of solutions of adjoint equations. Then, we can obtain our main results.

The paper is organized as follows. Section 2 is of assumptions and statement of the main result. Section 3 is destined for some auxiliary results. The proof of the main results is provided in Section 4.

2. Assumptions and statement of main results

Given a Banach space X, $v \in X$ and $r > 0$, we denote by $B_X(v,r)$ and $\bar{B}_X(v,r)$ the open ball and the closed ball with center v and radius r, respectively. In some cases, if no confusion caused, we write $B(v,r)$ and $\bar{B}(v,r)$. Also, we denote by $B_X, \bar{B}_X$ the open unit ball and the closed unit ball, respectively. As in the introductory section, we assume that $\bar{z} = (\bar{x},\bar{u})$ is a locally optimal solution of $(P(\bar{w}))$. For each parameter $w \in L^\infty([0,1],\mathbb{R}^k)$ and $R > 0$, we define

$$\Phi_R(w) = \Phi(w) \cap \bar{B}(\bar{z}, R),$$
$$S_R(w) = \{(y_w, u_w) \in \Phi_R(w) \mid I(y_w, u_w, w) = \inf_{(y,u)\in\Phi_R(w)} I(y,u,w)\}.$$

Given a vector function $\psi = \psi(t,x,u,w)$, the symbols $\psi[t]$ and $\psi[t,w]$ stand for $\psi(t,\bar{x}(t),\bar{u}(t), \bar{w}(t))$ and $\psi(t,\hat{x}_w(t),\hat{u}_w(t),w(t))$, respectively. Also, the notations $\psi_x[t]$, $\psi_u[t]$, $\psi_{xx}[t]$, $\psi_{xu}[t]$, $\psi_x[t,w]$, $\psi_u[t,w]$, ect, are used similarly. Given $x \in C([0,1],\mathbb{R}^n)$, the norm $\|x\|_0$ is defined by setting

$$\|x\|_0 = \max_{t\in[0,1]} |x(t)|.$$

For an $\varepsilon > 0$, we set

$$N(\bar{x},\varepsilon) = \{y \in \mathbb{R}^n \mid \inf_{t\in[0,1]} |y - \bar{x}(t)| < \varepsilon\},$$

and

$$N(\bar{w},\varepsilon) = \{w \in \mathbb{R}^k \mid \inf_{t\in[0,1]} |w - \bar{w}(t)| < \varepsilon\}.$$

Let L stand for φ_i $(i = 1,2,3)$, f_j $(j = 1,2)$ and g. We impose the following hypothesis on L.

$(H1)$ For a.e. $t \in [0,1]$ and $w \in N(\bar{w},\varepsilon)$, the mappings $\varphi_i(t,\cdot,w)$, $f_j(t,\cdot,w)$ and $g(t,\cdot,w)$ are twice continuously differentiable but $f_j(t,\cdot,\cdot)$ is of class C^1 on $N(\bar{x},\varepsilon) \times N(\bar{w},\varepsilon)$.

$(H2)$ For each $M > 0$, there exists a constant $k_{ML} > 0$ such that

$$|L(t,x_1,w_1) - L(t,x_2,w_2)| + |\nabla_x L(t,x_1,w_1) - \nabla_x L(t,x_2,w_2)|$$
$$+ |\nabla_x^2 L(t,x_1,w_1) - \nabla_x^2 L(t,x_2,w_2)| \le k_{ML}(|x_1 - x_2| + |w_1 - w_2|).$$

for a.e. $t \in [0,1]$, for all $x_i \in N(\bar{x},\varepsilon)$ and $w_i \in N(\bar{w},\varepsilon)$ satisfying $|x_i| \le M$ and $|w_i| \le M$, with $i = 1,2$. Furthermore, we require that $|L[\cdot]|$ and $|\nabla_x L[\cdot]|$ belong to $L^\infty([0,1],\mathbb{R})$.

$(H3)$ $f_{1x}[t] \le 0$, $f_2[t] \le 0$ and $g_x[t] \ge 0$ for a.e. $t \in [0,1]$, and there exists $\gamma > 0$ such that $\varphi_3(\bar{\mu}(t)) \ge \gamma$ for a.e. $t \in [0,1]$.

$(H4)$ $\inf_{t\in[0,1]} |g_i[t]| > 0$ for all $i = 1,2,...,m$.

In the sequel, we shall use the notations

$$\varphi(t,x,u,w) = \varphi_1(t,x,w) + \varphi_2(t,x,w)u + \varphi_3(w)|u|^2,$$
$$I(x,u,w) = \int_0^1 \varphi(t,x(t),u(t),w(t))dt,$$
$$\varphi_x[t,w] = \varphi_{1x}[t,w] + \varphi_{2x}[t,w]^T \hat{u}_w,\ \varphi_u[t,w] = \varphi_2[t,w] + 2\varphi_3(w)\hat{u}_w,$$
$$f(t,x,u,w) = f_1(t,x,w) + f_2(t,w)u,$$
$$f_x[t,w] = f_{1x}[t,w],\ f_u[t,w] = f_2[t,w].$$

Let us give some explanations on the assumptions. $(H1)$ and $(H2)$ make sure that each $w \in N(\bar{w},\varepsilon)$, $I(\cdot,\cdot,w)$ is of class C^2 around $(\bar{x},\bar{u})$. By a simple calculation, we have

$$\begin{aligned}
&\langle \nabla I(\hat{x}_w,\hat{u}_w,w),(x,u)\rangle \\
&= \int_0^1 \big((\varphi_{1x}(t,\hat{x}_w(t),w(t))x(t)+\varphi_{2x}(t,\hat{x}_w(t),w(t))\hat{u}_w(t)x(t)\big)dt \\
&\quad + \int_0^1 \big(\varphi_2(t,\hat{x}_w(t),w(t))u(t)+2\varphi_3(w(t))\hat{u}_w(t)u(t)\big)dt.
\end{aligned}$$

Assumption $(H3)$ guarantees that the Robinson constraint qualification is fulfilled. Meanwhile $(H4)$ is a technical assumption which help us to estimate $\|u_w-\bar{u}\|_\infty$.

Let us set

$$T_1(w)=\{t\in[0,1] \mid g[t,w]+\varepsilon\hat{u}_w(t)=0\}, \tag{2.1}$$

$$T_2(w)=\{t\in[0,1] \mid \hat{u}_w(t)=0\}. \tag{2.2}$$

Definition 2.1. A couple $(x,u)\in C([0,1],\mathbb{R}^n)\times L^2([0,1],\mathbb{R}^m)$ is said to be a critical direction of problem $(P(w))$ at $(\hat{x}_w,\hat{u}_w)$ if the following conditions are fulfilled:
(c_1) $\langle \nabla_z I(\hat{x}_w,\hat{u}_w,w),(x,u)\rangle \le 0$;
(c_2) $x(\cdot)=\int_0^{(\cdot)}\big((f_{1x}[s,w]x(s)+f_2[s,w]u(s)\big)ds$;
(c_3) $g_x[t,w]x(t)+\varepsilon u(t)\le 0$ a.e. $t\in T_1(w)$;
(c_4) $u(t)\le 0$ a.e. $t\in T_2(w)$.

We denote by $\mathscr{C}[(\hat{x}_w,\hat{u}_w)]$ the closed convex cone of critical directions of $(P(w))$ at $(\hat{y}_w,\hat{u}_w)$ and by $\mathscr{C}[(\bar{x},\bar{u})]$ the closed convex cone of critical directions of $(P(\bar{w}))$ at $(\bar{x},\bar{u})$.

Definition 2.2. A triple $(p_w,\theta_w,\vartheta_w)$, where $p_w\colon[0,1]\to\mathbb{R}^n$ is an absolutely continuous vector function, θ_w and ϑ_w belong to $L^2([0,1],\mathbb{R}^m)$, is said to be Lagrange multipliers of $(P(w))$ at $(\hat{x}_w,\hat{u}_w)$ if the following conditions are fulfilled:
(i) (the adjoint equation)

$$\begin{cases} \dot{p}_w(t)=-\varphi_x[t,w]-f_x[t,w]p_w(t)-\theta_w(t)g_x[t,w], \\ p_w(1)=0; \end{cases} \tag{2.3}$$

(ii) (the stationary condition in u)

$$\varphi_u[t,w]+f_u[t,w]p_w(t)+\varepsilon\theta_w(t)+\varepsilon\vartheta_w=0 \text{ a.e. } t\in[0,1]; \tag{2.4}$$

(iii) (the complimentary condition)

$$\theta_{wi}(t)\ge 0,\ \vartheta_{iw}(t)\ge 0 \text{ a.e. } t\in[0,1], \tag{2.5}$$

$$\theta_{wi}(t)(g_i[t,w]+\varepsilon\hat{u}_{wi}(t))=0,\ \text{a.e. } t\in[0,1], \tag{2.6}$$

$$\vartheta_{wi}w(t)(\varepsilon\hat{u}_{wi}(t))=0 \text{ a.e. } t\in[0,1], \tag{2.7}$$

where $i=1,2,...,m$; $\theta_w=(\theta_{w1},...,\theta_{wm})$ and $\vartheta_w=(\vartheta_{w1},...,\vartheta_{wm})$.

We shall denote by $\Lambda[(\hat{x}_w,\hat{u}_w)]$ and $\Lambda[(\bar{x},\bar{u})]$ the sets of multipliers of $(P(w))$ and $(P(\bar{w}))$, respectively. We are ready to state our main result.

Theorem 2.1. *Suppose that assumptions* $(H1)-(H3)$ *are fulfilled and there exists a triple* $(\bar p,\bar\theta,\bar\vartheta)\in\Lambda[(\bar x,\bar u)]$ *such that*

$$\int_0^1\big(\varphi_{1xx}[t]+\varphi_{2xx}[t]\bar u(t))x(t)^2+2\varphi_{2x}[t]x(t)u(t)+2\varphi_3(\bar w(t))|u(t)|^2\big)\,dt$$
$$+\int_0^1\big(\bar p(t)f_{1xx}[t]+\bar\theta(t)g_{xx}[t]x(t)^2\big)dt>0\ \forall(x,u)\in\mathscr{C}[(\bar x,\bar u)]\setminus\{(0,0)\}. \tag{2.8}$$

Then there exist numbers $R_*>0, l_*>0$ *and* $\varepsilon_*>0$ *such that, for each* $w\in B_W(\bar w,\varepsilon_*)$, $S_{R_*}(w)\subset S(w)$, *that is, any couple* $(\hat x_w,\hat u_w)\in S_{R_*}(w)$ *is a locally optimal solution of* $(P(w))$. *Moreover,* $(\hat y_w,\hat u_w)\in W^{1,\infty}([0,1],\mathbb{R}^n)\times L^\infty([0,1],\mathbb{R}^m)$ *and*

$$\|\hat x_w-\bar x\|_0+\|\hat u_w-\bar u\|_2\le l_*\|w-\bar w\|_\infty^{1/2}. \tag{2.9}$$

In addition, if $(H4)$ *is fulfilled, then there are numbers* $l_1>0$ *and* $\varepsilon_1>0$ *such that*

$$\|\hat x_w-\bar x\|_{1,\infty}+\|\hat u_w-\bar u\|_\infty\le l_1\|w-\bar w\|_\infty^{1/2}. \tag{2.10}$$

for all $(\hat x_w,\hat u_w)\in S_{R_*}(w)$ *and* $w\in B_W(\bar w,\varepsilon_1)$.

To prove Theorem 2.1, we need to establish some auxiliary results below.

3. Some auxiliary results

Let E_0,E,X,U and W be Banach spaces and $Z:=X\times U$. Let

$$J:X\times U\times W\to\mathbb{R},\ F:X\times U\times W\to E_0,\ G:X\times U\times W\to E$$

be given mappings and K be a nonempty closed convex set in E. For each $w\in W$, we consider the parametric optimal control problem of finding a control vector $u\in U$ and the corresponding state $x\in X$, which solve

$$(P'(w))\qquad\begin{cases}J(x,u,w)\to\min\\ F(x,u,w)=0,\\ G(x,u,w)\in K.\end{cases}$$

We denote by $\Phi'(w)$ the feasible set of $(P'(w))$ and define

$$D(w)=\{z=(x,u)\in Z\mid F(x,u,w)=0\}.$$

Then $(P'(w))$ can be written in the simple form

$$(P'(w))\qquad\begin{cases}J(z,w)\to\min\\ z\in\Phi'(w).\end{cases}$$

Fixing a parameter $w_0\in W$, we call $(P'(w_0))$ the unperturbed problem and assume that $z_0=(x_0,u_0)\in\Phi'(w_0)$.

Given a closed set C in Z and a point $z\in C$, the sets

$$T^\flat(C,z)=\{h\in Z|\forall t_n\to0^+,\exists h_n\to h,z+t_nh_n\in C\},$$

and

$$T(C,z)=\{h\in Z|\exists t_n\to0^+,\exists h_n\to h,z+t_nh_n\in C\}$$

are called *the adjacent tangent cone* and *the contingent cone* to C at z, respectively. These cones are closed and $T^\flat(C,z) \subseteq T(C,z)$. It is well known that if C is convex, then

$$T^\flat(C,z) = T(C,z) = \overline{\text{cone}}(C-z)$$

and the normal cone to C at z is given by

$$N(C,z) = \{z^* \in Z^* \mid \langle z^*, c-z\rangle \le 0,\ \forall c \in C\}.$$

We now impose hypothesis on J, F and G relating on z_0.

$(A1)$ There exist positive numbers r_1, r_1', r_1'' such that for any $w \in B_W(w_0, r_1'')$, the mapping $J(\cdot,\cdot,w)$, $F(\cdot,\cdot,w)$ and $G(\cdot,\cdot,w)$ are twice Fréchet differentiable on $B_Y(y_0,r_1)\times B_U(u_0,r_1')$. The mapping $F(\cdot,\cdot,\cdot)$ is continuously Fréchet differentiable on $B_Y(y_0,r_1)\times B_U(u_0,r_1')\times B_W(w_0,r_1'')$.

$(A2)$ There exist constants $k_G > 0$ and $k_J > 0$ such that

$$\|G(z_1,w_1)-G(z_2,w_2)\| + \|\nabla G(z_1,w_1)-\nabla G(z_2,w_2)\| \le k_G(\|z_1-z_2\|+\|w_1-w_2\|),$$
$$\|J(z_1,w_1)-J(z_2,w_2)\| + \|\nabla J(z_1,w_1)-\nabla J(z_2,w_2)\| \le k_J(\|z_1-z_2\|+\|w_1-w_2\|)$$

for all $z_1, z_2 \in B_Y(y_0,r_1)\times B_U(u_0,r_1')$ and $w_1,w_2 \in B_W(w_0,r_1'')$.

$(A3)$ The mapping $F_x(z_0,w_0)$ is bijective.

$(A4)$ $\nabla_z G(z_0,w_0)(T(D(w_0),z_0)) - \text{cone}(K - G(z_0,w_0)) = E$.

Conditions $(A3)$ and $(A4)$ make sure that the unperturbed problem satisfies the Robinson constraint qualification. According to [14, Theorem 2.5] (see also [25, Theorem 2.1]) condition $(A4)$ is amount to saying that there exists a number $\rho_0 > 0$ such that

$$B_E(0,\rho_0) \subset \nabla_z G(z_0,w_0)(T(D(w_0),z_0)\cap B_Z) - (K - G(z_0,w_0))\cap B_E.$$

The following lemma gives the Aubin property of $(\Phi'(w))$.

Lemma 3.1. ([16, Lemma 3.4]) *Suppose that $z_0 \in \Phi'(w_0)$ and assumptions $(A1)-(A4)$ are satisfied. Then the set-valued map $\Phi'(\cdot)$ has the Aubin property around (z_0,w_0), that is, there exists neighborhoods $B_Z(z_0,r_2')$, $B_W(w_0,r_2'')$ and a constant $k_\Psi > 0$ such that*

$$\Phi'(w)\cap B_Z(z_0,\alpha') \subset \Phi'(w') + k_{\Phi'}\|w-w'\|\bar{B}_Z$$

for all $w, w' \in B_W(w_0,r_2'')$.

From Lemma 3.1, we see that, for each $R > 0$, there exists a number r'' small enough such that

$$\Phi'(w)\cap B(z_0,R) \ne \emptyset,\ \forall w \in B_W(w_0,r'').$$

Putting $\Phi_R'(w) = \Phi'(w)\cap \bar{B}(z_0,R)$, we consider the problem

$$(P_R'(w)) \qquad \begin{cases} J(z,w) \to \min \\ z \in \Phi_R'(w). \end{cases}$$

We denote by $S_R'(w)$ the solution set of $(P_R'(w))$.

Recall that $z_0 \in \Phi'(w_0)$ is a locally strongly optimal solution of $(P'(w_0))$ if there exists constant $\gamma_0 > 0$ and $\alpha_0 > 0$ such that

$$J(z,w_0) \ge J(z_0,w_0) + \alpha_0\|z-z_0\|^2\ \forall z \in \Phi'(w_0)\cap B_Z(z_0,\gamma_0). \tag{3.1}$$

The following proposition is a sharper version of [4, Proposition 4.41] on the local stability of $(P'(w))$.

Proposition 3.1. *Suppose that* $(A1)-(A4)$ *are fulfilled and* $z_0 \in \Phi'(w_0)$ *is a locally strong solution of* $(P'(w_0))$. *Then there exist positive numbers* k_0, β_0 *and* R_0 *such that, for all* $w \in B(w_0, \beta_0)$ *and any* $(\bar{x}_w, \bar{u}_w) \in S'_{R_0}(w)$, $(\bar{x}_w, \bar{u}_w)$ *is a locally optimal solution of* $(P'(w))$ *and*

$$S'_{R_0}(w) \subset z_0 + k_0 \|w - w_0\|_W^{1/2} \bar{B}_Z.$$

Proof. By using Lemma 3.1, one finds that Φ' has Aubin property. Since z_0 is a locally strongly optimal solution of $(P'(w_0))$, the conclusion follows from [16, Theorem 3.2] or [12, Theorem 4.4] immediately. □

Let

$$\mathscr{L}(x,u,v^*,e^*,w) = J(x,u,w) + v^* F(x,u,w) + e^* G(x,u,w),$$

be the Lagrangian function corresponding to $(P'(w))$, where $v^* \in E_0^*$ and $e^* \in E^*$. Recall that a couple $(v_w^*, e_w^*) \in E_0^* \times E^*$ is said to be multipliers of $(P'(w))$ at $(\bar{x}_w, \bar{u}_w)$ if the following conditions are fulfilled

$$\nabla_x \mathscr{L}(\bar{x}_w, \bar{u}_w, v_w^*, e_w^*, w) = 0, \nabla_u \mathscr{L}(\bar{x}_w, \bar{u}_w, v_w^*, e_w^*, w) = 0, \tag{3.2}$$

$$e_w^* \in N(K, G(\bar{x}_w, \bar{u}_w, w)). \tag{3.3}$$

We shall denote by $\Lambda'[(\bar{x}_w, \bar{u}_w)]$ the set of multipliers of $(P'(w))$ at $(\bar{x}_w, \bar{u}_w)$. The following result show that $\Lambda'[(\bar{x}_w, \bar{u}_w)]$ is uniformly bounded when w varies around w_0. Its proof can be found in [2, Theorem 2.4] and [25, Theorem 2.6].

Proposition 3.2. *Suppose that* $(A1)-(A4)$ *are fulfilled. Then there are numbers* $r_3'' > 0$ *and* $M > 0$ *such that if* $(\bar{x}_w, \bar{u}_w)$ *is a locally optimal solution of* $(P'(w))$ *corresponding to* $w \in B_W(w_0, r_3'')$, *then* $\Lambda'[(\bar{x}_w, \bar{u}_w)]$ *are non-empty and*

$$\|e_w^*\| + \|v_w^*\| \le M, \ \forall (e_w^*, v_w^*) \in \Lambda'[(\bar{x}_w, \bar{u}_w)]. \tag{3.4}$$

4. The proof of Theorem 2.1

The idea for proving Theorem 2.1 is to reduce $(P(w))$ into $(P'(w))$ and then use Proposition 3.1 and Proposition 3.2. Let us put

$$X = C([0,1], \mathbb{R}^n), \ U = L^2([0,1], \mathbb{R}^m), W = L^\infty([0,1], \mathbb{R}^k),$$
$$Z = X \times U, E_0 = X, E = U \times U, \ K = K_1 \times K_2,$$

and

$$K_1 = K_2 = \{v \in L^2([0,1], \mathbb{R}^m) \mid v(t) \le 0\}.$$

We denote by $\|\cdot\|_0$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ the norm of X, U and W, respectively. We also use the Sobolev space $W^{1,p}([0,1], \mathbb{R}^n)$, which consists of absolutely continuous functions x with $\dot{x} \in L^p([0,1], \mathbb{R}^n)$, where $1 \le p \le \infty$. Given $x \in W^{1,p}([0,1], \mathbb{R}^n)$, one has $\|x\|_{1,p} := \|x\|_0 + \|\dot{x}\|_p$.

Let us define mappings

$$F(x,u,w) = x - \int_0^{(\cdot)} f(s, x(s), u(s), w(s)) ds,$$
$$G(x,u,w) = (G_1, G_2) = (g(\cdot, x, w) + \varepsilon u, \varepsilon u)$$

and

$$D(w) = \{z = (x,u) \in X \times U \mid F(x,u,w) = 0\}.$$

Then, for each $w \in W$, $(P(w))$ can be formulated in the form of $(P'(w))$:

$$(P(w)) \qquad \begin{cases} I(x,u,w) \to \min \\ F(x,u,w) = 0, \\ G(x,u,w) \in K. \end{cases}$$

We shall divide the proof of Theorem 2.1 into some steps.

• Step 1. Verification of assumptions $(A1)-(A4)$.

By $(H2)$, $\varphi_3(\cdot)$ is locally Lipschitz continuous. Hence $(H3)$ implies that there exists $\varepsilon'' \in (0,\varepsilon)$ so that, for all $w \in W$ with $\|w-\bar{w}\|_\infty < \varepsilon''$, $\varphi_3(w(t)) \geq \frac{\gamma}{2}$ for a.e. $t \in [0,1]$. By taking $r_1 = r_1' = \varepsilon$ and $r_1'' = \varepsilon''$, we see that $(H1)$ and $(H2)$ imply $(A1)$ and $(A2)$. Besides, for each $w \in B_W(\bar{w}, r_1'')$ and $\bar{z}_w \in B_X(\bar{x}, r_1) \times B_U(\bar{u}, r_1')$, we have

$$\langle \nabla_z I(\bar{z}_w, w), (x,u)\rangle =$$

$$= \int_0^1 (\varphi_{1x}[t,w]x(t) + \varphi_{2x}[t,w]^T \bar{u}_w(t)x(t) + \varphi_2[t,w]u(t) + 2\varphi_3(w(t))\bar{u}_w(t)u(t))dt \tag{4.1}$$

$$\langle \nabla_z F(\bar{z}_w, w), (x,u)\rangle = x - \int_0^{(\cdot)} (f_{1x}[s,w]x(s) + f_{2x}[s,w]\bar{u}_w(s)x(s) + f_2[s,w]u(s))ds \tag{4.2}$$

$$\nabla_z G_1(\bar{z}_w, w) = (g_x[\cdot,w], \varepsilon I_U),\ \nabla_z G_2(\bar{z}, w) = (0, \varepsilon I_U), \tag{4.3}$$

where $I_U : U \to U$ is the identical mapping.

In order to verify $(A3)$, we take $v \in E_0$ and consider equation $F_x(\bar{z},\bar{w})x = v$. This equation is equivalent to

$$x = \int_0^{(\cdot)} f_{1x}[s]x(s)ds + v.$$

By $(H2)$, we have $|f_{1x}[\cdot]| \in L^\infty([0,1],\mathbb{R})$. By [10, Lemma 1, p. 51], the equation has a unique solution $x \in X$. Hence $(A3)$ is valid.

Let us check $(A4)$. Since $\nabla_x F(\bar{z},\bar{w})$ is bijective, $\nabla F(\bar{z},\bar{w})\colon X \times U \to E_0$ is surjective. This implies that

$$T(D(\bar{w}); \bar{z}) = \{(x,u) \in Z : F_x(\bar{z},\bar{w})x + F_u(\bar{z},\bar{w})u = 0\}.$$

Therefore, in order to verify $(A4)$, we need to show that, for each $(e_1,e_2) \in E$, there exists $(x,u) \in Z$ and $(v_1,v_2) \in \text{cone}(K - G(\bar{z},\bar{w}))$ such that

$$\begin{aligned} &F_x(\bar{z},\bar{w})x + F_u(\bar{z},\bar{w})u = 0, \\ &e_1 = G_{1x}(\bar{z},\bar{w})x + G_{1u}(\bar{z},\bar{w})u - v_1, \\ &e_2 = G_{2x}(\bar{z},\bar{w})x + G_{2u}(\bar{z},\bar{w})u - v_2 \end{aligned}$$

or equivalently,

$$x(t) = \int_0^t (f_{1x}[s]x(s) + f_2[s]u(s))ds, \tag{4.4}$$

$$e_1 = g_x[\cdot]x + \varepsilon u - v_1. \tag{4.5}$$

$$e_2 = \varepsilon u - v_2. \tag{4.6}$$

We represent $e_i = e_{i1} - e_{i2}$, where $e_{i1}, e_{i_2} \in K_1 = K_2$ and $i = 1,2$. We choose u_1 and u_2 such that $\varepsilon u_1 = e_{11}$ and $\varepsilon u_2 = e_{21}$. Put $u = u_1 + u_2$. Since $e_{11}(t), e_{21}(t) \le 0$ a.e., then $u(t) \le 0$ a.e.

Lemma 4.1. *There exists a solution x of equation* (4.4) *such that $x(t) \le 0$ for all $t \in [0,1]$.*

Proof. For each $j \ge 1$, there exists a unique solution x_j satisfying

$$x_j(t) = -\frac{1}{j} + \int_0^t (f_{1x}[s]x_j(s) + f_2[s]u(s))ds. \tag{4.7}$$

Since $x_j(0) < 0$, there exists a number $t_j > 0$ such that $x_j(t) < 0$ for all $t \in (0,t_j)$. Since $f_{1x}[s] \ge 0$ and $f_2[s] \ge 0$, we have

$$f_{1x}[s]x_j(s) + f_2[s]u(s) \le 0$$

for all $s \in (0,t_j)$. Hence

$$x_j(t_j) = -\frac{1}{j} + \int_0^{t_j} (f_{1x}[s]x_j(s) + f_2[s]u(s))ds < 0.$$

Hence, there exists $t'_j > t_j$ such that $x_j(t) < 0$ for all $t \in (0,t'_j)$. Let $(0,t^*_j)$ be the maximal interval so that $x_j(t) < 0$ for all $t \in (0,t^*_j)$. If $t^*_j < 1$, then $x_j(t^*_j) < 0$. Therefore, there exists $\hat{t}_j > t^*_j$ such that $x_j(t) < 0$ for all $t \in (0,\hat{t}_j)$, which contradicts the fact that $(0,t^*_j)$ is the maximal interval. Consequently, $t^*_j = 1$ and $x_j(t) < 0$ for all $t \in [0,1]$. We now have

$$\begin{aligned} |x_j(t)| &\le 1 + \int_0^t (|f_{1x}[s]||x_j(s)| + |f_2[s]||u(s)|)ds \\ &\le \|f_{1x}[\cdot]\|_\infty \int_0^t |x_j(s)|ds + 1 + \int_0^1 |f_2[s]||u(s)|)ds. \end{aligned}$$

The Gronwall inequality (see [6, 18.1.i, p. 503]) implies that

$$\begin{aligned} |x_j(t)| &\le \big(1 + \int_0^1 |f_2[s]||u(s)|ds\big)\big(1 + t\|f_{1x}\|_\infty \exp(\|f_{1x}[\cdot]\|_\infty t)\big) \\ &\le \big(1 + \int_0^1 |f_2[s]||u(s)|ds\big)(1 + \|f_{1x}[\cdot]\|_\infty \exp(\|f_{1,}[\cdot]\|_\infty)) := M_1 \end{aligned}$$

for all $t \in [0,1]$. Since

$$\dot{x}_j(t) = f_{1x}[t]x_j(t) + f_2[t]u(t),$$

we have

$$|\dot{x}_j(t)| \le |f_{1x}[t]|M_1 + |f_2[t]||u(t)|.$$

Hence

$$\int_0^1 |\dot{x}_j(t)|^2 dt \le 2\int_0^1 (|f_{1x}[t]|^2 M_1^2 + |f_2[t]|^2|u(t)|^2)dt := M_2.$$

Thus $\{x_j\}$ is bounded in $W^{1,2}([0,1],\mathbb{R}^n)$. Hence we can assume that x_j converges weakly to a function $x \in W^{1,2}([0,1],\mathbb{R}^n)$. Since the embedding $W([0,1],\mathbb{R}^m) \hookrightarrow C([0,1],\mathbb{R}^n)$ is compact (see [5, Theorem 8.8]), $x_j \to x$ strongly in $C([0,1],\mathbb{R}^n)$, that is, $x_j \to x$ uniformly on $[0,1]$. By letting $j \to \infty$, we obtain from (4.7) that

$$x(t) = \int_0^t (f_{1x}[s]x(s) + f_2[s]u(s))ds, \ \forall t \in [0,1].$$

The lemma is proved. □

By $(H3)$, one has $g_x[t] \geq 0$. By Lemma 4.1, one has $x(t) \leq 0$. Hence $g_x[t]x(t) \leq 0$ for a.e. $t \in [0,1]$. It follows that

$$g_x[\cdot]x(\cdot) + \varepsilon u - e_{11} \in K_1,$$

and

$$\bar{u} + \varepsilon u - e_{21} \in K_2.$$

Define

$$d_1 = g[\cdot] + \varepsilon\bar{u} + g_x[\cdot]x + \varepsilon u - e_1,$$

and

$$d_2 = \varepsilon\bar{u} + \varepsilon u - e_2.$$

Then $d_1 \in K_1$ and $d_2 \in K_2$. This means that

$$e_1 = g_x[\cdot]y_u + \varepsilon u - (d_1 - g[\cdot] - \varepsilon\bar{u}),$$

and

$$e_2 = \varepsilon u - (d_2 - \varepsilon\bar{u}),$$

where (y_u, u) satisfies equation (4.4). By putting $v_1 = d_1 - g[\cdot] - \varepsilon\bar{u}$ and $v_2 = d_2 - \varepsilon\bar{u}$, we see that (4.4), (4.5) and (4.6) are valid. Hence $(A4)$ is satisfied.

• Step 2. Show that $(\bar{x}, \bar{u})$ is a locally strongly optimal solution of $(P(\bar{w}))$.

Lemma 4.2. *Under assumptions of Theorem 2.1, there exist numbers $\alpha_0 > 0$ and $\beta_0 > 0$ such that*

$$I(x,\bar{w}) \geq I(\bar{z},\bar{w}) + \alpha_0\|u-\bar{u}\|_2^2, \quad \forall z \in B_Z(\bar{z},\beta_0) \cap \Phi(\bar{w}).$$

In particular, $\bar{z} = (\bar{x},\bar{u})$ is a locally optimal solution of $(P(\bar{w}))$.

Proof. Suppose the conclusion of the lemma is false. Then we could find sequences $\{(x_k,u_k)\} \subset \Phi(\bar{w})$ and $\{c_k\}$ such that $(x_k,u_k) \to (\bar{x},\bar{u})$, $c_k \to 0^+$ and

$$I(x_k,u_k,\bar{w}) < I(\bar{x},\bar{u},\bar{w}) + c_k\|u_k-\bar{u}\|_2^2. \tag{4.8}$$

Define $t_k = \|u_k - \bar{u}\|_2$, $\hat{x}_k = \frac{x_k-\bar{x}}{t_k}$ and $\hat{u}_k = \frac{u_k-\bar{u}}{t_k}$. Then $t_k \to 0^+$, $\|\hat{u}_k\|_2 = 1$ and

$$I(x_k,u_k,\bar{w}) < I(\bar{x},\bar{u},\bar{w}) + 0(t_k^2). \tag{4.9}$$

Since $L^2([0,1],\mathbb{R}^m)$ is reflexive, we can assume that $\hat{u}_k \rightharpoonup \hat{u}$ as $k \to \infty$. We now claim that $\hat{x}_k$ converges uniformly to some $\hat{x}$ in $C([0,1],\mathbb{R}^n)$. Indeed, since $(x_k,u_k) \in \Phi(\bar{w})$, we have

$$x_k(t) = x_0 + \int_0^t f(s,x_k(s),u_k(s),\bar{w}(s))ds.$$

Since $x_k = \bar{x} + t_k\hat{x}_k$, we have

$$t_k\hat{x}_k(t) = \int_0^t (f(s,x_k(s),u_k(s),\bar{w}(s)) - f(s,\bar{x}(s),\bar{u}(s),\bar{w}(s))ds. \tag{4.10}$$

Since $x_k \to \bar{x}$ uniformly and $u_k \to \bar{u}$ in $L^2([0,1],\mathbb{R}^l)$, there exists a constant $\rho > 0$ such that $\|x_k\|_0 \leq \rho, \|u_k\|_2 \leq \rho$. By assumption $(H2)$, there exists $k_{f,\rho} > 0$ such that

$$|f(s,x_k(s),u_k(s),\bar{w}(s)) - f(s,\bar{x}(s),\bar{u}(s),\bar{w}(s))| \leq k_{f,\rho}(|x_k(s)-\bar{x}(s)| + |u_k(s)-\bar{u}(s)|)$$

for a.e. $s \in [0,1]$. Hence, we have from (4.10) that

$$|\hat{x}_k(t)| \leq \int_0^t k_{f,\rho}(|\hat{x}_k(s)| + |\hat{u}_k(s)|)ds$$

and

$$|\dot{\hat{x}}_k(t)| \le k_{f,\rho}(|\hat{x}_k(t)| + |\hat{u}_k(t)|). \tag{4.11}$$

It follows that

$$\begin{aligned}|\hat{x}_k(t)| &\le \int_0^t k_{f,\rho}|\hat{x}_k(s)|ds + \int_0^1 k_{f,\rho}|\hat{u}_k(s)|ds \\ &\le \int_0^t k_{f,\rho}|\hat{x}_k(s)|ds + k_{f,\rho}\left(\int_0^1 |\hat{u}_k(s)|^2 ds\right)^{1/2} \\ &\le \int_0^t k_{f,\rho}|\hat{x}_k(s)|ds + k_{f,\rho}, \qquad (\|\hat{u}_k\|_2 = 1).\end{aligned}$$

By the Gronwall Inequality (see [6, 18.1.i, p. 503]), we have

$$|\hat{x}_k(t)| \le k_{\varphi,\rho}\exp(k_{f,\rho}).$$

From this and (4.11), we see that

$$\begin{aligned}|\dot{\hat{x}}_k(t)|^2 &\le 2k_{f,\rho}^2\left(|\hat{x}_k(t)|^2 + |\hat{u}_k(t)|^2\right) \\ &\le 2k_{f,\rho}^2(k_{f,\rho}^2\exp(2k_{f,\rho}) + |\hat{u}_k|^2).\end{aligned}$$

Hence

$$\int_0^1 |\dot{\hat{x}}_k(t)|^2 dt \le 2k_{f,\rho}^2(k_{f,\rho}^2\exp(2k_{f,\rho}) + 1).$$

Consequently, $\{\hat{x}_k\}$ is bounded in $W^{1,2}([0,1],\mathbb{R}^n)$. By passing subsequence, we may assume that $\hat{x}_k \rightharpoonup \hat{x}$ weakly in $W^{1,2}([0,1],\mathbb{R}^n)$. Since the embedding $W^{1,2}([0,1],\mathbb{R}^n) \hookrightarrow C([0,1],\mathbb{R}^n)$ is compact, we have that $\hat{x}_k \to \hat{x}$ uniformly on $[0,1]$. The claim is justified.

For convenience, we write $I(\cdot), F(\cdot), G_1(\cdot)$ and $G_2(\cdot)$ for $I(\cdot,\bar{w}), F(\cdot,\bar{w}), G_1(\cdot,\bar{w})$ and $G_2(\cdot,\bar{w})$, respectively. Define the Lagrangian $\mathscr{L}(z,\bar{p},\bar{\theta},\bar{\vartheta})$ associated with $(P(\bar{w}))$ by setting

$$\mathscr{L}(z,\bar{p},\bar{\theta},\bar{\vartheta}) = I(z) + \langle \bar{p}, F(z)\rangle + \langle \bar{\theta}, G_1(z)\rangle + \langle \bar{\vartheta}, G_2(z)\rangle,$$

where

$$\begin{aligned}I(z) &= \int_0^1 \varphi(t,x(t),u(t),\bar{w}(t))dt, \\ \langle \bar{p}, F(z)\rangle &= \int_0^1 \dot{\bar{p}}(s)^T x(s)ds + \int_0^1 \bar{p}(s)^T f(s,x(s),u(s),\bar{w}(s))ds, \\ \langle \bar{\theta}, G_1(z,\bar{w})\rangle &= \int_0^1 (\bar{\theta}(s)g(s,x(s)) + \varepsilon u(s))ds, \\ \langle \bar{\vartheta}, G_2(z,\bar{w})\rangle &= \int_0^1 \varepsilon\bar{\vartheta}u(s))ds.\end{aligned}$$

Since $(\bar{p},\bar{\theta},\bar{\vartheta})$ satisfies condition (i) and (ii) of Defintion 2.2 w.r.t. $\bar{w}$, we have

$$\nabla_z\mathscr{L}(\bar{z},\bar{p},\bar{\theta},\bar{\vartheta}) = 0. \tag{4.12}$$

Also, the condition (iii) in Definition 2.2 implies that

$$\bar{\theta} \in N(K_1; G_1(\bar{z})), \bar{\vartheta} \in N(K_2; G_2(\bar{z})), \tag{4.13}$$

where $N(K_i; v)$ denotes the normal cone to K_i at v in $L^2([0,1],\mathbb{R}^m)$.

By a Taylor expansion, we have from (4.8) that

$$I_x(\bar{z})\hat{x}_k + I_u(\bar{z})\hat{u}_k + \frac{o(t_k)}{t_k} \le 0. \tag{4.14}$$

Note that $L_u[\cdot] \in L^\infty([0,1],\mathbb{R}^m)$ and $I_u(\bar{z})\colon L^2([0,1],\mathbb{R}^m) \to \mathbb{R}$ is a continuous linear mapping, where

$$\langle I_u(\bar{z}), u\rangle := \int_0^1 L_u[s]u(s)ds, \quad \forall u \in L^2([0,1],\mathbb{R}^m).$$

By [5, Theorem 3.10], $I_u(\bar{z})$ is weakly continuous on $L^2([0,1],\mathbb{R}^m)$. By letting $k \to \infty$ in (4.14), we get

$$I_x(\bar{z})\hat{x} + I_u(\bar{z})\hat{u} \le 0. \tag{4.15}$$

Since $F(\bar{z}) = 0, F(x_k,u_k) = 0$ and by a Taylor expansion, we have

$$F_x(\bar{z})\hat{x}_k + F_u(\bar{z})\hat{u}_k + \frac{o(t_k)}{t_k} = 0.$$

Using the same arguments as the above and letting $k \to \infty$, we obtain

$$F_x(\bar{z})\hat{x} + F_u(\bar{z})\hat{u} = 0.$$

This means

$$\hat{x}(t) = \int_0^t (f_{1x}[s]\hat{x}(s) + f_2[s]\hat{u}(s))ds. \tag{4.16}$$

Since $G(x_k,u_k) - G(\bar{x},\bar{u}) \in K - G(\bar{x},\bar{u})$ and by a Taylor expansion, we have

$$G_x(\bar{z})\hat{x}_k + G_u(\bar{z})\hat{u}_k + \frac{o(t_k)}{t_k} \in \text{cone}(K - G(\bar{x},\bar{u})) \subset T(K;G(\bar{x},\bar{u})), \tag{4.17}$$

where $T(K;G(\bar{x},\bar{u}))$ is the tangent cone to K at $G(\bar{x},\bar{u})$ in $L^2([0,1],\mathbb{R})$. It is easily seen that

$$T(K;G(\bar{x},\bar{u})) = T(K_1,G_1(\bar{z})) \times T(K_2,G_2(\bar{z})),$$

and

$$\begin{aligned} T(K_1,G_1(\bar{z})) &= \left\{v \in L^2([0,1],\mathbb{R}^m) : v(t) \in T((-\infty,0]^m; g[t] + \varepsilon\bar{u}(t)) \quad \text{a.e.}\right\} \\ &= \left\{v \in L^2([0,1],\mathbb{R}^m) : v(t) \in T_1(\bar{w}) \text{ a.e. }\right\}, \\ T(K_2,G_2(\bar{z})) &= \left\{v \in L^2([0,1],\mathbb{R}^m) : v(t) \in T((-\infty,0]^m; \varepsilon\bar{u}(t)) \quad \text{a.e.}\right\} \\ &= \{v \in L^2([0,1],\mathbb{R}^m) : v(t) \in T_2(\bar{w}) \text{ a.e.}\}. \end{aligned}$$

Here

$$T_1(\bar{w}) = \{t \in [0,1] : g[t] + \varepsilon\bar{u}(t) = 0\},\ T_2(\bar{w}) = \{t \in [0,1] : \varepsilon\bar{u}(t) = 0\}.$$

By passing the limit in (4.17) when $k \to \infty$, we obtain

$$G_x(\bar{z})\hat{x} + G_u(\bar{u})\hat{u} \in T(K_1,G_1(\bar{z})) \times T(K_2,G_2(\bar{z})).$$

It follows that

$$g_x[t]\hat{x}(t) + \varepsilon\hat{u}(t) \in T_1(\bar{w}),$$

and

$$\varepsilon\hat{u}(t) \in T_2(\bar{w}).$$

Combining this with (4.15) and (4.16), we get $(\hat{x},\hat{u}) \in \mathscr{C}(\bar{z})$. We now prove that $(\hat{x},\hat{u}) = (0,0)$. By a second-order Taylor expansion for $\mathscr{L}$ and (4.12), we get

$$\mathscr{L}(z_k,\bar{p},\bar{\theta},\bar{\vartheta}) - \mathscr{L}(\bar{z},\bar{p},\bar{\theta},\bar{\vartheta}) = \frac{t_k^2}{2}\nabla^2_{zz}\mathscr{L}(\bar{z},\bar{p},\bar{\theta},\bar{\vartheta})(\hat{z}_k,\hat{z}_k) + o(t_k^2), \quad (\hat{z}_k = (\hat{x}_k,\hat{u}_k)).$$

On the other hand, from (4.9) and (4.13), we have

$$\begin{aligned}
&\mathscr{L}(z_k,\bar{p},\bar{\theta},\bar{\vartheta}) - \mathscr{L}(\bar{z},\bar{p},\bar{\theta},\bar{\vartheta})\\
&= I(z_k) - I(\bar{z}) + \langle \bar{\theta}, G_1(z_k) - G_1(\bar{z})\rangle + \langle \bar{\vartheta}, G_2(z_k) - G_2(\bar{z})\rangle\\
&\le o(t_k^2)
\end{aligned}$$

due to the fact that $F(z_k) = F(\bar{z}) = 0$. Therefore,

$$\frac{t_k^2}{2}\nabla^2_{zz}\mathscr{L}(\bar{z},\bar{p},\bar{\theta},\bar{\vartheta})(\hat{z}_k,\hat{z}_k) + o(t_k^2) \le o(t_k^2),$$

or, equivalently,

$$\nabla^2_{zz}\mathscr{L}(\bar{z},\bar{p},\bar{\theta},\bar{\vartheta})(\hat{z}_k,\hat{z}_k) \le \frac{o(t_k^2)}{t_k^2}. \tag{4.18}$$

By letting $k \to \infty$, we obtain $\nabla^2_{zz}\mathscr{L}(\bar{z},\bar{p},\bar{\theta},\bar{\vartheta})(\hat{z},\hat{z}) \le 0$. By a simple calculation, we have

$$\begin{aligned}
&\int_0^1 \big(\varphi_{1xx}[t] + \varphi_{2xx}[t]\bar{u}(t))\hat{x}(t)^2 + 2\varphi_{2x}[t]\hat{x}(t)\hat{u}(t) + 2\varphi_3(\bar{w}(t))\hat{u}^2(t)\big)\,dt\\
&\quad + \int_0^1 (\bar{p}(t)f_{1xx}[t] + \bar{\theta}(t)g_{xx}[t])x(t)^2 dt\\
&= \nabla^2_{zz}\mathscr{L}(\bar{z},\lambda,\bar{p},\bar{\theta},\bar{\vartheta})(\hat{z},\hat{z})\\
&\le 0.
\end{aligned}$$

Combining this with (2.8), we must have $(\hat{x},\hat{u}) = (0,0)$. However, from (4.18) and $(H3)$, we have

$$\begin{aligned}
\frac{o(t_k^2)}{t_k^2} &\ge \nabla^2_{zz}\mathscr{L}(\bar{z},\bar{p},\bar{\theta},\bar{\vartheta})(\hat{z}_k,\hat{z}_k)\\
&= \int_0^1 \big(\varphi_{1xx}[t] + \varphi_{2xx}[t]\bar{u}(t))\hat{x}_k(t)^2 + 2\varphi_{2x}[t]\hat{x}_k(t)\hat{u}_k(t) + 2\varphi_3(\bar{w}(t))\hat{u}_k^2(t)\big)\,dt\\
&\quad + \int_0^1 (\bar{p}(t)f_{1xx}[t] + \bar{\theta}(t)g_{xx}[t])x_k(t)^2 dt\\
&\ge \int_0^1 (\varphi_{1xx}[t] + \varphi_{2xx}[t]\bar{u}(t))\hat{x}_k(t)^2 + 2\varphi_{2x}[t]\hat{x}_k(t)\hat{u}_k(t))dt + 2\gamma\|\hat{u}_k\|_2^2 +\\
&\quad + \int_0^1 (\bar{p}(t)f_{1xx}[t] + \bar{\theta}(t)g_{xx}[t])x_k(t)^2 dt.
\end{aligned}$$

By letting $k \to \infty$ and using the fact that $\hat{x}_k \to 0$, $\hat{u}_k \rightharpoonup 0$ together with $\|\hat{u}_k\|_2 = 1$, we obtain $0 \ge 2\gamma$, which is impossible. The proof of the lemma is complete. □

• Step 3. Prove upper Hölder continuity of solution sets and their regularity.

Let $F(x,u) := F(x,u,\bar{w})$. Since $F(\bar{x},\bar{u}) = 0$ and $F_x(\bar{x},\bar{u})$ is bijective, the implicit function theorem implies that there are numbers $\gamma_1 > 0$, $\gamma_2 > 0$ and a mapping $\zeta : B_U(\bar{u},\gamma_1) \to B_X(\bar{x},\gamma_2)$

of class C^1 such that $F(\zeta(u),u)=0$ for all $u\in B_U(\bar u,\gamma_1)$ and $\zeta(\bar u)=\bar x$. We can choose $\beta_1<\beta_0$, which is given in Lemma 4.2. Particularly, ζ is Lipschitz on $B_U(\bar u,\gamma_1)$. Hence there exists a number $k_\zeta>0$ such that $\|\zeta(u)-\bar x\|_0\le k_\zeta\|u-\bar u\|_2$ for all $u\in B_U(\bar u,\gamma_1)$. We now take any $(x,u)\in\Phi(\bar w)\cap(B_X(\bar x,\gamma_2)\times B_U(\bar u,\gamma_1))$. Then $x=\zeta(u)$ and $\|x-\bar x\|_0\le k_\zeta\|u-\bar u\|_2$. From this and Lemma 4.2, we have

$$\begin{aligned} I(x,\bar w)&\ge I(\bar z,\bar w)+\alpha_0\|u-\bar u\|_2^2\\ &=I(\bar z,\bar w)+\frac{\alpha_0}{2k_\zeta^2}(k_\zeta^2\|u-\bar u\|_2^2+k_\zeta^2\|u-\bar u\|_2^2)\\ &\ge I(\bar z,\bar w)+\frac{\alpha_0}{2k_\zeta^2}(k_\zeta^2\|u-\bar u\|_2^2+\|x-\bar x\|_0^2)\\ &\ge I(\bar z,\bar w)+\alpha_1(\|u-\bar u\|_2^2+\|x-\bar x\|_0^2)\end{aligned}$$

for some constant $\alpha_1>0$. Hence $(\bar x,\bar u)$ is a locally strongly optimal solution of $(P(\bar w))$. Thus we have shown that $(P(w))$ satisfies all conditions of Proposition 3.1. According to Proposition 3.1, there exist numbers $R_*>0, l_*>0$ and $\varepsilon_*>0$ such that

$$S_{R_*}(w)\subset(\bar x,\bar u)+l^*\|w-\bar w\|_\infty^{1/2}\bar B_Z\ \forall w\in B_W(\bar w,\varepsilon_*)$$

which implies (2.9). Moreover, for any $(\hat x_w,\hat u_w)\in S_{R_*}(w)$ with $w\in B_W(\bar w,\varepsilon_*)$, $(\hat x_w,\hat u_w)$ is a locally optimal solution of $(P(w))$. Here we can choose $\varepsilon_*>0$ small enough such that the conclusions Proposition 3.2 are fulfilled. Hence, for each $w\in B_W(\bar w,\varepsilon_*)$ and for any $(\hat x_w,\hat u_w)\in S_{R_*}(w)$, $\Lambda[(\hat x_w,\hat u_w)]$ is nonempty and uniformly bounded, that is, there exit $v_w^*\in E_0^*$ and $(\theta_w,\vartheta_w)\in E^*=L^2([0,1],\mathbb{R}^m)^2$ such that the following conditions are valid:

$$I_x(\hat z_w,w)+v_w^*F_x(\hat z_w,w)+\theta_wG_{1x}(\hat z_w,w)+\vartheta_wG_{2x}(\hat z_w,w)=0, \tag{4.19}$$
$$I_u(\hat z_w,w)+v_w^*F_u(\hat z_w,w)+\theta_wG_{1u}(\hat z_w,w)+\vartheta_wG_{2u}(\hat z_w,w)=0, \tag{4.20}$$
$$\theta_w\in N(K_1,G_1(\hat z_w,w))=N(K_1,g(\cdot,\hat x_w,w)+\varepsilon\hat u_w), \tag{4.21}$$
$$\vartheta_w\in N(K_2,G_2(\hat z_w,w))=N(K_2,\varepsilon\hat u_w), \tag{4.22}$$

and

$$\|v_w^*\|+\|\theta_w\|_2+\|\vartheta_w\|_2\le M_0 \tag{4.23}$$

for some absolute constant $M_0>0$. Here v_w^* is a signed Radon measure. By Riesz's Representation (see [10, Chapter 01, p. 19] and [9, Theorem 3.8, p. 73]), there exists a vector function of bounded variation ν, which is continuous from the right and vanish at zero such that

$$\langle v_w^*,y\rangle=\int_0^1 y(t)d\nu(t)\ \ \forall y\in E_0,$$

where $\int_0^1 y(t)d\nu(t)$ is the Riemann-Stieltjes integral. Besides, $\|v_w^*\|=V_0^1(\nu)$, which is the variation of ν. By [15, Lemma 2.4], we have

$$\begin{aligned}&N(K_i,G_i(\hat z_w,w))\\ &=\{v^*\in L^2([0,1],\mathbb{R}^m)\mid v^*(t)\in N((-\infty,0]^m,G_i(\hat z_w(t),w(t)))+u(t))\text{ a.e. }t\in[0,1]\}.\end{aligned}$$

Define $p_w\colon[0,1]\to\mathbb{R}^n$ by setting $p_w(t)=\nu(t)-\nu(1)$. Then $p_w(1)=0$. From (4.19)-(4.22) and using the similar arguments as the proof of [18, Theorem 2.2] (see also [3]), we can show that p_w is absolutely continuous and $(p_w,\theta_w,\vartheta_w)$ satisfies the following conditions $(i),(ii)$ and

(iii) of Definition 2.2. Since K_i are cones, (4.21) and (4.22) are equivalent to (iii) in Definition 2.2. Since p_w is absolute continuous on $[0,1]$, there is $t_w \in [0,1]$ such that

$$\begin{aligned}\|p_w\|_0 &= \max_{t\in[0,1]} |p_w(t)|\\ &= |p_w(t_w)|\\ &= |\nu(t_w) - \nu(1)|\\ &\le V_0^1(\nu).\end{aligned}$$

From this and (4.23), we have

$$\|p_w\|_0 + \|\theta_w\|_2 + \|\vartheta_w\|_2 \le M_0. \tag{4.24}$$

Let us show that $\hat{x}_w \in W^{1,\infty}([0,1],\mathbb{R}^n)$ and $\hat{u}_w \in L^\infty([0,1],\mathbb{R}^m)$. In fact, from (iii) in Definition 2.2, we have

$$(\varepsilon\theta_{wi}(t) + \varepsilon\vartheta_{wi}(t))(g_i(t,\hat{x}_w(t),w(t)) + \varepsilon\hat{u}_{wi}(t))(\varepsilon\hat{u}_i(t)) = 0 \text{ a.e. } t \in [0,1] \tag{4.25}$$

and so

$$(\varepsilon\theta_{wi}(t) + \varepsilon\vartheta_{wi}(t))^2(g_i(t,\hat{x}_w(t),w(t)) + \varepsilon\hat{u}_{wi}(t))(\varepsilon\hat{u}_i(t)) = 0 \text{ a.e. } t \in [0,1]. \tag{4.26}$$

We now consider two cases:
Case 1. $g_i(t,\hat{x}_w(t),w(t)) \ge 0$.
It is equivalent to

$$g_i(t,\hat{x}_w(t),w(t)) + \varepsilon\hat{u}_{wi}(t) \ge \varepsilon\hat{u}_{wi}(t).$$

If

$$g_i(t,\hat{x}_w(t),w(t)) + \varepsilon\hat{u}_{wi}(t) > \varepsilon\hat{u}_{wi}(t),$$

then

$$0 \ge g_i(t,\hat{x}_w(t),w(t)) + \varepsilon\hat{u}_{wi}(t) > \varepsilon\hat{u}_{wi}(t).$$

due to $(\hat{x}_w,\hat{u}_w) \in \Phi(w)$. From this and (4.25), we get

$$(\varepsilon\theta_{wi}(t) + \varepsilon\vartheta_{wi}(t))(g_i(t,\hat{x}_w(t),w(t)) + \varepsilon\hat{u}_{wi}(t)) = 0.$$

If

$$g_i(t,\hat{x}_w(t),w(t)) + \varepsilon\hat{u}_{wi}(t) = \varepsilon\hat{u}_{wi}(t)$$

then we have from (4.26) that

$$(\varepsilon\theta_{wi}(t) + \varepsilon\vartheta_{wi}(t))(g_i(t,\hat{x}_w(t),w(t)) + \varepsilon\hat{u}_{wi}(t)) = 0.$$

Consequently,

$$\varepsilon\theta_{iw}(t) + \varepsilon\vartheta_{wi}(t) \in N((-\infty,0], g_i(t,\hat{x}_w(t),w(t)) + \varepsilon\hat{u}_{wi}(t)).$$

Combining this with the condition (ii) of Definition 2.2 yields

$$-\varphi_{iu}[t,w] - f_{iu}[t,w]p_w(t) \in N((-\infty,0], g_i(t,\hat{x}_w(t),w(t)) + \varepsilon\hat{u}_{wi}(t))$$

or

$$-\varphi_{2i}[t,w] - 2\varphi_3(w(t))\hat{u}_{wi}(t) - f_{2i}[t,w]p_w(t) \in N((-\infty,0], g_i(t,\hat{x}_w(t),w(t)) + \varepsilon\hat{u}_{wi}(t)).$$

This implies that

$$-\frac{\varepsilon}{2\varphi_3(w(t))}\big(\varphi_{2i}[t,w] + f_{2i}[t,w]p_w(t)\big) - \varepsilon\hat{u}_{wi}(t) \in N((-\infty,0], g_i(t,\hat{x}_w(t),w(t)) + \varepsilon\hat{u}_{wi}(t)).$$

Note that $2\varphi_3(w(t)) > \gamma$. It follows that

$$-\frac{\varepsilon}{2\varphi_3(w(t))}\big(\varphi_{2i}[t,w] + f_{2i}[t,w]p_w(t)\big) + g_i(t,w) - g_i(t,w) - \varepsilon\hat{u}_{wi}(t)$$
$$\in N((-\infty,0], g_i(t,\hat{x}_w(t),w(t)) + \varepsilon\hat{u}_{iw}(t)).$$

Hence

$$g_i(t,w) + \varepsilon\hat{u}_{wi}(t) = P_{(-\infty,0]}\Big(-\frac{\varepsilon}{2\varphi_3(w(t))}\big(\varphi_{2i}[t,w] + f_{2i}[t,w]p_w(t)\big) + g_i(t,w)\Big), \tag{4.27}$$

where $P_{(-\infty,0]}(a)$ denotes the metric projection of a on $(-\infty,0]$ (see [5, Theorem 5.2]). Using non-expansive property of metric projections in Hilbert spaces (see, for instance [5, Proposition 5.3]) and the fact that $0 = P_{(-\infty,0]}(0)$, we have

$$\begin{aligned}\varepsilon|\hat{u}_{wi}(t)| &\le |g_i[t,w]| + \Big|\frac{\varepsilon}{2\varphi_3(w(t))}\big(\varphi_{2i}[t,w] + f_{2i}[t,w]p_w(t)\big) + g_i(t,w)\Big| \\ &\le 2|g_i[t,w]| + \frac{\varepsilon}{\gamma}\big(|\varphi_{2i}[t,w]| + |f_{2i}[t,w]|M_0\big).\end{aligned} \tag{4.28}$$

Case 2. $g_i(t,\hat{x}_w(t),w(t)) < 0$.

We have

$$g_i(t,\hat{x}_w(t),w(t)) + \varepsilon\hat{u}_{ui}(t) < \varepsilon\hat{u}_{ui}(t) \le 0.$$

It follows from (4.25) that

$$(\varepsilon\theta_{iw}(t) + \varepsilon\vartheta_{wi}(t))(\varepsilon\hat{u}_{wi}(t)) = 0.$$

Hence

$$\varepsilon\theta_{iw}(t) + \varepsilon\vartheta_{wi}(t)) \in N_{(-\infty,0]}(\varepsilon\hat{u}_{wi}(t)).$$

Combining this with (2.4) yields

$$-\varphi_{iu}[t,w] - f_{iu}[t,w]p_w(t) \in N_{(-\infty,0]}(\varepsilon\hat{u}_{wi}(t))$$

and so

$$-\varphi_{2i}[t,w] - 2\varphi_3(w(t))\hat{u}_{wi}(t) - f_{2i}[t,w]p_w(t) \in N_{(-\infty,0]}(\varepsilon\hat{u}_{wi}(t)).$$

This implies that

$$-\frac{\varepsilon}{2\varphi_3(w(t))}\big(\varphi_{2i}[t,w] + f_{2i}[t,w]p_w(t)\big) - \varepsilon\hat{u}_{wi}(t) \in N_{(-\infty,0]}(\varepsilon\hat{u}_{wi}(t)).$$

Hence

$$\varepsilon\hat{u}_{wi}(t) = P_{(-\infty,0]}\Big(-\frac{\varepsilon}{2\varphi_3(w(t))}\big(\varphi_{2i}[t,w] + f_{2i}[t,w]p_w(t)\big)\Big). \tag{4.29}$$

Using non-expansive property of metric projections, we get

$$\begin{aligned}\varepsilon|\hat{u}_{wi}(t)| &\le \Big|\frac{\varepsilon}{2\varphi_3(w(t))}\big(\varphi_{2i}[t,w] + f_{2i}[t,w]p_w(t)\Big| \\ &\le \frac{\varepsilon}{\gamma}(|\varphi_{2i}[t,w]| + |f_{2i}[t,w]|M_0).\end{aligned}$$

Combing this with (4.28) yields

$$\varepsilon|\hat{u}_{wi}(t)| \le 2|g_i[t,w]| + \frac{2\varepsilon}{\gamma}(|\varphi_{2i}[t,w]| + |f_{2i}[t,w]|M_0) \text{ a.e. } t \in [0,1]. \tag{4.30}$$

Since the terms of the right hand side are $L^\infty-$ functions, $\hat{u}_{wi} \in L^\infty([0,1],\mathbb{R})$. Since

$$\dot{x}_w(t) = f_1(t, \hat{x}_w(t), w(t)) + f_2(t, w(t))\hat{u}_w(t)$$

we have

$$|\dot{x}_w(t)| \le |f_1(t, \hat{x}_w(t), w(t))| + |f_2(t, w(t))||\hat{u}_w(t)|.$$

Because the function on the right hand side is of $L^\infty([0,1],\mathbb{R})$, one has $|\dot{x}_w| \in L^\infty([0,1],\mathbb{R})$. Hence $\hat{x}_w \in W^{1,\infty}([0,1],\mathbb{R}^n)$. To complete the proof of Theorem 2.1, we need the following lemma.

Lemma 4.3. *If $(H4)$ is valid, then there exist positive numbers l_1 and s_1 such that, for all $w \in B_W(\bar{w}, s_1)$ and $(\hat{x}_w, \hat{u}_w) \in S_{R_*}(w)$,*

$$\|\hat{x}_w - \bar{x}\|_{1,\infty} + \|\hat{u}_w - \bar{u}\|_\infty \le l_1 \|w - \bar{w}\|_\infty^{1/2}.$$

Proof. Let $w \in B_W(\bar{w}, \varepsilon_*)$ and $(\hat{x}_w, \hat{u}_w) \in S_{R_*}(w)$. By (2.9), we have

$$\|x_w - \bar{x}\|_0 \le l_* \|w - \bar{w}\|_\infty^{1/2} \le l_* \varepsilon_*^{1/2}. \tag{4.31}$$

It follows that

$$|\hat{x}_w(t)| \le |\hat{x}_w(t) - \bar{x}(t)| + \|\bar{x}\|_0 \le l_* \varepsilon_*^{1/2} + \|\bar{x}\|_0$$

and

$$|w(t)| \le |w(t) - \bar{w}(t)| + |\bar{w}(t)| \le \varepsilon_* + \|\bar{w}\|_\infty.$$

Putting

$$M = l_* \varepsilon_*^{1/2} + \|\bar{x}\|_0 + \varepsilon_* + \|\bar{w}\|_\infty, \tag{4.32}$$

we then have

$$|\hat{x}_w(t)| + |w(t)| \le M. \tag{4.33}$$

By $(H4)$, $0 < \inf_{t\in[0,1]} |g_i[t]| < +\infty$. Take a number σ such that $0 < \sigma < \inf_{t\in[0,1]} |g_i[t]|$ for all $i = 1, 2, ..., m$. Then, for all $t \in [0,1]$, $|g_i[t]| > \sigma$ for all $i = 1, 2, .., m$. This means $g_i[t] > \sigma$ or $g_i[t] < -\sigma$ for $i = 1, 2, .., m$. Using $(H2)$, we have for all $i = 1, 2, ..., m$ that

$$\begin{aligned} |g_i(t, \bar{x}(t), \bar{w}(t)) - g_i(t, \hat{x}_w(t), w(t)| &\le k_{g,M}(|\hat{x}_w(t) - \bar{x}(t)| + |w(t) - \bar{w}(t)|) \\ &\le k_{g,M}(l_* \|w - \bar{w}\|_\infty^{1/2} + \|w - \bar{w}\|_\infty) \\ &\le k_{g,M}(l_* \|w - \bar{w}\|_\infty^{1/2} + \varepsilon_*^{1/2} \|w - \bar{w}\|_\infty^{1/2}) \\ &\le k_{g,M}(l_* + \varepsilon_*^{1/2}) \|w - \bar{w}\|_\infty^{1/2} \\ &< \sigma, \end{aligned}$$

whenever $w \in B_W(\bar{w}, \varepsilon_1)$, where

$$\varepsilon_1 := \min \left\{ \frac{\sigma^2}{k_{g,M}^2 (l_* + \varepsilon_*^{1/2})^2}, \varepsilon_* \right\}.$$

Hence, if $g_i[t] > \sigma$, then

$$g_i(t, \hat{x}_w(t), w(t) \ge g_i(t, \bar{x}(t), \bar{w}(t)) - \sigma > 0. \tag{4.34}$$

If $g_i[t] < -\sigma$, then

$$g_i(t,\hat{x}_w(t),w(t)) \le g_i[t] + \sigma < 0. \tag{4.35}$$

Let us fix any $w \in B_W(\bar{w},\varepsilon_1)$ and $(\hat{x}_w,\hat{u}_w) \in S_{R_*}(w)$. Then $(\hat{y}_w,\hat{u}_w)$ is a locally optimal solution of $(P(w))$. Hence there is a triplet $(p_w,\theta_w,\vartheta_w) \in \Lambda[(\hat{x}_w,\hat{u}_w)]$, which satisfy the conditions $(i),(ii)$ and (iii) of Definition 2.2. Fixing any $i \in \{1,2,..,m\}$, we put

$$T_{1i} = \{t \in [0,1] : g_i[t] > \sigma\},\ T_{2i} = \{t \in [0,1] : g_i[t] < -\sigma\}. \tag{4.36}$$

Then $T_{1i} \cap T_{2i} = \emptyset$ and $[0,1] = T_{1i} \cup T_{2i}$. If $i \in T_{1i}$, then $g_i[t] > \sigma$ and $g_i(t,\hat{x}_w(t),\hat{u}_w(t)) > 0$ because of (4.34). Hence

$$0 \ge g_i[t] + \varepsilon\bar{u}_i(t) > \varepsilon\bar{u}_i(t),$$

and

$$0 \ge g_i(t,\hat{x}_w(t),\hat{u}_w(t),w(t)) + \varepsilon\hat{u}_{wi}(t) > \varepsilon\hat{u}_{wi}(t).$$

From this and (2.7), $\vartheta_i(t) = 0$ and $\vartheta_{wi}(t) = 0$. By (2.4), we have

$$\begin{aligned}\theta_{wi}(t) &= -\frac{1}{\varepsilon}(\varphi_{iu}[t,w] + f_{iu}[t,w]p_w(t) \\ &= -\frac{1}{\varepsilon}(\varphi_{2i}[t,w] + 2\varphi_3(w)\hat{u}_{wi}(t) + f_{2i}[t,w]p_w(t))\end{aligned}$$

and

$$\begin{aligned}\bar{\theta}_i(t) &= -\frac{1}{\varepsilon}(\varphi_{iu}[t] + f_{iu}[t]\bar{p}(t)) \\ &= -\frac{1}{\varepsilon}(\varphi_{2i}[t] + 2\varphi_3(\bar{w})\bar{u}_i(t) + f_{2i}[t]\bar{p}(t)).\end{aligned}$$

Inserting these into (2.3), we get

$$\begin{aligned}\dot{p}_{wi}(t) &= -\varphi_{ix}[t,w] - f_{ix}[t,w]p_w(t) + \frac{1}{\varepsilon}(\varphi_{2i}[t,w] + 2\varphi_3(w)\hat{u}_{wi}(t) + f_{2i}[t,w]p_w(t))g_{ix}[t,w] \\ &= -\varphi_{1ix}[t,w] - \varphi_{2ix}[t,w]\hat{u}_{wi}(t) - f_{ix}[t,w]p_w(t) \\ &\quad + \frac{1}{\varepsilon}(\varphi_{2i}[t,w] + 2\varphi_3(w)\hat{u}_{wi}(t) + f_{2i}[t,w]p_w(t))g_{ix}[t,w] \\ &= (\frac{1}{\varepsilon}f_{2i}[t,w]g_{ix}[t,w] - f_{ix}[t,w])p_w(t) \\ &\quad + \frac{1}{\varepsilon}\varphi_{2i}[t,w]g_{ix}[t,w] - \varphi_{1i}[t,w] + (2\varphi_3(w)(t)g_{ix}[t,w] - 2\varphi_{2i}[t,w])\hat{u}_{wi}(t)\end{aligned}$$

and

$$\begin{aligned}\dot{\bar{p}}_i(t) &= (\frac{1}{\varepsilon}f_{2i}[t]g_{ix}[t] - f_{ix}[t])\bar{p}(t) \\ &\quad + \frac{1}{\varepsilon}\varphi_{2i}[t]g_{ix}[t] - \varphi_{1i}[t] + (2\varphi_3(\bar{w})g_{ix}[t] - 2\varphi_{2i}[t])\bar{u}_i(t).\end{aligned}$$

This implies that

$$\begin{aligned}\dot{p}_{wi}(t)-\dot{\bar{p}}_i(t) &= (\frac{1}{\varepsilon}f_{2i}[t,w]g_{ix}[t,w]-f_{ix}[t,w])p_w(t)-(\frac{1}{\varepsilon}f_{2i}[t]g_{ix}[t]-f_{ix}[t])\bar{p}(t)\\ &\quad+\frac{1}{\varepsilon}\varphi_{2i}[t,w]g_{ix}[t,w]-\varphi_{1i}[t,w]+(2\varphi_3(w)g_{ix}[t,w]-2\varphi_{2i}[t,w])\hat{u}_{wi}(t)\\ &\quad-\frac{1}{\varepsilon}\varphi_{2i}[t]g_{ix}[t]+\varphi_{1i}[t]-(2\varphi_3(\bar{w})(t)g_{ix}[t]-2\varphi_{2i}[t])\bar{u}_i(t),\end{aligned}$$

which can rewrite in the form

$$\begin{aligned}\dot{p}_{wi}(t)-\dot{\bar{p}}_i(t) &= (\frac{1}{\varepsilon}f_{2i}[t]g_{ix}[t]-f_{ix}[t])(p_w(t)-\bar{p}(t))\\ &\quad+\big(\frac{1}{\varepsilon}f_{2i}[t,w]g_{ix}[t,w]-\frac{1}{\varepsilon}f_{2i}[t]g_{ix}[t]+f_{ix}[t]-f_{ix}[t,w]\big)p_w(t)\\ &\quad+\big(\frac{1}{\varepsilon}\varphi_{2i}[t,w]g_{ix}[t,w]-\frac{1}{\varepsilon}\varphi_{2i}[t]g_{ix}[t]-\varphi_{1i}[t,w]+\varphi_{1i}[t]\big)\\ &\quad+\big((2\varphi_3(w)g_{ix}[t,w]-2\varphi_3(\bar{w})g_{ix}[t]-2\varphi_{2i}[t,w]+2\varphi_{2i}[t]\big)\hat{u}_{wi}(t)\\ &\quad+(2\varphi_3(\bar{w})(t)g_{ix}[t]-2\varphi_{2i}[t])(\hat{u}_w(t)-\bar{u}_i(t))\\ &:= (\frac{1}{\varepsilon}f_{2i}[t]g_{ix}[t]-f_{ix}[t])(p_w(t)-\bar{p}(t))+H_{1i}(t).\end{aligned} \tag{4.37}$$

Similarly, if $t\in T_{2i}$, then $g_i[t]<-\sigma$ and $g_i(t,\hat{x}_w(t),\hat{u}_w(t))<0$. Hence

$$\begin{aligned}&g_i[t]+\varepsilon\bar{u}_i(t)<\varepsilon\bar{u}_i(t)\le 0,\\ &g_i(t,\hat{x}_w(t),\hat{u}_w(t),w(t))+\varepsilon\hat{u}_{wi}(t)<\varepsilon\hat{u}_{wi}(t)\le 0.\end{aligned}$$

These and (2.6) imply that $\theta_i(t)=0$ and $\theta_{wi}(t)=0$. By (2.4), we get

$$\begin{aligned}\dot{p}_{wi}(t) &= -\varphi_{ix}[t,w]-f_{ix}[t,w]p_w(t)\\ &= -\varphi_{1ix}[tw]-\varphi_{2ix}[t,w]\hat{u}_{wi}(t)-f_{ix}[t,w]p_w(t)\end{aligned}$$

and

$$\begin{aligned}\dot{\bar{p}}_i(t) &= -\varphi_{ix}[t]-f_{ix}[t]\bar{p}(t)\\ &= -\varphi_{1ix}[t]-\varphi_{2ix}[t]\bar{u}_i(t)-f_{ix}[t]\bar{p}(t)\end{aligned}$$

By taking difference, we have

$$\begin{aligned}\dot{p}_{wi}(t)-\dot{\bar{p}}_i(t) &= -\varphi_{1ix}[t,w]-\varphi_{2ix}[t,w]\hat{u}_{wi}(t)-f_{ix}[t,w]p_w(t)\\ &\quad+\varphi_{1ix}[t]+\varphi_{2ix}[t]\bar{u}_i(t)+f_{ix}[t]\bar{p}(t)\\ &= -f_{ix}[t](p_w(t)-\bar{p}(t))+(f_{ix}[t]-f_{ix}[t,w])p_w(t)+\\ &\quad+\varphi_{2ix}[t](\bar{u}_i(t)-\hat{u}_{wi}(t))+(\varphi_{2ix}[t]-\varphi_{2ix}[t,w])\hat{u}_{wi}(t)\\ &\quad+\varphi_{2ix}[t]-\varphi_{1ix}[t,w]\\ &:= -f_{ix}[t](p_w(t)-\bar{p}(t))+H_{2i}(t).\end{aligned} \tag{4.38}$$

We now define functions $h_i(t)$ and $H_i(t)$ which are given by

$$h_i(t)=\begin{cases}\frac{1}{\varepsilon}f_{2i}[t]g_{ix}[t]-f_{ix}[t], & \text{if } t\in T_{1i},\\ -f_{ix}[t], & \text{if } t\in T_{2i},\end{cases} \tag{4.39}$$

and

$$H_i(t) = \begin{cases} H_{1i}(t), & \text{if } t \in T_{1i}, \\ H_{2i}(t), & \text{if } t \in T_{2i}. \end{cases}$$

Then, form (4.37) and (4.38), we have

$$\begin{cases} \dot{p}_{wi}(t) - \dot{\bar{p}}_i(t) = h_i(t)(p_w(t) - \bar{p}(t)) + H_i(t) \text{ a.e. } t \in [0,1] \\ \dot{p}_{wi}(1) - \dot{\bar{p}}_i(1) = 0. \end{cases}$$

Define $q_w(t) = p_w(1-t)$ and $\bar{q}(t) = \bar{p}(1-t)$. From the above, we have

$$q_{wi}(t) - \bar{q}_i(t) = \int_0^t h_i(1-s)(q_w(s) - \bar{q}(s))ds + \int_0^t H_i(1-s)ds.$$

It follows that

$$|q_{wi}(t) - \bar{q}_i(t)| \le \|h_i\|_\infty \int_0^t |q_w(s) - \bar{q}(s)| + \int_0^1 |H_i(1-s)|ds. \tag{4.40}$$

From the definition of h_i, we have

$$\|h_i\|_\infty \le \frac{1}{\varepsilon}\|f_{2i}[\cdot]g_{ix}[\cdot]\|_\infty + 2\|f_{ix}[\cdot]\|_\infty \le \frac{1}{\varepsilon}\|f_2[\cdot]g_x[\cdot]\|_\infty + 2\|f_x[\cdot]\|_\infty := M_1. \tag{4.41}$$

Also,

$$\begin{aligned} \int_0^1 |H_i(1-s)|ds &= \int_{1-T_{1i}} |H_{1i}(1-s)|ds + \int_{1-T_{2i}} |H_{2i}(1-s)|ds \\ &\le \int_0^1 |H_{1i}(t)|dt + \int_0^1 |H_{2i}(t)|dt. \end{aligned} \tag{4.42}$$

From definition of H_{1i} in (4.37), $(H3)$, (2.9) and (4.24), we have

$$\begin{aligned} |H_{1i}(t)| \le\ & \frac{C_1}{\varepsilon} k_{M,f} k_{M,g}(|\hat{x}_w(t) - \bar{x}(t)| + |w(t) - \bar{w}(t)|)M_0 \\ & + \frac{C_2}{\varepsilon} k_{M,\varphi} k_{M,g}(|\hat{x}_w(t) - \bar{x}(t)| + |w(t) - \bar{w}(t)|) \\ & + C_3 k_{M,\varphi_3} k_{M,g} k_{M,\varphi_2}(|\hat{x}_w(t) - \bar{x}(t)| + |w(t) - \bar{w}(t)|)|\hat{u}_w(t)| \\ & + C_4|\hat{u}_w(t) - \bar{u}(t)| \\ \le\ & C_5\|w - \bar{w}\|_\infty^{1/2} + C_6\|w - \bar{w}\|_\infty^{1/2}|\hat{u}_w(t)| + C_4|\hat{u}_w(t) - \bar{u}(t)|. \end{aligned}$$

Since $\|\hat{u}_w - \bar{u}\|_2 \le l_*\|w - \bar{w}\|_\infty^{1/2}$, we have

$$\int_0^1 |H_{1i}(t)|dt \le C_7\|w - \bar{w}\|_\infty^{1/2} \tag{4.43}$$

for some constant $C_7 > 0$, which is independent of w. By the same argument, we obtain from definition of H_{2i} in (4.38) that

$$\int_0^1 |H_{2i}(s)|ds \le C_8\|w - \bar{w}\|_\infty^{1/2}.$$

Combining this with (4.42) and (4.43) yields

$$\int_0^1 |H(1-s)|ds \le C_9\|w - \bar{w}\|_\infty^{1/2}.$$

From this and (4.40), we get

$$|q_{wi}(t)-\bar{q}_i(t)| \le M_1 \int_0^t |q_w(s)-\bar{q}(s)| + C_9\|w-\bar{w}\|_\infty^{1/2}.$$

Summing on i, we have

$$|q_w(t)-\bar{q}(t)| \le nM_1 \int_0^t |q_w(s)-\bar{q}(s)| + nC_9\|w-\bar{w}\|_\infty^{1/2}.$$

The Gronwall inequality implies that

$$|q_w(t)-\bar{q}(t)| \le nC_9\|w-\bar{w}\|_\infty^{1/2}(1+M_1\exp(M_1)).$$

Hence, there exists a constant $C_{10}>0$ such that

$$|p_w(t)-\bar{p}(t)| \le C_{10}\|w-\bar{w}\|_\infty^{1/2}, \quad \forall t\in[0,1]. \tag{4.44}$$

Let us consider the following possibilities.

(i) The case $t\in T_{1i}$.

In this case, we have $g_i[t]>\sigma$ and $g_i(t,\hat{x}_w(t),w(t))>0$. By the same argument as in *Case 1*, we have from (4.27) that

$$\varepsilon\hat{u}_{wi}(t) = -g_i[t,w] + P_{(-\infty,0]}\Big(-\frac{\varepsilon}{2\varphi_3(w(t))}\big(\varphi_{2i}[t,w]+f_{2i}[t,w]p_w(t)\big)+g_i(t,w)\Big),$$

and

$$\varepsilon\bar{u}_i(t) = -g_i[t] + P_{(-\infty,0]}\Big(-\frac{\varepsilon}{2\varphi_3(\bar{w}(t))}\big(\varphi_{2i}[t]+f_{2i}[t]\bar{p}(t)\big)+g_i[t]\Big).$$

Using the non-expansive property of metric projections, $(H2)$ and (4.44), we have

$$\begin{aligned}
&\varepsilon|\hat{u}_{wi}(t)-\bar{u}_i(t)| \le 2|g_i[t,w]-g_i[t]| \\
&+\Big|-\frac{\varepsilon}{2\varphi_3(w(t))}\big(\varphi_{2i}[t,w]+f_{2i}[t,w]p_w(t)\big)+\frac{\varepsilon}{2\varphi_3(\bar{w}(t))}\big(\varphi_{2i}[t]+f_{2i}[t]\bar{p}(t)\big)\Big| \\
&\le 2k_{M,g}(|\hat{x}_w(t)-\bar{x}(t)|+|w(t)-\bar{w}(t)|) \\
&+\frac{\varepsilon}{2\varphi_3(w(t))\varphi_3(\bar{w}(t))}\big|\varphi_3(\bar{w}(t))(\varphi_{2i}[t,w]+f_{2i}[t,w]p_w(t)-\varphi_{2i}[t]-f_{2i}[t]\bar{p}(t))\big| \\
&+\frac{\varepsilon}{2\varphi_3(w(t))\varphi_3(\bar{w}(t))}\big|\big(\varphi_{2i}[t]+f_{2i}[t]\bar{p}(t)\big)(\varphi_3(w(t))-\varphi_3(\bar{w}(t))\big| \\
&\le 2k_{M,g}(l_*\|w-\bar{w}\|_\infty^{1/2}+\|w-\bar{w}\|_\infty) \\
&+\frac{\varepsilon}{\gamma^2}\|\varphi_3[\bar{w}]\|_\infty\big(k_{M,f_2}|w(t)-\bar{w}(t)|M_0+\|f_{2i}[\cdot]\|_\infty|p_w(t)-\bar{p}(t)|+k_{M,\varphi_2}|w(t)-\bar{w}(t)|\big) \\
&+\frac{\varepsilon}{\gamma^2}|\|\varphi_{2i}[\cdot]+f_{2i}[\cdot]\bar{p}(\cdot)\|_\infty|w(t)-\bar{w}(t)|.
\end{aligned}$$

Hence

$$\varepsilon|\hat{u}_{wi}(t)-\bar{u}_i(t)| \le C_{11}\|w-\bar{w}\|_\infty^{1/2} \tag{4.45}$$

for some constant $C_{11}>0$.

(ii) The case $t\in T_{2i}$.

In this case, $g_i[t] < -\sigma$ and $g_i(t,\hat{x}_w(t),w(t)) < 0$. By the same argument as in *Case 2*, we have

$$\varepsilon\hat{u}_{wi}(t) = P_{(-\infty,0]}\big(-\frac{\varepsilon}{2\varphi_3(w(t))}\big(\varphi_{2i}[t,w] + f_{2i}[t,w]p_w(t)\big)\big).$$

and

$$\varepsilon\bar{u}_i(t) = P_{(-\infty,0]}\big(-\frac{\varepsilon}{2\varphi_3(\bar{w}(t))}\big(\varphi_{2i}[t] + f_{2i}[t]\bar{p}(t)\big)\big).$$

It follows that

$$\begin{aligned}
&\varepsilon|\hat{u}_{wi}(t) - \bar{u}_i(t)| \\
&\leq \big|\frac{\varepsilon}{2\varphi_3(w(t))}\big(\varphi_{2i}[t,w] + f_{2i}[t,w]p_w(t)\big) - \frac{\varepsilon}{2\varphi_3(\bar{w}(t))}\big(\varphi_{2i}[t] + f_{2i}[t]\bar{p}(t)\big)\big| \\
&\leq \frac{\varepsilon}{2\varphi_3(w(t))\varphi_3(\bar{w}(t))}\big|\varphi_3(\bar{w}(t))(\varphi_{2i}[t,w] + f_{2i}[t,w]p_w(t) - \varphi_{2i}[t] - f_{2i}[t]\bar{p}(t))\big| \\
&\leq \frac{\varepsilon}{2\varphi_3(w(t))\varphi_3(\bar{w}(t))}\big|(\varphi_3(w(t)) - \varphi_3(\bar{w}(t)))(\varphi_{2i}[t] - f_{2i}[t]\bar{p}(t))\big| \\
&\leq \frac{\varepsilon}{\gamma^2}(\|\varphi_3(\bar{w})\|_\infty k_{M,\varphi_2}(|\hat{x}_w(t) - \bar{x}(t)| + |w(t) - \bar{w}(t)|) \\
&\quad + k_{M,f_2}(|\hat{x}_w(t) - \bar{x}(t)| + |w(t) - \bar{w}(t)|) + \|f_2[\cdot]\|_\infty|p_w(t) - \bar{p}(t)|) \\
&\quad + \frac{\varepsilon}{\gamma^2}k_{M,\varphi_3}|w(t) - \bar{w}(t)|\|\varphi_{2i}[\cdot] - f_{2i}[\cdot]\bar{p}(\cdot))\|_\infty.
\end{aligned}$$

From this, (4.44) and the inequality $\|\hat{x}_w - \bar{x}\|_0 \leq l_*\|w - \bar{w}\|_\infty^{1/2}$, we get

$$\varepsilon|\hat{u}_{wi}(t) - \bar{u}_i(t)| \leq C_{12}\|w - \bar{w}\|_\infty^{1/2}. \tag{4.46}$$

Combining this with (4.45), we get

$$\varepsilon|\hat{u}_{wi}(t) - \bar{u}_i(t)| \leq (C_{11} + C_{12})\|w - \bar{w}\|_\infty^{1/2} \text{ a.e. } t \in [0,1].$$

This implies that

$$\|\hat{u}_{wi} - \bar{u}_i\|_\infty \leq C_{13}\|w - \bar{w}\|_\infty^{1/2}.$$

Summing on $i = 1,2,..,m$, we obtain

$$\|\hat{u}_w - \bar{u}\|_\infty \leq C_{14}\|w - \bar{w}\|_\infty^{1/2}. \tag{4.47}$$

Finally, since $(\hat{x}_w,\hat{u}_w) \in \Phi(w)$ and $(\bar{x},\bar{u}) \in \Phi(\bar{w})$, we have

$$\begin{aligned}
&\dot{x}_w(t) = f_1(t,\hat{x}_w(t),w(t)) + f_2(t,w(t))\hat{u}_w(t), \\
&\dot{\bar{x}}(t) = f_1[t] + f_2[t]\bar{u}(t).
\end{aligned}$$

Using $(H2)$, (2.9), (4.47) and the boundedness of $\hat{u}_w$, we get

$$\begin{aligned}
|\dot{x}_w(t) - \dot{\bar{x}}(t)| &\le |f_1(t,\hat{x}_w(t),w(t)) - f_1[t]| + |f_2(t,w(t))\hat{u}_w(t) - f_2[t]\bar{u}(t)| \\
&\le k_{M,f_1}(|\hat{x}_w(t) - \bar{x}(t)| + |w(t) - \bar{w}(t)|) \\
&\quad + |f_2(t,w(t)) - f_2[t]||\hat{u}_w(t)| + |f_2[t]||\hat{u}_w(t) - \bar{u}(t)| \\
&\le k_{M,f_1}(l_*\|w-\bar{w}\|_\infty^{1/2} + \varepsilon_1^{1/2}\|w-\bar{w}\|_\infty^{1/2}) \\
&\quad + k_{M,f_2}(|w(t) - \bar{w}(t)|)|\hat{u}_w(t)| + \|f_2[\cdot]\|_\infty C_{14}\|w-\bar{w}\|_\infty^{1/2} \\
&\le C_{15}\|w-\bar{w}\|_\infty^{1/2}
\end{aligned}$$

for some constant $C_{15} > 0$. Hence

$$\|\dot{x}_w - \dot{\bar{x}}\|_\infty \le C_{15}\|w-\bar{w}\|_\infty^{1/2}.$$

Combining this with (2.9) and (4.47), we obtain

$$\|\hat{x}_w - \bar{x}\|_0 + \|\dot{x}_w - \dot{\bar{x}}\|_\infty + \|\hat{u}_w - \bar{u}\|_\infty \le (l_* + C_{14} + C_{15})\|w-\bar{w}\|_\infty^{1/2},$$

which is (2.10). The proof of the lemma is complete. □

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