J. Appl. Numer. Optim. 2 (2020), No. 2, pp. 199-211
Available online at http://jano.biemdas.com
https://doi.org/10.23952/jano.2.2020.2.06

ON VARIATIONAL ANALYSIS FOR GENERAL DISTANCE FUNCTIONS

LUONG V. NGUYEN[1], XIAOLONG QIN[2,*]

[1]*Department of Natural Sciences, Hong Duc University, Thanh Hoa, Vietnam*
[2]*General Education Center, National Yunlin University of Science and Technology, Douliou 64002, Taiwan*

Abstract. In this paper, we present some observations about variational analysis of minimal time functions. Some new results on generalized differentiation of directional minimal time functions are provided.

Keywords. Minimal time function; Distance function; Fréchet subdifferentials; Fréchet singular subdifferentials; Generalized differentiation.

1. INTRODUCTION

Let F be a nonempty subset and let Ω be a nonempty closed subset of a normed space X. The *minimal time function* to the *target* Ω with the *constant dynamics* F is defined as

$$T_\Omega^F(x) := \inf\{t \geq 0 : (x+tF) \cap \Omega \neq \emptyset\}, \qquad x \in X. \tag{1.1}$$

This function is a natural generalization of the classical distance function. In fact, if the dynamics F is the closed unit ball $\mathbb{B}$ of X, then the minimal time function is reduced to the classical distance function to Ω which is formed by

$$d(x;\Omega) := \inf_{y\in\Omega} ||y-x||, \qquad x \in X, \tag{1.2}$$

whose subdifferential properties have been extensively studied (see, e.g., [1, 5, 15, 16, 17] and references therein). Let $G : Z \rightrightarrows X$ be a set-valued mapping between normed spaces. Another extension of the classical distance function is formed by

$$\rho(z,x) = \inf_{y\in G(z)} ||y-x|| = d(x;G(z)), \qquad (z,x) \in Z \times X, \tag{1.3}$$

The generalized differentiation of this general distance function with applications to the Lipschitz stability was investigated by Mordukhovich and Nam [16, 17]. We note that some estimates of the subgradients of the general distance function ρ in [16] require the Lipschitz continuity of ρ around reference points. In [14], Jiang and He studied some subdifferential properties of the so-called minimal time function with a moving target set which is defined by

$$\Gamma(z,x) := \inf\{t \geq 0 : (x+tF) \cap G(z) \neq \emptyset\}, \qquad (z,x) \in Z \times X. \tag{1.4}$$

In [14], an estimate for the Fréchet subdifferentials of Γ at points outside the graph of G also requires the calmness of Γ at the reference points. Function Γ was also studied in [2, 27].

*Corresponding author.
E-mail addresses: nguyenvanluong@hdu.edu.vn (L.V. Nguyen), qinxl@yuntech.edu.tw (X. Qin).
Received May 2, 2020; Accepted July 6, 2020.

Observe that the minimal time function with a moving target set (1.4) is an extension of general distance function (1.3). If $G(z) \equiv \Omega$, then Γ is reduced to minimal time function T_{Ω}^{F}. In Section 3, we show that the minimal time function with a moving target set (1.4) is actually a special class of the minimal time function (1.1).

Variational analysis and generalized differentiation of the minimal time function associated with a convex dynamics set containing the origin in its interior, in a Hilbert space, was initially studied by Colombo and Wolenski [6, 8]. Since then, the minimal time function has been investigated by many researchers in various ways and for different purposes; see, e.g., [3, 4, 7, 9, 12, 13, 18, 19, 22, 25, 26, 29]. The applications of variational analysis and the generalized differentiation of the minimal time function to generalized Sylvester problems and to generalized Fermat - Terricelli problems were presented in [19, 20, 21, 22, 23, 24, 26] and references therein. We note that all above-mentioned papers deal with convex dynamics except the paper [9], where the authors considered the dynamics F being a subset of the unit sphere of X. We refer the reader to, e.g., [10, 11] for applications of the minimal time function with this nonconvex dynamics. Notice that most results in [9] require that cone$F$ is convex. In this manuscript, we will present a nice observation showing that if cone$F$ is convex, then T_{Ω}^{F} coincides with the minimal time function associated with a convex dynamics. This could allow us to improve various results in literature. Here, we mainly focus on some results about generalized differentiation of the minimal time function. Finally, we obtain some subdifferential formulas for the minimal time function without requiring the lower calmness. The results significantly improve corresponding results in [26].

2. Notations and Definitions

We recall some notations and definitions from [5, 15]. Let X be a real normed space with norm $||\cdot||$. The dual space of X is denoted by X^*, the norm in X^* is also denoted by $||\cdot||$ and the pairing between $x^* \in X^*$ and $x \in X$ is denoted by $\langle x^*, x\rangle$, i.e., $\langle x^*, x\rangle := x^*(x)$. The closed ball and open ball centered at $\bar{x} \in X$ with radius $r > 0$ are denoted by $\mathbb{B}(\bar{x}, r)$ and $\mathbb{B}^o(\bar{x}, r)$, respectively. The closed unit ball of X and X^* are denoted by $\mathbb{B}$ and $\mathbb{B}^*$, respectively. The unit sphere of X is denoted by $\mathbb{S}$. For a nonempty subset C of X, we denote by $||C||$ the valued $\sup\{||x|| : x \in C\}$ and by cone$C$ the cone generated by C.

Let Ω be a subset of X. We use the notation $u \xrightarrow{\Omega} x$ to denote that $u \to x$ and $u \in \Omega$. For any $x \in \Omega$ and $\varepsilon \geq 0$, the set of $\varepsilon-$normals to Ω at x is defined by

$$\widehat{N}_{\varepsilon}(x;\Omega) := \left\{ x^* \in X^* : \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x\rangle}{||u - x||} \leq \varepsilon \right\}.$$

The set $\widehat{N}(x;\Omega) := \widehat{N}_0(x;\Omega)$ is called the Fréchet normal cone to Ω at x. If $x \notin \Omega$, then we put $\widehat{N}_{\varepsilon}(x;\Omega) := \emptyset$ for all $\varepsilon \geq 0$.

Given an extended real-valued function $f : X \to (-\infty, \infty]$, with the domain $\text{dom}(f) := \{x \in X : f(x) < \infty\}$ and epigraph $\text{epi}(f) := \{(x, \beta) \in X \times \mathbb{R} : f(x) \leq \beta\}$ and given $\varepsilon \geq 0$, the $\varepsilon-$ Fréchet subdifferential (or the set of $\varepsilon-$ Fréchet subgradients) of f at a point $\bar{x} \in \text{dom}(f)$ is defined by

$$\widehat{\partial}_{\varepsilon} f(\bar{x}) := \left\{ x^* \in X^* : \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^*, x - \bar{x}}{||x - \bar{x}||} \geq -\varepsilon \right\}.$$

If $\bar{x} \notin \text{dom}(f)$, we set $\widehat{\partial}_\varepsilon f(\bar{x}) = \emptyset$. In the case that $\varepsilon = 0$, we use the notation $\widehat{\partial} f(\bar{x})$ instead of $\widehat{\partial}_0 f(\bar{x})$ and we call $\widehat{\partial} f(\bar{x})$ the Fréchet subdifferential of f at $\bar{x}$. Equivalently,

$$\widehat{\partial} f(\bar{x}) = \left\{ x^* \in X^* : (x^*, -1) \in \widehat{N}((\bar{x}, f(\bar{x})); \text{epi}(f)) \right\}.$$

The Fréchet singular subdifferential of f at $\bar{x} \in \text{dom}(f)$ is the set

$$\widehat{\partial}^\infty f(\bar{x}) := \left\{ x^* \in X^* : (x^*, 0) \in \widehat{N}((\bar{x}, f(\bar{x})); \text{epi}(f)) \right\}.$$

Let $\psi : X \to (-\infty, \infty]$ be a given function and $\bar{x} \in \text{dom}(\psi)$. The function ψ is said to be

- calm at $\bar{x}$ if there exist $k > 0$ and $\delta > 0$ such that
$$|\psi(x) - \psi(\bar{x}| \leq k||x - \bar{x}||, \quad \forall x \in B(\bar{x}, \delta).$$
- lower calm at $\bar{x}$ if there exist constants $\ell \in \mathbb{R}$ and $\delta > 0$ such that
$$\psi(x) - \psi(\bar{x}) \geq \ell ||x - \bar{x}|| \quad \forall x \in B(\bar{x}, \delta).$$

For a subset K of X, the support function $\sigma_K : X^* \to (-\infty, \infty]$ of K is defined by

$$\sigma_K(x^*) := \sup_{x \in K} \langle x^*, x \rangle, \qquad \text{forall } x^* \in X^*.$$

Let X, Z be two normed spaces and $G : Z \rightrightarrows X$ be a set-valued map. The graph of G is the set

$$\text{gph}G := \{(z, x) \in Z \times X : x \in G(z)\}.$$

3. Main results

Let T^F_Ω, ρ and Γ be defined as in Section 1. Recall that $G : Z \rightrightarrows X$ is a set-valued mapping between normed spaces.

Proposition 3.1. *The function Γ (1.4) is actually the minimal time function to the target* gph$G$ *with the dynamics* $\{0\} \times F$.

Proof. For all $(z, x) \in Z \times X$, we have

$$\begin{aligned} \Gamma(z, x) &= \inf\{t \geq 0 : (x + tF) \cap G(z) \neq \emptyset\} \\ &= \inf\{t \geq 0 : ((z, x) + t\{0\} \times F) \cap \text{gph}G \neq \emptyset\} \\ &= T^{\{0\} \times F}_{\text{gph}G}(z, x). \end{aligned}$$

That is, the minimal time function Γ with the moving target $G(\cdot)$ and the constant dynamics F is actually the minimal time function with the fixed target gph$G$ and the constant dynamics $\{0\} \times F$ in $Z \times X$. □

It follows from the proposition that the general distance function ρ (1.3) is the minimal time function to the target gph$G$ with the dynamic $\{0\} \times \mathbb{B}$. Furthermore, for any $(z^*, x^*) \in Z^* \times X^*$, the support function of $\{0\} \times F$ at (z^*, x^*) is computed as

$$\sigma_{\{0\} \times F}(y^*, x^*) = \sup_{(u,v) \in \{0\} \times F} \langle (y^*, x^*), (u, v) \rangle = \sup_{v \in F} \langle x^*, v \rangle = \sigma_F(x^*).$$

If $F = \mathbb{B}$, then

$$\sigma_{\{0\} \times \mathbb{B}}(y^*, x^*) = ||x^*||.$$

For $t>0$, let $G_t : Z \rightrightarrows X$ be defined by $G_t(z) := \{x \in X : \Gamma(z,x) \leq t\}$. It follows that

$$\begin{aligned}\mathrm{gph}G_t &= \{(z,x) \in Z \times X : x \in G_t(z)\} = \{(z,x) \in Z \times X : \Gamma(z,x) \leq t\} \\ &= \left\{(z,x) \in Z \times X : T^{\{0\}\times F}_{\mathrm{gph}G}(z,x) \leq t\right\}.\end{aligned}$$

Above discussions allow us to obtain several properties of general distance function ρ from the properties of minimal time functions. For example, we present here an improvement of a result for general distance function ρ in [16] based on recent results for minimal time functions.

Denote by set $\Omega_t = \{x \in X : T^F_\Omega(x) \leq t\}$ for $t>0$. We recall the following results.

Theorem 3.1. [13, 29] *Assume that F is closed, bounded and convex. Let $\bar{x} \notin \Omega$ with $r := T^F_\Omega(\bar{x}) < \infty$. Then*

$$\widehat{\partial} T^F_\Omega(\bar{x}) = \widehat{N}(\bar{x};\Omega_r) \cap \{x^* \in X^* : \sigma_F(-x^*) = 1\}.$$

The latter result is generalized as follows.

Theorem 3.2. [19, 25] *Assume that F is closed, bounded and convex. Let $\bar{x} \notin \Omega$ with $r := T^F_\Omega(\bar{x}) < \infty$.*

(i) *For any $\varepsilon \geq 0$,*

$$\widehat{\partial}_\varepsilon T^F_\Omega(\bar{x}) \subset \widehat{N}_\varepsilon(\bar{x},\Omega_r) \cap \{x^* \in X^* : 1-\varepsilon||F|| \leq \sigma_F(-x^*) \leq 1+\varepsilon||F||\}.$$

(ii) *For any $x^* \in \widehat{N}_\varepsilon(\bar{x};\Omega_r) \cap \{x^* \in X^* : 1-\varepsilon||F|| \leq \sigma_F(-x^*) \leq 1+\varepsilon||F||\}$ and $\varepsilon \geq 0$ satisfying $1-2\varepsilon||F|| > 0$, there exists a constant $\ell := 1+2\kappa||F||$ with $\kappa > ||x^*||$ such that $x^* \in \widehat{\partial}_{\ell\varepsilon} T^F_\Omega(\bar{x})$.*

Thus, by the above discussion, we have the following improvement for the theorem 3.2 and the corollary 3.3 in [16]. In fact, we do not require the Lipschitz continuity of ρ around the reference point.

Theorem 3.3. *Let $(\bar{z},\bar{x}) \in Z \times X$ with $0 < r = d(\bar{x},G(\bar{z})) < \infty$. Then*

(i) *for any $(z^*,x^*) \in \widehat{N}_\varepsilon((\bar{z},\bar{x});\mathrm{gph}G_r)$ and $\varepsilon \geq 0$ satisfying $1-2\varepsilon > 0$, there exists a constant $\ell > 1$ such that $(z^*,x^*) \in \widehat{\partial}_{\ell\varepsilon}\rho(\bar{z},\bar{x})$,*

(ii) *in particular,*

$$\widehat{\partial}\rho(\bar{z},\bar{x}) = \left\{(z^*,x^*) \in \widehat{N}((\bar{z},\bar{x});\mathrm{gph}G_r),\ ||x^*|| = 1\right\},$$

where $G_r : Z \rightrightarrows X$ defined by $G_r(z) = \{x \in X : d(x;G(z)) \leq r\}$.

By the above observation, one can also improve [14, Theorem 3.2(b)] by removing the calmness at the reference point of the function Γ - the minimal time function with a moving target. However, we will not state its improvement here.

It is well-known that if the dynamics F is nonempty, closed and convex, then the minimal time function T^F_Ω coincides with the minimum time function to the target Ω for the differential inclusion

$$\begin{cases} \dot{y}(t) & \in \quad F, \qquad \text{a.e.}\, t>0, \\ y(0) & = \quad x \in X \end{cases} \tag{3.1}$$

in control theory, which is defined by

$$\mathscr{T}(x) := \inf\{t \geq 0 : \exists y(\cdot) \text{ satisfying (3.1) and } y(t) \in \Omega\}. \tag{3.2}$$

For the study of variational analysis and generalized differentiation of the minimum time function $\mathscr{T}$ for more general control systems in finite dimensional setting, we refer the reader to, e.g., [28, 30] and references therein. When F is nonconvex, then T_Ω^F and $\mathscr{T}$ are different functions.

Example 3.1. Let $X = \mathbb{R}^2$, $F = \{(1,0),(0,1)\}$ and $\Omega = \{(1,1)\}$. One can easily compute that

$$T_\Omega^F(x_1,x_2) = \begin{cases} 1-x_1, & \text{if } x_1 \le 1, x_2 = 1,\\ 1-x_2, & \text{if } x_1 = 1, x_2 \le 1,\\ \infty, & \text{otherwise,} \end{cases}$$

and

$$\mathscr{T}(x_1,x_2) = \begin{cases} 2-x_1-x_2, & \text{if } x_1 \le 1, x_2 \le 1,\\ \infty, & \text{otherwise.} \end{cases}$$

Thus, $\mathscr{T}$ and T_Ω^F are not the same.

To the best of our knowledge, the paper [9] is the only one, which studied the minimal time function with nonconvex dynamics F. In fact, it studied the minimal time function with the dynamics being a subset of the unit sphere. The applications of this function to optimization problems can be found in, e.g., [11, 10]. We next present another observation which could allows us to improve various results in the literature for T_Ω^F, i.e., various results are still valid when the dynamics is not necessarily closed and/or convex. Before proceeding, we introduce the following notation. For a nonempty subset C of X, we denote by the set $\widehat{C}$

$$\widehat{C} = \{\alpha x : x \in C \text{ and } \alpha \in [0,1]\}.$$

From the definition of $\widehat{C}$, we see that $0 \in \widehat{C}$, $C \subset \widehat{C}$ and $||C|| = ||\widehat{C}||$. Moreover, we have the following relations between C and $\widehat{C}$.

Lemma 3.1. *Let C be a nonempty subset of X. Then*

(i) *C is bounded if and only if $\widehat{C}$ is bounded.*
(ii) *If C is bounded and closed then $\widehat{C}$ is bounded and closed.*
(ii) *If C is convex, then $\widehat{C}$ is convex. The converse is not true. If C is a subset of $\mathbb{S}$ and* cone$C$ *is convex, then $\widehat{C}$ is convex.*

Proof. (i) It is obvious. We now prove (ii). It is enough to prove that $\widehat{C}$ is closed whence C is both bounded and closed. Assume $\{\widehat{x}_n\}$ is a sequence in $\widehat{C}$ converging to some $\widehat{x} \in X$. Since $\widehat{x}_n \in \widehat{C}$, there exist $\alpha_n \in [0,1]$ and $x_n \in C$ such that $\widehat{x}_n = \alpha_n x_n$. Since $\alpha_n \in [0,1]$, there exists a subsequence of $\{\alpha_n\}$, which is still denoted by $\{\alpha_n\}$, converging to some $\alpha \in [0,1]$. If $\alpha = 0$, then $\widehat{x}_n \to 0$ as $n \to \infty$ since $\{x_n\}$ is bounded. Hence, $\widehat{x} = 0 \in \widehat{C}$. If $\alpha \neq 0$, then

$$\left|\left|x_n - \frac{\widehat{x}}{\alpha}\right|\right| = \frac{1}{\alpha}||\alpha x_n - \alpha_n x_n + \alpha_n x_n - \widehat{x}|| \le \frac{1}{\alpha}||\alpha_n - \alpha||||x_n|| + \frac{1}{\alpha}||\widehat{x}_n - \widehat{x}||,$$

which implies that $x_n \to \widehat{x}/\alpha$ as $n \to \infty$. Since $x_n \in C$ and C is closed, $\widehat{x}/\alpha =: x \in C$. Thus $\widehat{x} = \alpha x \in \widehat{C}$. Therefore, $\widehat{C}$ is closed.

We now in a position to prove (iii). Assume that C is convex. Let $\widehat{x}, \widehat{y} \in \widehat{C}$ and $\gamma \in (0,1)$. There exist $\alpha, \beta \in [0,1]$ and $x, y \in C$ such that $\widehat{x} = \alpha x$ and $\widehat{y} = \beta y$. We show that $\gamma\widehat{x} + (1-\gamma)\widehat{y} \in \widehat{C}$. If

$\alpha=\beta=0$, then there is nothing to show since $\widehat{x}=\widehat{y}=0$. Assume that $\alpha^2+\beta^2\neq 0$. Then

$$\begin{aligned}\gamma\widehat{x}+(1-\gamma)\widehat{y} &= \gamma\alpha x+(1-\gamma)\beta y\\ &= [\gamma\alpha+(1-\gamma)\beta]\left(\frac{\gamma\alpha}{\gamma\alpha+(1-\gamma)\beta}x+\frac{(1-\gamma)\beta}{\gamma\alpha+(1-\gamma)\beta}y\right).\end{aligned}$$

Since $x,y\in C$ and C is convex, one has

$$\frac{\gamma\alpha}{\gamma\alpha+(1-\gamma)\beta}x+\frac{(1-\gamma)\beta}{\gamma\alpha+(1-\gamma)\beta}y\in C.$$

Thus $\gamma\widehat{x}+(1-\gamma)\widehat{y}\in\widehat{C}$ as $\gamma\alpha+(1-\gamma)\beta\in[0,1]$. Therefore, $\widehat{C}$ is convex. The inverse is not true. For example, let $X=\mathbb{R}^2$ and $C=\{x\in X: 1\leq ||x||\leq 2\}$. Then C is not convex, but $\widehat{C}=\{x\in X: ||x||\leq 2\}$ is convex. The last conclusion is obvious. □

Remark 3.1. The closedness of C and the closedness of $\widehat{C}$ are independent. For example, let $C=(0,1]$, then $\widehat{C}=[0,1]$. The latter is closed while not the former. Consider $X=\mathbb{R}^2$, endowed with the usual norm and let

$$C=\left\{n\left(\cos\left(\frac{\pi}{n}\right),\sin\left(\frac{\pi}{n}\right)\right):\ n\in\mathbb{N}\right\}.$$

Then C is a closed subset of $\mathbb{R}^2$. However, the set $\widehat{C}$ is not closed. Indeed, considering the sequence $\{x_n\}$ defined by: for each $n\in\mathbb{N}$

$$x_n=\left(\cos\left(\frac{\pi}{n}\right),\sin\left(\frac{\pi}{n}\right)\right),$$

we have $x_n\in\widehat{C}$ as

$$\left(\cos\left(\frac{\pi}{n}\right),\sin\left(\frac{\pi}{n}\right)\right)=\frac{1}{n}\times n\left(\cos\left(\frac{\pi}{n}\right),\sin\left(\frac{\pi}{n}\right)\right).$$

Since

$$\lim_{n\to\infty}\left(\cos\left(\frac{\pi}{n}\right),\sin\left(\frac{\pi}{n}\right)\right)=(1,0)\notin\widehat{C},$$

the set $\widehat{C}$ is not closed.

It is known that T_Ω^F is convex when F and Ω are convex (see, e.g., [6, 22]) or when Ω is convex and F is a subset of the unit sphere with the cone generated by F being convex (see [11]). There is no other result about convexity of T_Ω^F for a more complicated set F.

Example 3.2. Consider the minimal time function T_Ω^F with $\Omega=\{(2,2)\}$ and the dynamics F is defined as

$$F=\{(x_1,x_2): x_1\leq 1, x_2\leq 1, x_1+x_2>1\}\cup\{(x_1,x_2): x_1\geq 0, x_1+x_2\leq 1, x_1<x_2\}.$$

By existing results, we do not know whether T_Ω^F is convex or not since F is not convex nor a subset of the unit sphere. One can compute that: for $x=(x_1,x_2)\in\mathbb{R}^2$,

$$T_\Omega^F(x)=\begin{cases}2-x_2, & \text{if } \ x_2\leq x_1\leq 2,\\ 2-x_1, & \text{if } \ x_1\leq x_2\leq 2,\\ +\infty, & \text{otherwise},\end{cases}$$

and see that function T_Ω^F is convex. The question is that: can we get the convexity of T_Ω^F for this kind of dynamics without computing the function? Notice that in this case $\widehat{F}=\{(x_1,x_2): 0\leq x_1\leq 1, 0\leq x_2\leq 1\}$ is convex.

We have an affirmative answer for the latter question using the following result.

Proposition 3.2. *Let F and Ω be nonempty subsets of X. Then*

$$T_\Omega^F(x) = T_\Omega^{\widehat{F}}(x), \quad \text{for all } x \in X.$$

Proof. Since $F \subset \widehat{F}$, it is known that $T_\Omega^{\widehat{F}}(x) \le T_\Omega^F(x)$ for all $x \in X$. We now show, for all $x \in X$, that

$$T_\Omega^F(x) \le T_\Omega^{\widehat{F}}(x). \tag{3.3}$$

Let $x \in X$ be arbitrary. If $T_\Omega^{\widehat{F}}(x) = \infty$, then (3.3) holds. Assume that $t := T_\Omega^{\widehat{F}}(x) < \infty$. Then, for any $\varepsilon > 0$, there exist $t_\varepsilon \in (t, t+\varepsilon)$ and $\widehat{f} \in \widehat{F}$ such that $x + t_\varepsilon \widehat{f} \in \Omega$. Since $\widehat{f} \in \widehat{F}$, there exist $\alpha \in [0,1]$ and $f \in F$ such that $\widehat{f} = \alpha f$. Thus $x + t_\varepsilon \alpha f \in \Omega$. This implies that

$$T_\Omega^F(x) \le \alpha t_\varepsilon < t_\varepsilon < t + \varepsilon.$$

Letting $\varepsilon \to 0^+$, we get $T_\Omega^F(x) \le t = T_\Omega^{\widehat{F}}(x)$. This ends the proof. □

In Example 3.2, the set $\widehat{F}$ is convex and by existing results $T_\Omega^{\widehat{F}}$ is convex. Using Proposition 3.2, we can conclude that T_Ω^F is convex without computing the function. By Proposition 3.2, we have the following improvement.

Proposition 3.3. *If $\widehat{F}$ and Ω are convex, then T_Ω^F is convex.*

Moreover, since $0 \in \widehat{F}$, *we can always assume that* $0 \in F$ *when we deal with minimal time function* T_Ω^F. Proposition 3.2 also allows us to improve several results on minimal time functions. Given $\bar{x} \in X$ with $T_\Omega^F(\bar{x}) < \infty$, the minimal time projection of $\bar{x} \in X$ on the target set Ω is defined by

$$\Pi_F(\bar{x};\Omega) := \big(\bar{x} + T_\Omega^F(\bar{x})F\big) \cap \Omega. \tag{3.4}$$

We also have the following improvement of Proposition 3.4 in [19].

Proposition 3.4. *Assume that $\widehat{F}$ is nonempty, closed, bounded and convex. Let $\bar{x} \notin \Omega$ and let $\bar{\omega} \in \Pi_F(\bar{x};\Omega)$. Then, for any $\alpha \in [0,1]$,*

$$T_\Omega^F(\alpha\bar{\omega} + (1-\alpha)\bar{x}) = (1-\alpha)T_\Omega^F(\bar{x}). \tag{3.5}$$

The latter result can apply to the function T_Ω^F in Example 3.2, where F is not convex nor closed. We now improve Theorem 3.1 to the cases in which F is not necessarily convex nor closed. However, this is not straightforward like Proposition 3.3 or Proposition 3.4. We need the following results.

Proposition 3.5. *Assume that F is bounded. Let $\bar{x} \notin \Omega$ be such that $r := T_\Omega^F(\bar{x}) < \infty$. If $x^* \in \widehat{N}(\bar{x};\Omega_r)$, then $\sigma_F(-x^*) \ge 0$.*

Proof. Since $x^* \in \widehat{N}(\bar{x};\Omega_r)$, for any $\varepsilon > 0$, one sees that there exists $\delta > 0$ such that, for any $y \in \mathbb{B}^o(\bar{x},\delta) \cap \Omega_r$,

$$\langle x^*, y - \bar{x}\rangle \le \varepsilon||y - \bar{x}||. \tag{3.6}$$

Let $\eta > 0$ be such that $\eta < \max\{r, \delta/||F||\}$. By the definition of T_Ω^F, for any $0 < \gamma < \eta$, there exist $t \in (r, r+\gamma)$, $w \in \Omega$, and $u \in F$ such that $w = \bar{x} + tu$. Observe that $\bar{x} + \eta u \in \mathbb{B}^o(\bar{x},\delta)$. Moreover,

$$w = \bar{x} + tu = \bar{x} + \eta u + (t-\eta)u \in \bar{x} + \eta u + (t-\eta)F.$$

Thus $T^F_\Omega(\bar{x}+\eta u) \le t-\eta < r+\gamma-\eta \le r$, i.e., $\bar{x}+\eta u \in \Omega_r$. From (3.6), we have

$$\langle x^*, \eta u\rangle \le \varepsilon ||\eta u|| \le \varepsilon\eta ||F||,$$

or, equivalently, $\langle x^*, u\rangle \le \varepsilon ||F||$. Letting $\varepsilon \to 0+$, we get $\langle x^*, u\rangle \le 0$. Hence $\sigma_F(-x^*) \ge 0$. □

Proposition 3.6. *Assume that F is bounded. Let $x \in X$ be such that $0 < r := T^F_\Omega(x) < \infty$. Then*

$$\sigma_F(-\zeta) = \sigma_{\widehat{F}}(-\zeta), \quad \forall \zeta \in \widehat{N}(x;\Omega_r).$$

Proof. Let $\zeta \in \widehat{N}(x;\Omega_r)$. Since $F \subset \widehat{F}$, we have

$$\sigma_F(-\zeta) \le \sigma_{\widehat{F}}(-\zeta).$$

Since $T^F_\Omega \equiv T^{\widehat{F}}_\Omega$, we find from Proposition 3.5 that $\sigma_{\widehat{F}}(-\zeta) \ge 0$. If $\sigma_{\widehat{F}}(-\zeta) = 0$, then $\sigma_F(-\zeta) = \sigma_{\widehat{F}}(-\zeta) = 0$. Assume now that $\sigma_{\widehat{F}}(-\zeta) > 0$. For any $0 < \varepsilon < \sigma_{\widehat{F}}(-\zeta)$, there exists $\widehat{f} \in \widehat{F}$ such that

$$\sigma_{\widehat{F}}(-\zeta) < \langle -\zeta, \widehat{f}\rangle + \varepsilon.$$

Then $\langle -\zeta, \widehat{f}\rangle > 0$ and there exist $\alpha \in (0,1]$ and $f \in F$ such that $\widehat{f} = \alpha f$. Thus

$$\sigma_{\widehat{F}}(-\zeta) < \alpha\langle -\zeta, f\rangle + \varepsilon \le \langle -\zeta, f\rangle + \varepsilon \le \sigma_F(-\zeta) + \varepsilon.$$

Letting $\varepsilon \to 0^+$, we get $\sigma_{\widehat{F}}(-\zeta) \le \sigma_F(-\zeta)$. This completes the proof. □

Combining Theorem 3.1, Proposition 3.5, 3.6 and Lemma 3.1, we have the following improvement of Theorem 3.1.

Theorem 3.4. *Assume that $\widehat{F}$ is nonempty, closed, bounded and convex. Let $\bar{x} \notin \Omega$ with $r := T^F_\Omega(\bar{x}) < \infty$. Then, one has*

$$\widehat{\partial} T^F_\Omega(\bar{x}) = \widehat{N}(\bar{x};\Omega_r) \cap \{x^* \in X^* : \sigma_F(-x^*) = 1\}.$$

Concerning the Fréchet singular subdifferential of the minimal time function at a point $\bar{x} \notin \Omega$ with $T^F_\Omega(\bar{x}) = r < \infty$, it was shown in [25] that if F is nonempty closed bounded and convex and $\bar{x} \notin \Omega$ with $T^F_\Omega(\bar{x}) = r < \infty$, then

$$\widehat{\partial}^\infty T^F_\Omega(\bar{x}) = \widehat{N}(\bar{x};\Omega_r) \cap \{x^* \in X^* : \langle x^*, q\rangle \ge 0 \text{ for all } q \in F\}. \tag{3.7}$$

By using above results, we can refine formula (3.7) as follows.

Theorem 3.5. *Assume that $\widehat{F}$ is nonempty, closed, bounded and convex. Let $\bar{x} \notin \Omega$ with $r := T^F_\Omega(\bar{x}) < \infty$. Then*

$$\widehat{\partial}^\infty T^F_\Omega(\bar{x}) = \widehat{N}(\bar{x};\Omega_r) \cap \{x^* \in X^* : \sigma_F(-x^*) = 0\}.$$

Theorem 3.5 improves the corresponding result in [25] in the following ways. First, the dynamics F is not necessarily convex nor closed. Second, the inequality in (3.7) now is an equality.

Concerning the minimal time projection, we have the following new results.

Proposition 3.7. *Assume that $\widehat{F}$ is nonempty, closed, bounded and convex. Let $\bar{x} \notin \Omega$ with $r := T^F_\Omega(\bar{x}) < \infty$ be such that $\Pi_F(\bar{x};\Omega) \neq \emptyset$. For any $\varepsilon \ge 0$, $\alpha \in [0,1]$ and $\bar{\omega} \in \Pi_F(\bar{x};\Omega)$, we have*

$$\widehat{N}_\varepsilon(\bar{x};\Omega_r) \subset \widehat{N}_\varepsilon(\alpha\bar{x} + (1-\alpha)\bar{\omega});\Omega_{r\alpha}) \tag{3.8}$$

and

$$\widehat{\partial}_\varepsilon T_\Omega^F(\bar{x}) \subset \widehat{\partial}_\varepsilon T_\Omega^F(\alpha\bar{x} + (1-\alpha)\bar{\omega}). \tag{3.9}$$

Proof. We will prove the inclusion (3.8). The inclusions (3.9) can be proved in a similar way. Fix $\varepsilon \geq 0, \alpha \in [0,1)$ and $\bar{\omega} \in \Pi_F(\bar{x};\Omega)$. Set $\bar{y} = \alpha\bar{x} + (1-\alpha)\bar{\omega}$. Then $T_\Omega^F(\bar{y}) = \alpha T_\Omega^F(\bar{x}) = r\alpha$ (see Proposition 3.4). Let $\bar{u} \in F$ be such that $\bar{x} + r\bar{u} = \bar{\omega}$. Observe that $\bar{y} = \bar{x} + (1-\alpha)r\bar{u}$. Let $x^* \in \widehat{N}_\varepsilon(\bar{x};\Omega_r)$. Then for any $\eta > 0$, there exists $\delta > 0$ such that

$$\langle x^*, x - \bar{x}\rangle \leq (\varepsilon + \eta)||x - \bar{x}|| \qquad \text{for all } x \in \Omega_r \cap \mathbb{B}^o(\bar{x},\delta). \tag{3.10}$$

For any $y \in \Omega_{r\alpha} \cap \mathbb{B}^o(\bar{y},\delta)$, set $x = y - (1-\alpha)r\bar{u}$. Then by principle of optimality, we have

$$T_\Omega^F(y) \leq T_\Omega^F(y) + (1-\alpha)r \leq r\alpha + (1-\alpha)r = r.$$

Moreover,

$$||x - \bar{x}|| = ||y - (1-\alpha)r\bar{u} - (\bar{y} - (1-\alpha)r\bar{u})|| = ||y - \bar{y}|| < \delta.$$

Thus $x \in \Omega_r \cap \mathbb{B}^o(\bar{x},\delta)$. It follows from (3.10) that

$$\langle x^*, y - \bar{y}\rangle = \langle x^*, x - \bar{x}\rangle \leq (\varepsilon + \eta)||x - \bar{x}|| = (\varepsilon + \eta)||y - \bar{y}||,$$

which means that $x^* \in \widehat{N}_\varepsilon(\alpha\bar{x} + (1-\alpha)\bar{\omega});\Omega_{r\alpha})$. □

From Theorem 3.4 and Proposition 3.7, we have the following result which improves Proposition 5.2 (ii) in [9] and the first conclusion of Theorem 3.16 in [26].

Proposition 3.8. *Assume that $\widehat{F}$ is nonempty, closed, bounded and convex.. Let $\bar{x} \notin \Omega$ be such that $r := T_\Omega^F(\bar{x}) < \infty$ be such that $\Pi_F(\bar{x};\Omega) \neq \emptyset$. For any $\alpha \in [0,1]$ and $\bar{\omega} \in \Pi_F(\bar{x};\Omega)$, we have*

$$\widehat{\partial} T_\Omega^F(\bar{x}) \subset \widehat{N}(\alpha\bar{x} + (1-\alpha)\bar{\omega};\Omega_{r\alpha}) \cap \{x^* \in X^* : \sigma_F(-x^*) = 1\}. \tag{3.11}$$

The following example shows that the inclusion (3.11) is strict.

Example 3.3. Consider the minimal time function associated with the dynamics $F := [0,1] \times \{1\} \subset \mathbb{R}^2$ and the target $\Omega := \{(2,2)\}$. Then the support function of F is computed by: for $x = (x_1,x_2) \in \mathbb{R}^2$,

$$\sigma_F(x) = \begin{cases} x_2, & \text{if } x_1 <, 0 \\ x_1 + x_2, & \text{if } x_1 \geq 0. \end{cases}$$

The domain of the minimal time function T_Ω^F is $D := \text{dom}(T_\Omega^F) = \{(x_1,x_2) : x_2 \leq x_1 \leq 2\}$. For $(x_1,x_2) \in \mathbb{R}^2$,

$$T_\Omega^F(x) = \begin{cases} 2 - x_2, & \text{if } x \in D, \\ +\infty, & \text{if } x \notin D. \end{cases}$$

Let $\bar{x} = (1,0)$. It is easy to see that $\widehat{\partial} T_\Omega^F(\bar{x}) = \{(0,-1)\}$. We have $\bar{\omega} = (2,2) = \Pi_F(\bar{x};\Omega)$ and $\widehat{N}(\bar{\omega};\Omega) = \mathbb{R}^2$. Moreover,

$$\{x^* \in \mathbb{R}^2 : \sigma_F(-x^*) = 1\} = [0,\infty) \times \{-1\} \cup \{(x_1^*,x_2^*) : x_1^* + x_2^* = -1, x_1^* \leq 0\}.$$

Thus, $\widehat{\partial} T_\Omega^F(\bar{x})$ is a strict subset of $\{x^* : \sigma_F(-x^*) = 1\} \cap \widehat{N}(\bar{\omega};\Omega)$.

However, if F is a singleton, then (3.11) becomes an equality. More precisely, we have the following results, which improves Theorem 3.16 in [26] by relaxing the lower calmness of T_Ω^F at $\bar{x}$.

Theorem 3.6. *Assume that $F = \{v\} \neq \{0\}$ and Ω is a nonempty closed subset of X. Let $\bar{x} \notin \Omega$ with $r := T_\Omega^F(\bar{x}) < \infty$. For any $\alpha \in [0,1]$, one has*

$$\widehat{\partial} T_\Omega^F(\bar{x}) = \{x^* \in X^* : \sigma_F(-x^*) = 1\} \cap \widehat{N}(\alpha\bar{x} + (1-\alpha)\bar{\omega}; \Omega_{r\alpha}), \tag{3.12}$$

where $\bar{\omega} := \Pi_F(\bar{x}; \Omega)$. Moreover,

$$\widehat{\partial} T_\Omega^F(\bar{x}) = \widehat{\partial} T_\Omega^F(\alpha\bar{x} + (1-\alpha)\bar{\omega}) \quad \textit{for all} \;\; \alpha \in (0,1].$$

Proof. It is enough to show that

$$\{x^* \in X^* : \sigma_F(-x^*) = 1\} \cap \widehat{N}(\bar{\omega}; \Omega) \subset \widehat{\partial} T_\Omega^F(\bar{x}).$$

Let $x^* \in \widehat{N}(\bar{\omega}; \Omega)$ be such that $\langle x^*, v\rangle = -1$. We attempt to show that $x^* \in \widehat{\partial} T_\Omega^F(\bar{x})$. Let $\sigma > 0$ and set

$$c := \min\left\{1, \frac{1}{2||v||}, \frac{\sigma}{1+||v||.||x^*||}, \frac{\sigma}{1+2||v||+2||v||.||x^*||}\right\}.$$

Let $\sigma_1 \in (0,c)$. Since $x^* \in \widehat{N}(\bar{\omega}; \Omega)$, one has that there exists $\eta_1 > 0$ such that, for all $\omega \in \mathbb{B}^o(\bar{\omega}, \eta_1) \cap \Omega$, it holds

$$\langle x^*, \omega - \bar{\omega}\rangle \leq \sigma_1 ||\omega - \bar{\omega}||. \tag{3.13}$$

Take $\sigma_2 \in (0, \eta_1/(1+||v||))$. Since T_Ω^F is lower semicontinuous at $\bar{x}$, we have that there exists $\eta_2 > 0$ such that

$$T_\Omega^F(\bar{x}) - T_\Omega^F(y) \leq \sigma_2, \quad \text{for all } y \in \mathbb{B}^o(\bar{x}, \eta_2). \tag{3.14}$$

Let $\eta \in (0, c_1)$ with

$$c_1 := \min\left\{\eta_2, \eta_1 - \sigma_2||v||, \frac{\eta_1}{1+||v||.||x^*||}\right\}.$$

Assume that $x^* \notin \widehat{\partial} T_\Omega^F(\bar{x})$. Then, there exists $\bar{y} \in \mathbb{B}^o(\bar{x}, \eta) \cap \operatorname{dom}(T_\Omega^F)$ such that

$$T_\Omega^F(\bar{y}) - T_\Omega^F(\bar{x}) - \langle x^*, \bar{y} - \bar{x}\rangle < -\sigma||\bar{y} - \bar{x}||. \tag{3.15}$$

Setting $\bar{\theta} := \bar{y} + T_\Omega^F(\bar{y})v \in \Omega$, we have

$$\begin{aligned} ||\bar{\theta} - \bar{\omega}|| &= ||\bar{y} + T_\Omega^F(\bar{y})v - \bar{x} - T_\Omega^F(\bar{x})v|| \\ &\leq ||\bar{y} - \bar{x}|| + |T_\Omega^F(\bar{y}) - T_\Omega^F(\bar{x})|.||v||. \end{aligned} \tag{3.16}$$

Since $\langle x^*, v\rangle = -1$, we have

$$\begin{aligned} \langle x^*, \bar{\theta} - \bar{\omega}\rangle &= \langle x^*, \bar{y} + T_\Omega^F(\bar{y})v - \bar{x} - T_\Omega^F(\bar{x})v\rangle \\ &= \langle x^*, \bar{y} - \bar{x}\rangle + T_\Omega^F(\bar{x}) - T_\Omega^F(\bar{y}). \end{aligned}$$

Thus,

$$T_\Omega^F(\bar{y}) - T_\Omega^F(\bar{x}) - \langle x^*, \bar{y} - \bar{x}\rangle = -\langle x^*, \bar{\theta} - \bar{\omega}\rangle. \tag{3.17}$$

We have three possible cases: (i) $T_\Omega^F(\bar{y}) = T_\Omega^F(\bar{x})$, (ii) $T_\Omega^F(\bar{y}) > T_\Omega^F(\bar{x})$, and (iii) $T_\Omega^F(\bar{y}) < T_\Omega^F(\bar{x})$.

Case (i): $T_\Omega^F(\bar{y}) = T_\Omega^F(\bar{x})$. Then, by (3.16), one has

$$||\bar{\theta} - \bar{\omega}|| \leq ||\bar{y} - \bar{x}|| < \eta < \eta_1,$$

i.e., $\bar{\theta} \in \mathbb{B}^o(\bar{\omega}, \eta_1) \cap \Omega$. Thus, by (3.13) and (3.17), we get

$$\begin{aligned} T_\Omega^F(\bar{y}) - T_\Omega^F(\bar{x}) - \langle x^*, \bar{y} - \bar{x}\rangle &\geq -\sigma_1||\bar{\theta} - \bar{\omega}|| \geq -\sigma_1||\bar{y} - \bar{x}|| \\ &\geq -\sigma||\bar{y} - \bar{x}|| \end{aligned}$$

which contradicts to (3.15).

Case (ii): $T^F_\Omega(\bar{y}) > T^F_\Omega(\bar{x})$. It follows from (3.15) that

$$0 < T^F_\Omega(\bar{y}) - T^F_\Omega(\bar{x}) \le ||x^*||.||\bar{y}-\bar{x}||.$$

Then, using (3.16), we have

$$\begin{aligned} ||\bar{\theta}-\bar{\omega}|| &\le ||\bar{y}-\bar{x}|| + ||x^*||.||v||.||\bar{y}-\bar{x}|| = (1+||x^*||.||v||)||\bar{y}-\bar{x}|| \\ &< (1+||x^*||.||v||)\eta < \eta_1, \end{aligned}$$

i.e., $\bar{\theta} \in \mathbb{B}^o(\bar{\omega}, \eta_1) \cap \Omega$. Using (3.13), one has

$$\begin{aligned} \langle x^*, \bar{\theta}-\bar{\omega}\rangle &\le \sigma_1 ||\bar{\theta}-\bar{\omega}|| \\ &\le \sigma_1(||\bar{y}-\bar{x}|| + ||v||.|T^F_\Omega(\bar{y}) - T^F_\Omega(\bar{x})|) \\ &\le \sigma_1(1+||v||.||x^*||)||\bar{y}-\bar{x}||. \end{aligned}$$

Then, one has from (3.17) that

$$\begin{aligned} T^F_\Omega(\bar{y}) - T^F_\Omega(\bar{x}) - \langle x^*, \bar{y}-\bar{x}\rangle &\ge -\sigma_1(1+||v||.||x^*||)||\bar{y}-\bar{x}|| \\ &\ge -\sigma||\bar{y}-\bar{x}||, \end{aligned}$$

which contradicts (3.15).

Case (iii): $T^F_\Omega(\bar{y}) < T^F_\Omega(\bar{x})$. Then, $0 < T^F_\Omega(\bar{x}) - T^F_\Omega(\bar{y}) \le \sigma_2$. It follows from (3.16) that

$$||\bar{\theta}-\bar{\omega}|| \le \eta + \sigma_2||v|| < \eta_1,$$

i.e., $\bar{\theta} \in \mathbb{B}^o(\bar{\omega}, \eta_1) \cap \Omega$. Hence,

$$\begin{aligned} \langle x^*, \bar{\theta}-\bar{\omega}\rangle &\le \sigma_1 ||\bar{\theta}-\bar{\omega}|| \\ &\le \sigma_1||\bar{y}-\bar{x}|| + \sigma_1||v||(T^F_\Omega(\bar{x}) - T^F_\Omega(\bar{y})). \end{aligned}$$

Moreover, we obtain from (3.17) that

$$\begin{aligned} T^F_\Omega(\bar{x}) - T^F_\Omega(\bar{y}) &= \langle x^*, \bar{\theta}-\bar{\omega}\rangle - \langle x^*, \bar{y}-\bar{x}\rangle \\ &\le \sigma_1||\bar{y}-\bar{x}|| + \sigma_1||v||(T^F_\Omega(\bar{x}) - T^F_\Omega(\bar{y})) + ||x^*||.||\bar{y}-\bar{x}|| \\ &= (\sigma_1+||x^*||)||\bar{y}-\bar{x}|| + \sigma_1||v||(T^F_\Omega(\bar{x}) - T^F_\Omega(\bar{y})) \\ &\le (1+||x^*||)||\bar{y}-\bar{x}|| + \frac{1}{2}(T^F_\Omega(\bar{x}) - T^F_\Omega(\bar{y})), \end{aligned}$$

which leads to

$$T^F_\Omega(\bar{x}) - T^F_\Omega(\bar{y}) \le 2(1+||x^*||)||\bar{y}-\bar{x}||.$$

Again, from (3.17), one has

$$\begin{aligned} T^F_\Omega(\bar{y}) - T^F_\Omega(\bar{x}) - \langle x^*, \bar{y}-\bar{x}\rangle &= -\langle x^*, \bar{\theta}-\bar{\omega}\rangle \\ &\ge -\sigma_1||\bar{y}-\bar{x}|| - \sigma_1||v||(T^F_\Omega(\bar{x}) - T^F_\Omega(\bar{y}) \\ &\ge -\sigma_1(1+2||v||+2||v||.||x^*||)||\bar{y}-\bar{x}|| \\ &\ge -\sigma||\bar{y}-\bar{x}||, \end{aligned}$$

which contradicts (3.15). This ends the proof. □

Remark 3.2. (i) Theorem 3.6 still holds true if we replace the Fréchet subdifferential and Fréchet normal cone by s- Hölder subdifferential and s- Hölder normal cone, respectively. Thus, we obtain a result which also improves Theorem 3.14 in [26] by relaxing the lower calmness of T_{Ω}^{F}.

(ii) One can also relax the lower calmness of the minimal time function in [26, Theorem 3.19].

Acknowledgements

This paper was s was supported by the Feng Tay project of National Yunlin University of Science and Technology, Taiwan. The authors are grateful to the referees for useful suggestions which improved this paper.

References

[1] M. Bounkhel, L. Thibault, On various notions of regularity of sets in nonsmooth analysis, Nonlinear Anal. 48 (2002), 223-246.
[2] M. Bounkhel, Subdifferential properties of minimal time functions associated with set-Valued mappings with closed convex graphs in Hausdorff topological vector spaces, J. Funct. Spaces Appl. 2013 (2013), Art. ID 707603.
[3] M. Bounkhel, Directional Lipschitzness of minimal time functions in Hausdorff topological vector spaces, Set-Valued Var. Anal. 22 (2014), 221–245.
[4] M. Bounkhel, On subdifferentials of a minimal time function in Hausdorff topological vector spaces, Appl. Anal. 93 (2014), 1761–1791.
[5] F.H. Clarke, Yu. S. Ledyaev, R.J. Stern, P.R. Wolenski, Nonsmooth Analysis and Control Theory, Graduate Texts in Mathematics, 178, Springer, New York, 1998.
[6] G. Colombo, P.R. Wolenski, The subgradient formula for the minimal time function in the case of constant dynamics in Hilbert space, J. Global Optim. 28 (2004), 269-282.
[7] G. Colombo, V. Goncharov, B. Mordukhovich, Well-posedness of minimal time problems with constant dynamics in Banach spaces, Set-Valued Var. Anal. 18 (2010), 349-372.
[8] G. Colombo, P.R. Wolenski, Variational analysis for a class of minimal time functions in Hilbert spaces, J. Convex Anal. 11 (2004), 335-361.
[9] M. Durea, M. Pantiruc, R. Strugariu, Minimal time function with respect to a set of directions: basic properties and applications, Optim. Method Softw. 31 (2016), 535-561.
[10] M. Durea, R. Strugariu, Vectorial penalization for generalized functional constrained problems, J. Global Optim. 68 (2017), 899–923.
[11] M. Durea, M. Pantiruc, R. Strugariu, A new type of directional regularity for mappings and applications to optimization, SIAM J. Optim. 27 (2017), 1204-1229.
[12] Y. He, K.F. Ng, Subdifferentials of a minimum time function in Banach spaces, J. Math. Anal. Appl. 321 (2006), 896-910.
[13] Y. Jiang, Y. He, Subdifferentials of a minimum time function in normed spaces, J. Math. Anal. Appl. 358 (2009), 410-418.
[14] Y. Jiang, Y. He, Subdifferential properties for a class of minimal time functions with moving target sets in normed spaces, Appl. Anal. 91 (2012), 491-502.
[15] B. S. Mordukhovich, Variational Analysis and Generalized Differentiation, I: Basic Theory, II: Applications, Springer, Berlin, 2006.
[16] B. S. Mordukhovich, N. M. Nam, Subgradients of distance functions with applications to Lipschitzian stability, Math. Program. 104 (2005), 635-668.
[17] B. S. Mordukhovich, N. M. Nam, Subgradients of distance functions at out-of-set points, Taiwanese J. Math. 10 (2006), 299-326.

[18] B.S. Mordukhovich, N.M. Nam, Limiting subgradients of minimal time functions in Banach spaces, J. Global Optim. 46 (2010), 615-633
[19] B.S. Mordukhovich, N.M. Nam, Subgradients of minimal time functions under minimal assumptions, J. Convex Anal. 18 (2011), 915-947.
[20] B. Mordukhovich, N.M. Nam, Applications of variational analysis to a generalized Fermat - Torricelli problem, J. Optim. Theory Appl. 148 (2011), 431-454.
[21] B. Mordukhovich, N.M. Nam, J. Salinas, Applications of variational analysis to a generalized Heron problem, Appl. Anal. 91 (2012), 1915-1942.
[22] N.M. Nam, N.T. An, C. Villalobos, Minimal time functions and the smallest intersecting ball problem with unbounded dynamics, J. Optim. Theory Appl. 154 (2012), 768-791.
[23] N.M. Nam, T.A. Nguyen, R.B. Rector, J. Sun, Nonsmooth algorithms and Nesterov's smoothing techniques for generalized Fermat-Torricelli problems, SIAM J. Optim. 24 (2014), 1815-1839.
[24] N.M. Nam, N. Hoang, A generalized Sylvester problem and a generalized Fermat-Torricelli problem, J. Convex Anal. 20 (2013), 669-687.
[25] N.M. Nam, D. V. Cuong, Subgradients of minimal time functions without calmness, J. Convex Anal. 26 (2019), 189-200.
[26] N.M. Nam, C. Zălinescu, Variational analysis of directional minimal time functions and applications to location problems, Set-Valued Var. Anal. 21 (2013), 405-430.
[27] N.M. Nam, G. Lafferriere, Lipschitz properties of nonsmooth functions and set-valued mappings via generalized differentiation, Nonlinear Anal. 89 (2013), 110-120.
[28] L. V. Nguyen, Variational analysis and regularity of the minimum time function for differential inclusions, SIAM J. Control Optim. 54 (2016), 2235-2258.
[29] S. Sun, Y. He, Exact characterztion for subdifferentials a speccial optimal value function, Optim Lett. 12 (2018), 519-534.
[30] P.R. Wolenski, Z. Yu, Proximal analysis and the minial time function. SIAM J. Control Optim. 36 (1998), 1048-1072.