J. Appl. Numer. Optim. 2 (2020), No. 2, pp. 187-197
Available online at http://jano.biemdas.com
https://doi.org/10.23952/jano.2.2020.2.05

WEAK CONVERGENCE OF A PRIMAL-DUAL ALGORITHM FOR SPLIT COMMON FIXED-POINT PROBLEMS IN HILBERT SPACES

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Abstract. In this paper, we use the dual variable to propose a new iterative algorithm for solving the split common fixed-point problem of quasi-nonexpansive mappings in real Hilbert spaces. Under suitable conditions, we establish a weak convergence theorem of the proposed algorithm and obtain a related result for the split common fixed-point problem of firmly quasi-nonexpansive mappings. Some numerical experiments are given to illustrate the efficiency of the proposed iterative algorithm.

Keywords. Iterative algorithm; Quasi-nonexpansive mapping; Split common fixed-point problem; Weak convergence.

1. INTRODUCTION

Let H_1 and H_2 be two real Hilbert spaces. The problem under consideration in this paper is formulated as finding

$$x^* \in F(U) \text{ such that } Ax^* \in F(T), \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $F(U)$ and $F(T)$ stand for the fixed point sets of $U : H_1 \to H_1$ and $T : H_2 \to H_2$, respectively. Problem (1.1) is called the split common fixed point problem (shortly, SCFP), which was introduced by Censor and Segal [5] in finite dimensional Hilbert spaces. In recent years, there has been growing interest in the SCFP due to its applications in the inverse problem of intensity-modulated radiation therapy and in the dynamic emission tomographic image reconstruction [2, 6, 10].

In particular, if U and T are projection operators, then the SCFP is reduced to the well-known split feasibility problem (SFP) [2, 7], which consists in finding

$$x^* \in C \text{ such that } Ax^* \in Q, \tag{1.2}$$

where C and Q are nonempty closed convex subsets of H_1 and H_2, respectively. Such problems arise in the field of intensity-modulated radiation therapy when one attempts to describe physical dose constraints and equivalent uniform dose constraints within a single model (see [6]). Since the SFP in finite dimensional Hilbert spaces is introduced first by Censor and Elfving [7], many algorithms have been proposed to solve the SFP (see [3, 4, 8, 13, 14, 16, 18, 21] and references therein).

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Received October 20, 2019; Accepted May 11, 2020.

For solving the SFP (1.2), Censor and Elfving [7] used multidistance method to obtain iterative algorithms, which involved matrix inverses at each step. In order to avoid usage of the inverse, Byrne [2] proposed an iterative method called CQ algorithm that involves only the orthogonal projections onto C and Q. The CQ algorithm is defined as follows:

$$x_{k+1} = P_C(I - \gamma A^*(I - P_Q)A)x_k \tag{1.3}$$

for each $k \in \mathbb{N}$, where P_C and P_Q are the (orthogonal) projection onto C and Q, respectively, and $\gamma \in (0, \frac{2}{\lambda})$ with λ being the spectral radius of the operator A^*A.

Note that, if split feasibility problem (1.2) is consistent (i.e., (1.2) has a solution), then it is no hard to see that x^* solves the SFP (1.2) if and only if it solves the fixed point equation:

$$P_C(I - \gamma A^*(I - P_Q)A)x^* = x^*,$$

where $\gamma > 0$ and A^* denotes the adjoint of A. This implies that we can use fixed point algorithms (see [1, 15, 19, 22, 23]) to solve the SFP.

For solving the SCFP (1.1) of directed operators (i.e. firmly quasi-nonexpansive mapping), Censor and Segal [5] proposed and proved, in finite dimensional spaces, the convergence of the following iterative scheme:

$$x_{k+1} = U(x_k - \gamma A^*(I - T)Ax_k), \tag{1.4}$$

where γ is chosen in the interval $(0, \frac{2}{\lambda})$ with λ being the spectral radius of the operator A^tA(A^t stands for matrix transposition). Many authors have introduced various algorithms to solve the SCFP (1.1) (see [9, 12, 17, 20, 24, 25]).

For solving the SCFP (1.1) of quasi-nonexpansive mappings, Moudafi [17] introduced the following relaxed algorithm:

$$x_{k+1} = (1 - \alpha_k)u_k + \alpha_k U(u_k), \;\; k \in N, \tag{1.5}$$

where $u_k = x_k + \gamma\beta A^*(T - I)Ax_k$, $\beta \in (0,1)$, $\alpha_k \in (0,1)$ and $\gamma \in (0, \frac{1}{\eta\beta})$ with η being the spectral radius of the operator A^*A. Moudafi proved the weak convergence of the algorithm in Hilbert spaces.

In [11], Chen, Huang and Zhang considered to minimize the sum of two proper lower semi-continuous convex functions, i.e.,

$$x^* = \arg\min_{x \in \mathbb{R}^n} f_1(x) + f_2(x), \tag{1.6}$$

where $f_1, f_2 \in \Gamma_0(\mathbb{R}^n)$ (all proper lower semi-continuous convex functions from $\mathbb{R}^n$ to $(-\infty, +\infty]$) and f_2 is differentiable on $\mathbb{R}^n$ with $1/\beta$-Lipschitz continuous gradient for some $\beta \in (0, +\infty)$. To solve convex separable problem (1.6), they obtained the following fixed point formulation: the point x^* is a solution of (1.6) if and only if there exists $v^* \in \mathbb{R}^m$ such that

$$\begin{cases} v^* = (I - prox_{\frac{\gamma}{\lambda} f_1})(x^* - \gamma \nabla f_2(x^*) + (1 - \lambda)v^*), \\ x^* = x^* - \gamma \nabla f_2(x) - \lambda v^*, \end{cases}$$

where λ and γ are two positive numbers. They introduced the following Picard iterative sequence:

$$\begin{cases} v_{k+1} = (I - prox_{\frac{\gamma}{\lambda} f_1})(x_k - \gamma \nabla f_2(x_k) + (1 - \lambda)v_k), \\ x_{k+1} = x_k - \gamma \nabla f_2(x_k) - \lambda v_{k+1}. \end{cases}$$

It was shown [11] that, under appropriate conditions, the sequence $\{x_k\}$ converges to a solution of problem (1.6). Since x is the primal variable related to (1.6), it is very natural to ask what role the variable v plays in above algorithm. Indeed, v is actually the dual variable of the primal-dual form related to (1.6).

Inspired and motivated by the works mentioned above, we use the dual variable to propose a new iterative algorithm for the SCFP governed by quasi-nonexpansive mappings. The organization of this paper is as follows. Some useful definitions and results are listed for the convergence analysis of the proposed iterative algorithms in Section 2. In Section 3, we introduce new iterative algorithm by the primal-dual method and the weak convergence theorem of the proposed iterative algorithm is obtained. We give a corollary for the SCFP (1.1) governed by firmly quasi-nonexpansive mappings. In Section 4, we also give some numerical experiments to illustrate the efficiency of the proposed iterative method.

2. Preliminaries

In this paper, we denote the inner product by $\langle\cdot,\cdot\rangle$ and the norm by $\|\cdot\|$. We use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively. We use $\omega_w(x_k) = \{x : \exists x_{k_j} \rightharpoonup x\}$ to stand for the weak ω-limit set of $\{x_k\}$. We use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively.

Definition 2.1. A mapping $T : H \to H$ is said to be

(i) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in H$;

(ii) firmly nonexpansive if $2T - I$ is nonexpansive or, equivalently,

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2$$

for all $x, y \in H$. Alternatively, a mapping $T : H \to H$ is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S),$$

where I denotes the identity mapping on H and $S : H \to H$ is a nonexpansive mapping;

(iii) quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - q\| \leq \|x - q\|$$

for all $x \in H$ and $q \in F(T)$;

(iv) firmly quasi-nonexpansive (also called directed operator) if $F(T) \neq \emptyset$ and

$$\|Tx - q\|^2 \leq \|x - q\|^2 - \|x - Tx\|^2$$

for all $x \in H$ and $q \in F(T)$.

Definition 2.2. A mapping $T : H \to H$ is said to be demiclosed at the origin if, for any sequence $\{x_n\}$ which converges weakly to x, the sequence $\{Tx_n\}$ converges strongly to 0, then $Tx = 0$.

Lemma 2.1. [17] *Let H be a real Hilbert space, and let $T : H \to H$ be a quasi-nonexpansive mapping. Set $T_\alpha = (1-\alpha)I + \alpha T$ for $\alpha \in [0,1)$. Then the following properties are reached, for all $(x,q) \in H \times F(T)$,*

(i) $\langle x - Tx, x - q\rangle \geq \frac{1}{2}\|x - Tx\|^2$ *and* $\langle x - Tx, q - Tx\rangle \leq \frac{1}{2}\|x - Tx\|^2$;
(ii) $\|T_\alpha x - q\|^2 \leq \|x - q\|^2 - \alpha(1-\alpha)\|Tx - x\|^2$;

(iii) $\langle x - T_\alpha x, x - q\rangle \geq \frac{\alpha}{2}\|x - Tx\|^2$.

Remark 2.1. If $T_\alpha = (1-\alpha)I + \alpha T$, where $T : H \to H$ is a quasi-nonexpansive mapping and $\alpha \in (0,1)$, then $F(T_\alpha) = F(T)$ and $\|T_\alpha x - x\|^2 = \alpha^2\|Tx - x\|^2$. It follows from (ii) of Lemma 2.1 that $\|T_\alpha x - q\|^2 \leq \|x - q\|^2 - \frac{1-\alpha}{\alpha}\|T_\alpha x - x\|^2$, which implies that T_α is firmly quasi-nonexpansive when $\alpha = \frac{1}{2}$. On the other hand, if $\hat{T}$ is a firmly quasi-nonexpansive mapping, we can easily obtain $\hat{T} = \frac{1}{2}I + \frac{1}{2}T$, where T is quasi-nonexpansive.

It follows from (iii) of Lemma 2.1 that the following result is easily obtained.

Proposition 2.1. *Let T be a quasi-nonexpansive mapping and $\alpha \in [0,1)$. If $T_\alpha = (1-\alpha)I + \alpha T$, then*

$$\|(I - T_\alpha)x\|^2 \leq 2\alpha\langle x - q, (I - T_\alpha)x\rangle$$

for all $(x,q) \in H \times F(T)$.

Lemma 2.2. [22] *Let K be a nonempty convex closed subset of real Hilbert space H. Let $\{x_k\}$ be a bounded sequence which satisfies the following properties:*

(a) *every weak limit point of $\{x_k\}$ lies in K;*
(b) $\lim_{k\to\infty}\|x_k - x\|$ *exists for every $x \in K$.*

Then $\{x_k\}$ converges weakly to a point in K.

3. Weak convergence results

In this section, we assume that the SCFP (1.1) is always consistent and its solution set is denoted by Γ, i.e.,

$$\Gamma = \{x \in F(U) : Ax \in F(T)\}.$$

We always assume the H_1 and H_2 are two real Hilbert spaces and $A : H_1 \to H_2$ is a bounded linear operator. Assume $U : H_1 \to H_1$ and $T : H_2 \to H_2$ be quasi-nonexpansive mappings and let $U_{\alpha_1} = (1-\alpha_1)I + \alpha_1 U$ and $T_{\alpha_2} = (1-\alpha_2)I + \alpha_2 T$, where $\alpha_1, \alpha_2 \in (0,1)$.

Now, we use the dual variable to propose the new iterative algorithm for the SCFP (1.1) governed by quasi-nonexpansive mapping.

Algorithm 3.1. Let $x_0, v_0 \in H_1$ be arbitrarily chosen, and $\lambda \in (0,1]$. For $k \geq 1$, let

$$\begin{cases} y_k = x_k - \gamma_k A^*(I - T_{\alpha_2})Ax_k, \\ v_{k+1} = (I - U_{\alpha_1})(y_k + (1-\lambda)v_k), \\ x_{k+1} = y_k - \lambda v_{k+1}, \end{cases}$$

where the stepsize γ_k satisfies

$$0 < \liminf_{k\to\infty}\gamma_k \leq \limsup_{k\to\infty}\gamma_k < \frac{1}{\alpha_2\|A\|^2}.$$

Theorem 3.1. *Let $\alpha_1 \in (0, \frac{1}{2}]$, and $I-U$ and $I-T$ be demiclosed at origin. Let $\{(v_k, x_k)\}$ be the sequence generated by Algorithm 3.1. Then $\{x_k\}$ converges weakly to a solution $x^* \in \Gamma$ and the sequence $\{(v_k, x_k)\}$ weakly converges to $(0, x^*)$.*

Proof. First, we show that $\lim_{k\to\infty} \|x_k - x^*\|$ exists for any $x^* \in \Gamma$. Taking $x^* \in \Gamma$, we have $x^* \in F(U)$ and $Ax^* \in F(T)$. By Algorithm 3.1 and Proposition 2.1, we have

$$\begin{aligned}
\|v_{k+1}\|^2 &= \|(I-U_{\alpha_1})(y_k + (1-\lambda)v_k)\|^2 \\
&= \|(I-U_{\alpha_1})(y_k + (1-\lambda)v_k) - (I-U_{\alpha_1})x^*\|^2 \\
&\le 2\alpha_1 \langle v_{k+1}, y_k - x^* + (1-\lambda)v_k\rangle
\end{aligned}$$

and

$$\begin{aligned}
\|x_{k+1} - x^*\|^2 &= \|y_k - \lambda v_{k+1} - x^*\|^2 \\
&= \|y_k - x^*\|^2 - 2\lambda\langle y_k - x^*, v_{k+1}\rangle + \lambda^2\|v_{k+1}\|^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\|x_{k+1} - x^*\|^2 + \lambda\|v_{k+1}\|^2 \\
&= \|y_k - x^*\|^2 - 2\lambda\langle y_k - x^*, v_{k+1}\rangle + \lambda^2\|v_{k+1}\|^2 + \lambda\|v_{k+1}\|^2 \\
&= \|y_k - x^*\|^2 - 2\lambda\langle y_k - x^*, v_{k+1}\rangle + \frac{1}{\alpha_1}\lambda\|v_{k+1}\|^2 - \lambda\left(\frac{1}{\alpha_1} - 1 - \lambda\right)\|v_{k+1}\|^2 \\
&\le \|y_k - x^*\|^2 - 2\lambda\langle y_k - x^*, v_{k+1}\rangle + 2\lambda\langle y_k - x^* + (1-\lambda)v_k, v_{k+1}\rangle \\
&\quad - \lambda\left(\frac{1}{\alpha_1} - 1 - \lambda\right)\|v_{k+1}\|^2 \\
&= \|y_k - x^*\|^2 + 2\lambda(1-\lambda)\langle v_k, v_{k+1}\rangle - \lambda\left(\frac{1}{\alpha_1} - 1 - \lambda\right)\|v_{k+1}\|^2.
\end{aligned} \tag{3.1}$$

Observe that

$$2\langle v_{k+1}, v_k\rangle = \|v_{k+1}\|^2 + \|v_k\|^2 - \|v_{k+1} - v_k\|^2. \tag{3.2}$$

From (3.1) and (3.2), we obtain

$$\begin{aligned}
\|x_{k+1} - x^*\|^2 + \lambda\|v_{k+1}\|^2 \le{}& \|y_k - x^*\|^2 + \lambda(1-\lambda)\|v_k\|^2 \\
&- \lambda(1-\lambda)\|v_{k+1} - v_k\|^2 - \lambda\left(\frac{1}{\alpha_1} - 2\right)\|v_{k+1}\|^2.
\end{aligned}$$

In view of

$$\begin{aligned}
\langle x_k - x^*, A^*(I-T_{\alpha_2})Ax_k\rangle &= \langle Ax_k - Ax^*, (I-T_{\alpha_2})Ax_k - (I-T_{\alpha_2})Ax^*\rangle \\
&\ge \frac{1}{2\alpha_2}\|(I-T_{\alpha_2})Ax_k - (I-T_{\alpha_2})Ax^*\|^2 \\
&= \frac{1}{2\alpha_2}\|(I-T_{\alpha_2})Ax_k\|^2,
\end{aligned}$$

we have

$$\begin{aligned}\|y_k - x^*\|^2 &= \|x_k - \gamma_k A^*(I - T_{\alpha_2})Ax_k - x^*\|^2\\ &\le \|x_k - x^*\|^2 - 2\gamma_k\langle x_k - x^*, A^*(I - T_{\alpha_2})Ax_k\rangle + \gamma_k^2\|A\|^2\|(I - T_{\alpha_2})Ax_k\|^2\\ &\le \|x_k - x^*\|^2 - \frac{1}{\alpha_2}\gamma_k\|(I - T_{\alpha_2})Ax_k\|^2 + \gamma_k^2\|A\|^2\|(I - T_{\alpha_2})Ax_k\|^2\\ &= \|x_k - x^*\|^2 - \gamma_k\Big(\frac{1}{\alpha_2} - \gamma_k\|A\|^2\Big)\|(I - T_{\alpha_2})Ax_k\|^2.\end{aligned}$$

It follows that

$$\begin{aligned}&\|x_{k+1} - x^*\|^2 + \lambda\|v_{k+1}\|^2\\ &\le \|x_k - x^*\|^2 - \gamma_k\Big(\frac{1}{\alpha_2} - \gamma_k\|A\|^2\Big)\|(I - T_{\alpha_2})Ax_k\|^2 + \lambda(1-\lambda)\|v_k\|^2\\ &\quad - \lambda(1-\lambda)\|v_{k+1} - v_k\|^2 - \lambda\Big(\frac{1}{\alpha_1} - 2\Big)\|v_{k+1}\|^2\\ &= \|x_k - x^*\|^2 + \lambda\|v_k\|^2 - \gamma_k\Big(\frac{1}{\alpha_2} - \gamma_k\|A\|^2\Big)\|(I - T_{\alpha_2})Ax_k\|^2 - \lambda^2\|v_k\|^2\\ &\quad - \lambda(1-\lambda)\|v_{k+1} - v_k\|^2 - \lambda\Big(\frac{1}{\alpha_1} - 2\Big)\|v_{k+1}\|^2.\end{aligned} \tag{3.3}$$

Let

$$s_k = \|x_k - x^*\|^2 + \lambda\|v_k\|^2.$$

By the assumptions on γ_k, λ, α_1, α_2 and (3.3), we obtain that $s_{k+1} \le s_k$, which implies that sequence $\{s_k\}$ is non-increasing. Since $\{s_k\}$ is lower bounded by 0, we have that $\lim_{k\to\infty} s_k$ exists. Thus it follows that $\{s_k\}$ is bounded. Hence $\{x_k\}$ is bounded.

Moreover, from (3.3), we also have

$$\gamma_k\Big(\frac{1}{\alpha_2} - \gamma_k\|A\|^2\Big)\|(I - T_{\alpha_2})Ax_k\|^2 + \lambda^2\|v_k\|^2 \le s_k - s_{k+1},$$

which implies that

$$\lim_{k\to\infty}\|(I - T_{\alpha_2})Ax_k\| = 0 \tag{3.4}$$

and

$$\lim_{k\to\infty}\|v_k\| = 0.$$

So, we obtain that $\lim_{k\to\infty}\|x_k - x^*\|^2 = \lim_{k\to\infty}(s_k - \lambda\|v_k\|^2) = \lim_{k\to\infty} s_k$ exists. In view of

$$\|x_k - y_k\| = \|-\gamma_k A^*(I - T_{\alpha_2})Ax_k\|,$$

we obtain from (3.4) that $\lim_{k\to\infty}\|x_k - y_k\| = 0$. This shows that

$$\lim_{k\to\infty}\|x_k - (y_k + (1-\lambda)v_k)\| = 0. \tag{3.5}$$

Next, we prove that $\omega_w(x_k) \subseteq \Gamma$. Taking $\bar{x} \in \omega_w(x_k)$, i.e., there exists a subsequence $\{x_{k_j}\}$ of $\{x_k\}$ such that $x_{k_j} \rightharpoonup \bar{x}$, we have $Ax_{k_j} \rightharpoonup A\bar{x}$ as $j \to \infty$. Using (3.5), we have $y_{k_j} + (1-\lambda)v_{k_j} \rightharpoonup \bar{x}$ as $j \to \infty$. Moreover,

$$\lim_{k\to\infty}\|v_{k+1}\| = \lim_{k\to\infty}\|(I - U_{\alpha_1})(y_k + (1-\lambda)v_k)\| = 0. \tag{3.6}$$

Note that $I-U$ and $I-T$ are demiclosed at 0. From $I-U_{\alpha_1}=(1-\alpha_1)(I-U)$ and $I-T_{\alpha_2}=(1-\alpha_2)(I-T)$, we have that $I-U_{\alpha_1}$ and $I-T_{\alpha_2}$ are also demiclosed at 0. Using (3.4) and (3.6), we have $\bar{x}\in F(U_{\alpha_1})=F(U)$ and $A\bar{x}\in F(T_{\alpha_2})=F(T)$, which imply that $\bar{x}\in\Gamma$. So $\omega_w(x_k)\subseteq\Gamma$ is proved.

Finally, by using Lemma 2.2, we have $x_k\rightharpoonup x^*$, where x^* is a solution of the SCFP (1.1). It follows from $v_k\to 0$ that $(v_k,x_k)\rightharpoonup(0,x^*)$. This completes the proof. □

Remark 3.1. For the particular case, "$\lambda=1$", Algorithm 3.1 becomes the following iterative algorithm for solving the SCFP (1.1) of quasi-nonexpansive mappings:

$$x_{k+1}=U_{\alpha_1}(x_k-\gamma_k A^*(I-T_{\alpha_2})x_k), \tag{3.7}$$

where $\alpha_1\in(0,\frac{1}{2}]$ and the stepsize γ_k satisfies

$$0<\liminf_{k\to\infty}\gamma_k\le\limsup_{k\to\infty}\gamma_k<\frac{1}{\alpha_2\|A\|^2}.$$

Algorithm 3.7 can be rewritten as

$$x_{k+1}=\big((1-\alpha_1)I+\alpha_1 U\big)(x_k-\gamma_k\alpha_2 A^*(I-T)x_k),$$

which becomes algorithm (1.5) proposed by Moudafi [17] for solving the SCFP (1.1) of quasi-nonexpansive mappings.

We now turn our attention to the application of the proposed algorithm to the SCFP (1.1) governed by firmly quasi-nonexpansive mappings. Since any firmly quasi-nonexpansive mapping is quasi-nonexpansive, we can straightly obtain Algorithm 3.1 for solving the SCFP (1.1) of firmly quasi-nonexpansive mappings. From Remark 2.1, we know that any firmly quasi-nonexpansive mapping can be expressed be the $\frac{1}{2}$-relaxed operator of quasi-nonexpansive mapping. Algorithm 3.1 takes the following equivalent form for solving the SECFP (1.1) of firmly quasi-nonexpansive mappings U and T.

Algorithm 3.2. Let $x_0,v_0\in H_1$ be arbitrarily chosen, and $\lambda\in(0,1]$. For $k\ge 1$, let

$$\begin{cases} y_k=x_k-\gamma_k A^*(I-T)Ax_k,\\ v_{k+1}=(I-U)(y_k+(1-\lambda)v_k),\\ x_{k+1}=y_k-\lambda v_{k+1},\end{cases}$$

where the stepsize γ_k satisfies

$$0<\liminf_{k\to\infty}\gamma_k\le\limsup_{k\to\infty}\gamma_k<\frac{2}{\|A\|^2}.$$

Corollary 3.1. *Let $U:H_1\to H_1$ and $T:H_2\to H_2$ be firmly quasi-nonexpansive mappings. Assume that $I-U$ and $I-T$ are demiclosed at origin. Let the sequence $\{(v_k,x_k)\}$ be generated by Algorithm 3.2. Then the sequence $\{x_k\}$ converges weakly to a solution $x^*\in\Gamma$ and the sequence $\{(v_k,x_k)\}$ weakly converges to $(0,x^*)$.*

Remark 3.2. For the particular case, "$\lambda=1$", Algorithm 3.2 becomes the following iterative algorithm for solving the SCFP (1.1) of firmly quasi-nonexpansive mappings:

$$x_{k+1}=U(x_k-\gamma_k A^*(I-T)x_k), \tag{3.8}$$

where the stepsize γ_k satisfies

$$0 < \liminf_{k\to\infty} \gamma_k \le \limsup_{k\to\infty} \gamma_k < \frac{2}{\|A\|^2}.$$

Algorithm (3.8) becomes algorithm (1.4) which was introduced by Censor and Segal [5] for solving the SCFP (1.1) of firmly quasi-nonexpansive mappings.

4. Numerical experiments

In this section, we provide some numerical experiments and show the performance of the proposed primal-dual iterative Algorithm 3.1 for solving the SCFP (1.1). All the codes are written in MATLAB and are performed on a personal Lenovo computer with Intel(R) Core(TM)i7-6500U CPU @ 2.5GHz and RAM 8.00GB.

Example 4.1. Let R^2 be the two dimensional Euclidean space with inner product

$$\langle x,y\rangle = x^{(1)}y^{(1)} + x^{(2)}y^{(2)}$$

and the norm $\|x\| = \sqrt{(x^{(1)})^2 + (x^{(2)})^2}$ for all $x = (x^{(1)}, x^{(2)})^T$, $y = (y^{(1)}, y^{(2)})^T \in R^2$. Defined $U : R^2 \to R^2$ by

$$U : x = (x^{(1)}, x^{(2)})^T \mapsto (x^{(1)}, \sin x^{(2)})^T.$$

Obviously, U is quasi-nonexpansive mapping and the set of fixed points of U, denoted by $F(U) = \{(x^{(1)}, 0) | x^{(1)} \in R\}$, is not empty. Let the nonempty closed convex set $Q_1 = \{y = (y^{(1)}, y^{(2)})^T \in R^2 |\ 1 \le y_j \le 2, j = 1,2\}$ and $Q_2 = \{y \in R^2 | \|y-(2,1.5)^T\| \le 1\}$. Let the matrix

$$A = \begin{pmatrix} 2 & -1 \\ 2 & 2 \end{pmatrix}.$$

We consider the following problem:

$$\text{finding } \ x^* \in F(U) \ \text{ such that } \ Ax^* \in Q_1 \bigcap Q_2. \tag{4.1}$$

We know the metric projections onto the sets Q_1 and Q_2 have closed-form expressions. We can compute the projection onto the set Q_1 with

$$P_{Q_1}(x) = \{z = (z^{(1)}, z^{(2)})^T | z^{(i)} = max\{1, min\{x^{(i)}, 2\}\}, i = 1,2\}, x = (x^{(1)}, x^{(2)})^T \in R^2.$$

Since the set Q_2 is a closed ball, the projection onto the set Q_2 can be computed with

$$P_{Q_2}(x) := \begin{cases} c + \frac{r}{\|x-c\|^2}(x-c), & \|x-c\| > r, \\ x, & \|x-c\| \le r, \end{cases}$$

where $c = (2,1.5)^T$ and $r = 1$. Taking $T = P_{Q_2}P_{Q_1}$, we have T is quasi-nonexpansive mapping and

$$F(T) = F(P_{Q_2}) \bigcap F(P_{Q_1}) = Q_1 \bigcap Q_2,$$

where $F(T)$ is the nonempty set of fixed points of T. The problem (4.1) becomes the following SCFP:

$$\text{finding } \ x^* \in F(U) \ \text{ such that } \ Ax^* \in F(T). \tag{4.2}$$

Now we turn to realizing primal-dual iterative Algorithm 3.1 for approximating a solution of problem (4.2) by using Theorem 3.1. We take $\alpha_1 = \alpha_2 = \frac{1}{2}$ and

$$p_k = \|x_k - Ux_k\| + \|Ax_k - TAx_k\| < \varepsilon = 10^{-5}$$

TABLE 1. Numerical results for solving Example 4.1 with different λ.

$x_0 = (-5,-4)^T$, $v_0 = (0,0)^T$			
λ	k	t	p_k
0.1	13	0.0001	9.5973×10^{-6}
0.2	13	0.0001	7.5569×10^{-7}
0.6	12	0.0001	5.9308×10^{-7}
1(algorithm (3.7))	2756	0.2500	9.9974×10^{-6}

TABLE 2. Numerical results for solving Example 4.1 with different λ.

$x_0 = (-5,0)^T$, $v_0 = (1,0)^T$			
λ	k	t	p_k
0.6	10	0.0001	2.7678×10^{-6}
0.7	10	0.0001	2.3466×10^{-7}
0.8	1827	0.1719	9.9978×10^{-6}
1(algorithm (3.7))	3581	0.2656	9.9981×10^{-6}

TABLE 3. Numerical results for solving Example 4.1 with different λ.

$x_0 = (10,0)^T$, $v_0 = (0,0)^T$			
λ	k	t	p_k
0.3	14	0.0001	2.0679×10^{-6}
0.6	13	0.0001	1.7704×10^{-6}
0.9	2184	0.1406	9.9973×10^{-6}
1(algorithm (3.7))	2682	0.2188	9.9973×10^{-6}

as the stopping criterion. We take $x_0 = (x_0^{(1)}, x_0^{(2)})^T$ and $v_0 = (v_0^{(1)}, v_0^{(2)})^T$ as initial points.

In all the tables below, 'k' and 't'denote the number of iterations and the total computing time in seconds. We take $\gamma_k = \frac{1.95}{\|A\|^2}$ and

$$p_k = \|x_k - Ux_k\| + \|Ax_k - TAx_k\|$$

as error estimation of our algorithm. We take different values of λ, x_0 and v_0 for solving this example in Table 1-3. When the parameter $\lambda = 1$, the Algorithm 3.1 becomes algorithm (3.7) which was proposed by Moudafi [17] for solving the SCFP (1.1) of quasi-nonexpansive mappings. We take $x_0 = (-5,-4)$, $v_0 = (0,0)$; $x_0 = (-5,0)$, $v_0 = (1,0)$; $x_0 = (10,0)$, $v_0 = (0,0)$ in numerical experiments. We compare our proposed Algorithm 3.1 with algorithm (3.7).

We can see from Table 1-3 that Algorithm 3.1 is efficient and behaved better than algorithm (3.7) if we choose a suitable parameter λ for solving Example 4.1. So, the proper choice of the parameter $\lambda \in (0,1]$ may accelerate the convergence.

5. The conclusion

In this paper, we may take parameter $\lambda \in (0,1]$. If $\lambda = 1$, then Algorithm 3.1 is reduced to algorithm (3.7), which was proposed by Moudafi [17] for solving the SCFP (1.1) of quasi-nonexpansive mappings. Similarly, if $\lambda = 1$, then Algorithm 3.1 is reduced to algorithm (3.8) (i.e. (1.4)), which was introduced by Censor and Segal [5] for solving the SCFP (1.1) of firmly quasi-nonexpansive mappings.

Acknowledgements
The authors are grateful the anonymous referee for useful comments which improved this paper. This paper was supported by the Scientific Research Project of Tianjin Municipal Education Commission (No. 2018KJ253).

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