J. Appl. Numer. Optim. 2 (2020), No. 1, pp. 101-107
Available online at http://jano.biemdas.com
https://doi.org/10.23952/jano.2.2020.1.07

INEXACT ORBITS OF SET-VALUED NONEXPANSIVE MAPPINGS WITH SUMMABLE ERRORS

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Abstract. Given a set-valued nonexpansive mapping which acts on a metric space, we study the convergence of its inexact iterates to its attracting sets in the case where the errors are summable. In particular, we extend results which are known to hold for single-valued nonexpansive self-mappings of metric spaces.

Keywords. Attracting set; Inexact iteration; Metric space; Nonexpansive mapping.

1. Introduction

During more than fifty-five years now, there has been a lot of activity regarding the fixed point theory of nonexpansive (that is, 1-Lipschitz) mappings. See, for example, [3, 9, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 25, 26] and references cited therein. This activity stems from Banach's classical theorem [1] regarding the existence of a unique fixed point for a strict contraction. It also concerns the convergence of (inexact) iterates of a nonexpansive mapping to one of its fixed points. Since that seminal result, many developments have taken place in this field including, in particular, studies of feasibility and common fixed point problems, which find important and diverse applications in the physical, engineering and medical sciences [8, 10, 22, 23, 24, 25, 26].

For example, in [5], it was shown that if every exact orbit of a nonexpansive mapping converges to one of its fixed points, then this convergence property also holds for its inexact orbits with summable errors. This result was obtained for a single-valued nonexpansive self-mapping of a complete metric space.

In this paper, we establish three variants of this result for inexact orbits of *set-valued* nonexpansive mappings with summable errors.

Such results have already found interesting applications. They are, for instance, an important ingredient in the study of superiorization and perturbation resilience of algorithms. See [2, 4, 6, 7] and references mentioned therein. The superiorization methodology works by taking an iterative algorithm, investigating its perturbation resilience, and then using proactively such perturbations in order to "force" the perturbed algorithm to do, in addition to its original task, something useful.

This methodology can be explained by the following result on convergence of inexact iterates.

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Received August 14, 2019; Accepted December 15, 2019.

Assume that $(X, \|\cdot\|)$ is a Banach space, the mapping $A: X \to X$ is nonexpansive, and that for each $x \in X$, the sequence $\{A^n x\}_{n=1}^{\infty}$ converges in the norm topology. Assume further that the point x_0 belongs to X, $\{\beta_k\}_{k=0}^{\infty}$ is a sequence of positive numbers satisfying

$$\sum_{k=0}^{\infty} \beta_k < \infty, \tag{1.1}$$

$\{v_k\}_{k=0}^{\infty} \subset X$ is a norm bounded sequence and for each integer $k \geq 0$,

$$x_{k+1} = A(x_k + \beta_k v_k). \tag{1.2}$$

Then it was shown in [5] that the sequence $\{x_k\}_{k=0}^{\infty}$ converges in the norm topology of X and its limit is a fixed point of A. In this case the mapping A is said to be bounded perturbation resilient (see [6]). In other words, if all the exact iterates of a nonexpansive mapping converge, then its inexact iterates with bounded summable perturbations converge too.

Now assume that the point $x_0 \in X$ and the sequence $\{\beta_k\}_{k=0}^{\infty}$ satisfying (1.1) are given, and that we need to find an (approximate) fixed point of A. In order to meet this goal, we construct a sequence $\{x_k\}_{k=1}^{\infty}$ defined by (1.2). Under an appropriate choice of the norm bounded sequence $\{v_k\}_{k=0}^{\infty}$, the sequence $\{x_k\}_{k=1}^{\infty}$ also possesses some useful property. For example, the sequence $\{f(x_k)\}_{k=1}^{\infty}$ can be decreasing, where f is a given function.

We remark in passing that the convergence of perturbed iterates of set-valued mappings was also studied in [20].

Let (X, ρ) be a metric space. For each $x \in X$ and each $r > 0$, set

$$B(x,r) := \{y \in X : \rho(x,y) \leq r\}.$$

For each $x \in X$ and each nonempty set $A \subset X$, define

$$\rho(x,A) := \inf\{\rho(x,y) : y \in A\}.$$

Finally, for each pair of nonempty sets $A, B \subset X$, define

$$H(A,B) := \max\{\sup_{x \in A} \rho(x,B), \sup_{y \in B} \rho(x,A)\}.$$

Assume that a mapping $T: X \to 2^X \setminus \{\emptyset\}$ is such that $T(x)$ is closed for each $x \in X$, for each $x, y \in X$, we have

$$H(Tx, Ty) \leq \rho(x,y) \tag{1.3}$$

and that $F \subset X$ is a nonempty closed set.

In this paper, we obtain three results (see Theorems 2.1, 3.1 and 4.1 below). They are stated and proved in Sections 2, 3 and 4, respectively.

2. The first result

Theorem 2.1. *Assume that the following property holds:*

(a) for each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that

$$x_{i+1} \in T(x_i),\ i = 0, 1, \ldots,$$

the relation

$$\lim_{i \to \infty} \rho(x_i, F) = 0$$

is true.

Let $\{\gamma_i\}_{i=1}^{\infty} \subset (0,\infty)$ *satisfy*

$$\sum_{i=1}^{\infty} \gamma_i < \infty$$

and let $\{x_i\}_{i=0}^{\infty} \subset X$ *satisfy, for each integer* $i \geq 0$,

$$\rho(x_{i+1}, Tx_i) \leq \gamma_{i+1}. \tag{2.1}$$

Then

$$\lim_{i\to\infty} \rho(x_i, F) = 0.$$

Proof. Let $\varepsilon > 0$. Choose a natural number n_0 such that

$$\sum_{i=n_0}^{\infty} \gamma_i < \varepsilon/6.$$

Set

$$y_{n_0} = x_{n_0}.$$

Assume that $n \geq n_0$ is an integer, that a finite sequence $\{y_i\}_{i=n_0}^{n} \subset X$ has already been defined and that for each natural number i satisfying $n_0 < i \leq n$, we have

$$\rho(x_i, y_i) \leq 3 \sum_{j=n_0+1}^{i} \gamma_j. \tag{2.2}$$

We now proceed to define $y_{n+1} \in X$. In view of (1.1), we have

$$H(Tx_n, Ty_n) \leq \rho(x_n, y_n). \tag{2.3}$$

From (2.1), we see that there exists a point

$$z \in Tx_n \tag{2.4}$$

such that

$$\rho(x_{n+1}, z) \leq 2\gamma_{n+1}. \tag{2.5}$$

It follows from (2.3) and (2.4) that

$$\rho(z, Ty_n) \leq \rho(x_n, y_n).$$

Hence, there exists a point

$$y_{n+1} \in Ty_n$$

such that

$$\rho(z, y_{n+1}) \leq \rho(x_n, y_n) + \gamma_{n+1}. \tag{2.6}$$

In view of (2.5) and (2.6), we have

$$\begin{aligned}\rho(x_{n+1}, y_{n+1}) &\leq \rho(x_{n+1}, z) + \rho(z, y_{n+1}) \\ &\leq 2\gamma_{n+1} + \rho(x_n, y_n) + \gamma_{n+1}\end{aligned}$$

and

$$\rho(x_{n+1}, y_{n+1}) \leq \rho(x_n, y_n) + 3\gamma_{n+1}.$$

From (2.2), we see that the assumption made for n also holds for $n+1$. Thus we have constructed by induction a sequence $\{y_i\}_{i=n_0}^{\infty}$ such that (2.2) holds for all integers $i > n_0$. It follows from (2.2) that, for all integers $k \geq n_0$,

$$\rho(x_k, y_k) \leq 3 \sum_{i=n_0}^{k} \gamma_i < \varepsilon/2. \tag{2.7}$$

and

$$y_{k+1} \in T y_k.$$

Property (a) implies that

$$\lim_{i \to \infty} \rho(y_i, F) = 0. \tag{2.8}$$

In view of (2.7) and (2.8), we have, for all sufficiently large natural numbers k,

$$\rho(x_k, F) < \varepsilon/2.$$

Since ε is an arbitrary positive number, we conclude that

$$\lim_{k \to \infty} \rho(x_k, F) = 0.$$

This completes the proof of Theorem 2.1. □

3. The second result

Theorem 3.1. *Assume that the following property holds:*

(b) for each sequence $\{x_i\}_{i=0}^{\infty} \subset X$ such that

$$x_{i+1} \in T(x_i),\ i = 0, 1, \ldots,$$

the relation

$$\liminf_{i \to \infty} \rho(x_i, F) = 0$$

is true.

Let $\{\gamma_i\}_{i=1}^{\infty} \subset (0, \infty)$ satisfy

$$\sum_{i=i}^{\infty} \gamma_i < \infty$$

and let $\{x_i\}_{i=0}^{\infty} \subset X$ satisfy, for each integer $i \geq 0$,

$$\rho(x_{i+1}, T x_i) \leq \gamma_{i+1}.$$

Then

$$\liminf_{i \to \infty} \rho(x_i, F) = 0.$$

Proof. Let $\varepsilon > 0$. Choose a natural number n_0 such that

$$\sum_{i=n_0}^{\infty} \gamma_i < \varepsilon/6.$$

Arguing as in the proof of Theorem 2.1, we construct a sequence $\{y_i\}_{i=n_0}^{\infty} \subset X$ such that

$$y_{n_0} = x_{n_0},$$

$$y_{i+1} \in T(y_i) \text{ for all integer } i \geq n_0 \tag{3.1}$$

and

$$\rho(x_i, y_i) \le \varepsilon \text{ for all integer } i \ge n_0. \tag{3.2}$$

Property (b) and (3.1) imply that

$$\liminf_{i\to\infty} \rho(y_i, F) = 0. \tag{3.3}$$

In view of (3.2) and (3.3), we have

$$\liminf_{i\to\infty} \rho(x_i, F) \le \varepsilon.$$

Since ε is an arbitrary positive number, we conclude that

$$\liminf_{k\to\infty} \rho(x_k, F) = 0.$$

Theorem 2.1 is proved. □

4. The third result

For every set A, we denote by Card(A) its cardinality.

Theorem 4.1. *Assume that the following property holds:*

(c) for each sequence $\{z_i\}_{i=0}^{\infty} \subset X$ such that

$$z_{i+1} \in T(z_i),\ i = 0, 1, \dots,$$

and each $\varepsilon > 0$, we have

$$\lim_{n\to\infty} Card(\{i \in \{0, \dots, n-1\} :\ \rho(z_i, F) > \varepsilon\})n^{-1} = 0.$$

Let $\{\gamma_i\}_{i=1}^{\infty} \subset (0, \infty)$ satisfy

$$\sum_{i=i}^{\infty} \gamma_i < \infty$$

and let $\{x_i\}_{i=0}^{\infty} \subset X$ satisfy for each integer $i \ge 0$,

$$x_{i+1} \in T(x_i)$$

and

$$\rho(x_{i+1}, Tx_i) \le \gamma_{i+1}.$$

Then, for each $\varepsilon > 0$,

$$\lim_{n\to\infty} Card(\{i \in \{0, \dots, n-1\} :\ \rho(x_i, F) > \varepsilon\})n^{-1} = 0.$$

Proof. Let $\varepsilon > 0$. Choose a natural number n_0 such that

$$\sum_{i=n_0}^{\infty} \gamma_i < \varepsilon/6.$$

Arguing as in the proof of Theorem 2.1, we construct a sequence $\{y_i\}_{i=n_0}^{\infty} \subset X$ such that

$$y_{n_0} = x_{n_0},$$

$$y_{i+1} \in T(y_i) \text{ for all integer } i \ge n_0 \tag{4.1}$$

and

$$\rho(x_i, y_i) \le \varepsilon/2 \text{ for all integer } i \ge n_0. \tag{4.2}$$

Property (c) and (4.1) imply that

$$\lim_{k\to\infty} \mathrm{Card}(\{i \in \{n_0,\dots,k-1\} : \rho(y_i,F) > \varepsilon/2\})k^{-1} = 0. \tag{4.3}$$

Let $n > n_0$ be an integer. By (4.2) and (4.3), we have

$$\begin{aligned}&\{i \in \{n_0,\dots,n-1\} : \rho(x_i,F) > \varepsilon\}\\ &\subset \{i \in \{n_0,\dots,n-1\} : \rho(y_i,F) > \varepsilon/2\}\end{aligned}$$

and

$$\begin{aligned}&n^{-1}\mathrm{Card}(\{i \in \{0,\dots,n-1\} : \rho(x_i,F) > \varepsilon\})\\ &\le n^{-1}(n_0 + \mathrm{Card}(\{i \in \{n_0,\dots,n-1\} : \rho(y_i,F) > \varepsilon/2\})\\ &\to 0\end{aligned}$$

as $n \to \infty$. This completes the proof of Theorem 4.1. □

Remark 4.1. It is not difficult to see that the above result remains true if we replace the n^{-1} factor by t_n^{-1}, where $\{t_n\}_{n=1}^{\infty}$ is any strictly increasing sequence of natural numbers.

Acknowledgements

The first author was partially supported by the Israel Science Foundation (Grant No. 820/17), by the Fund for the Promotion of Research at the Technion and by the Technion General Research Fund. Both authors thank the three referees for their helpful comments and suggestions.

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