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THE ROLE OF NONLINEAR SCALARIZATION FUNCTIONS IN CHARACTERIZING GENERALIZED CONVEX VECTOR FUNCTIONS

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Abstract. The aim of this paper is to present new characterizations of cone-convex and explicitly cone-quasiconvex vector functions with respect to a proper closed solid convex cone of a real linear topological space. These characterizations are given in terms of classical convexity and explicit quasiconvexity of certain real-valued functions, defined by means of the nonlinear scalarization function introduced by Gerstewitz (Tammer) in 1983.

Keywords. Cone-convexity; Cone-quasiconvexity; Explicit cone-quasiconvexity; Nonlinear scalarization; Linear perturbation.

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1. INTRODUCTION

Generalized convex functions play an important role in scalar/vector/set optimization (see, e.g., Avriel *et al.* [2], Bagdasar and Popovici [3], Cambini and Martein [6], Göpfert *et al.* [10], Günther and Tammer [13], Jahn [14], Khan, Tammer and Zălinescu [15], La Torre and Popovici [17], Luc [19], Popovici [21], and the references therein).

In this paper, we study three concepts of generalized convexity for vector functions (defined on a nonempty convex subset D of a real linear space X and taking values in a real linear topological space Y, endowed with a convex cone C), namely, the C-convexity, C-quasiconvexity and explicit C-quasiconvexity.

These concepts were introduced by Luenberger [20], Borwein [5], Luc [19], and Popovici [22], respectively. They are natural extensions of the classical notions of convexity, quasiconvexity and explicit quasiconvexity of real-valued functions, since, in the particular framework of the finite-dimensional real Euclidean space $Y = \mathbb{R}^m$, partially ordered by the standard cone $C = \mathbb{R}^m_+$, a vector function $f = (f_1, \dots, f_m)$ is C-convex (C-quasiconvex, explicitly C-quasiconvex) if and only if its scalar components are convex (quasiconvex, explicit quasiconvex) in classical sense.

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However, since such a componentwise approach cannot be used in general spaces, an interesting topic related to scalarization in vector optimization (initiated by Luc [19]) is to characterize generalized convex vector functions in terms of classical generalized convexity of certain real-valued functions. Indeed, it is nowadays known that, under appropriate assumptions, C-convex functions can be characterized by means of the elements of the polar cone C^+ of C (see, e.g., Luc [19]), while C-quasiconvex and explicitly C-quasiconvex functions can be characterized by means of extreme directions of C^+ (see, e.g., Benoist, Borwein, Popovici [4] and Luc [19], and Günther and Popovici [11], respectively).

Another approach is to use the nonlinear scalarization functions, introduced by Gerstewitz (Tammer) in [8], instead of the linear functionals from the polar cone. The first characterization of this type has been obtained by Luc [19] for C-quasiconvex vector functions (see Proposition 3.1). La Torre, Popovici and Rocca [18] established a similar characterization for C-convex vector functions, assuming that Y is a normed space and C satisfies a special property somewhat more general than polyhedrality (see Theorem 3.1). In our recent paper [11], we also proved that such a characterization holds for explicitly C-quasiconvex vector functions in the particular framework of $Y = \mathbb{R}^m$ and a polyhedral cone C. The principal aim of this paper is to show that this characterization of explicit C-quasiconvexity is still valid for general spaces and cones (see Theorem 3.3). Moreover, by using a vectorial variant of Crouzeix's characterization of convex functions by quasiconvexity of their linear perturbations, obtained by Kuroiwa, Popovici and Rocca [16] (see Proposition 2.10), we establish a new characterization of C-convex functions by combining the nonlinear scalarization functions and the linear perturbations (see Theorem 3.2).

The rest of the paper is organized as follows. In Section 2, we recall the definition of the nonlinear scalarization functions and some of their fundamental properties. Then we present some notions of generalized convexity for scalar and vector functions that will be used in the sequel. Section 3 contains our main results concerning the characterization of generalized convex vector functions by means of nonlinear scalarization functions. This paper concludes with an outlook to future work in Section 4.

2. Preliminaries

Throughout this paper, we assume that X is a real linear space, Y is a real topological linear space, $D \subseteq X$ is a nonempty convex set, and $C \subseteq Y$ is a convex cone, i.e., $0 \in C = \mathbb{R}_+ \cdot C = C + C$, where 0 stands for the origin of Y while $\mathbb{R}_+$ is the set of all nonnegative real numbers. Moreover, we assume that this cone is proper, closed and solid, i.e., $C = \operatorname{cl} C \neq Y$ and $\operatorname{int} C \neq \emptyset$.

2.1. **Nonlinear scalarization functions.** Given a point $e \in \operatorname{int} C$, we denote by $\sigma : Y \to \mathbb{R}$ the nonlinear scalarization function in the sense of Gerstewitz (Tammer) [8], defined by, for all $y \in Y$,

$$\sigma(y) := \min\{s \in \mathbb{R} \mid y \in se - C\}. \tag{2.1}$$

We also associate to every point $a \in Y$ a function $\sigma_a : Y \to \mathbb{R}$, defined by, for all $y \in Y$,

$$\sigma_a(y) = \sigma(y - a),$$

the initial function σ being recovered as σ_0. These functions have found many applications in vector/set optimization, being also known under different names in mathematical economics (see, e.g., Göpfert *et al.* [10], Khan, Tammer and Zălinescu [15], Luc [19], Tammer and

Zălinescu [23] and the references therein). The next two results gather several useful properties of the nonlinear scalarization functions (see, e.g., Gerstewitz (Tammer) [8], Gerth (Tammer) and Weidner [9], and Göpfert *et al.* [10, Th. 2.3.1]).

Proposition 2.1. *The following representations hold:*

$$\{y \in Y \mid \sigma(y) \le 0\} = -C, \tag{2.2}$$

$$\{y \in Y \mid \sigma(y) < 0\} = -\operatorname{int} C. \tag{2.3}$$

Moreover, for any $\lambda \in \mathbb{R}$ and $y \in Y$, we have

$$\sigma(y - \lambda e) = \sigma(y) - \lambda. \tag{2.4}$$

Proposition 2.2. *For any point $a \in Y$, the nonlinear scalarization function $\sigma_a : Y \to \mathbb{R}$ is convex and C-increasing, i.e., for any points $y, y' \in Y$,*

$$y \in y' - C \implies \sigma_a(y) \le \sigma_a(y').$$

The nonlinear scalarization function σ_a can be described explicitly for certain cones, as shown below. In what follows, for any $n \in \mathbb{N}$ it will be convenient to denote

$$I_n := \{1, \ldots, n\}.$$

Example 2.1. Consider the particular framework of a finite-dimensional real Euclidean space $Y = \mathbb{R}^m$ endowed with a polyhedral cone (see Aliprantis and Tourky [1]) of type

$$C = \{v \in \mathbb{R}^m \mid \langle v, u^i \rangle \ge 0,\ \forall i \in I_p\},$$

where $p \in \mathbb{N}$ and $u^1, \ldots, u^p \in \mathbb{R}^m$ satisfy the conditions

$$0 \notin \operatorname{conv}\{u^1, \ldots, u^p\} \quad \text{and} \quad u^i \notin \sum_{j \in I_p \setminus \{i\}} \mathbb{R}_+ \cdot u^j,\ \forall i \in I_p,$$

the latter sum of sets being empty when $p = 1$. As shown by us in [11], C is proper and solid. Moreover, for an arbitrary fixed point $e \in \operatorname{int} C$ and any $a \in \mathbb{R}^m$ the nonlinear scalarization function $\sigma_a : \mathbb{R}^m \to \mathbb{R}$ admits the following explicit representation (see, e.g., Günther and Popovici [11, 12]):

$$\sigma_a(y) = \max_{i \in I_p} \frac{\langle y - a, u^i \rangle}{\langle e, u^i \rangle}$$

for all $y \in \mathbb{R}^m$. Notice that, when $p = m$ and $u^1 = (1, 0, \ldots, 0), \ldots, u^m = (0, \ldots, 0, 1)$ are the canonical unit vectors in $\mathbb{R}^m$, we recover the standard ordering cone $C = \mathbb{R}^m_+$. Then, by choosing $e = (1, \ldots, 1)$, we conclude that for any point $a = (a_1, \ldots, a_m) \in \mathbb{R}^m$ the function σ_a is given by

$$\sigma_a(y) = \max_{i \in I_m} (y_i - a_i)$$

for all $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$.

2.2. **Generalized convex scalar functions.** We recall some classical concepts of generalized convexity that play an important role in scalar optimization (see, e.g., Avriel *et al.* [2]).

Definition 2.1. A function $\varphi : D \to \mathbb{R}$ is said to be convex if, for all $x, x' \in D$ and $t \in]0,1[$,

$$\varphi((1-t)x + tx') \leq (1-t)\varphi(x) + t\varphi(x').$$

Proposition 2.3. *For any function $\varphi : D \to \mathbb{R}$, the following assertions are equivalent :*

1° *φ is convex.*
2° *For any $\lambda, \lambda' \in \mathbb{R}$, $x, x' \in D$ and $t \in]0,1[$,*

$$\varphi(x) \leq \lambda \text{ and } \varphi(x') \leq \lambda' \implies \varphi((1-t)x + tx') \leq (1-t)\lambda + t\lambda'.$$

Definition 2.2. A function $\varphi : D \to \mathbb{R}$ is said to be quasiconvex if, for all $x, x' \in D$ and $t \in]0,1[$,

$$\varphi((1-t)x + tx') \leq \max\{\varphi(x), \varphi(x')\}. \tag{2.5}$$

Proposition 2.4. *For any function $\varphi : D \to \mathbb{R}$, the following assertions are equivalent :*

1° *φ is quasiconvex.*
2° *For any $\lambda \in \mathbb{R}$, $x, x' \in D$ and $t \in]0,1[$,*

$$\varphi(x) \leq \lambda \text{ and } \varphi(x') \leq \lambda \implies \varphi((1-t)x + tx') \leq \lambda.$$

Definition 2.3. A function $\varphi : D \to \mathbb{R}$ is called explicitly quasiconvex if φ is quasiconvex and strict inequality holds in (2.5) whenever $\varphi(x) \neq \varphi(x')$.

Proposition 2.5. *For any function $\varphi : D \to \mathbb{R}$, the following assertions are equivalent :*

1° *φ is explicitly quasiconvex.*
2° *For any $\lambda \in \mathbb{R}$, $x, x' \in D$ and $t \in]0,1[$,*

$$\varphi(x) \leq \lambda \text{ and } \varphi(x') < \lambda \implies \varphi((1-t)x + tx') < \lambda.$$

Remark 2.1. The following facts concerning scalar functions are well-known:

a) Convex functions are explicitly quasiconvex, hence quasiconvex.

b) Quasiconvexity does not imply explicit quasiconvexity (as shown by the following example).

Example 2.2. Consider the function $\varphi : \mathbb{R} \to \mathbb{R}$, defined for all $x \in \mathbb{R}$ by

$$\varphi(x) = \min\{x, 0\}.$$

Obviously φ is quasiconvex, as being monotone. However, f is not explicitly quasiconvex.

Remark 2.2. In view of Popositions 2.3, 2.4 and 2.5, it is easy to check the following

a) If a function $\varphi : D \to \mathbb{R}$ is convex (quasiconvex, explicitly quasiconvex), then for any numbers $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}$ the function

$$\alpha\varphi + \beta$$

is also convex (quasiconvex, explicitly quasiconvex, respectively).

b) If several functions $\varphi_1 : D \to \mathbb{R}, \ldots, \varphi_n : D \to \mathbb{R}$ ($n \geq 2$) are convex (quasiconvex, explicitly quasiconvex), then the function

$$\max_{i \in I_n} \varphi_i$$

is also convex (quasiconvex, explicitly quasiconvex, respectively). Moreover, when the functions $\varphi_1, \ldots, \varphi_n$ are convex their sum

$$\sum_{i \in I_n} \varphi_i$$

is convex as well. However, the sum of quasiconvex (even explicitly quasiconvex) functions is not necessarily quasiconvex, as shown by the following example.

Example 2.3. Let $D = X = \mathbb{R}$, and let $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ be defined for all $x \in \mathbb{R}$ by

$$\varphi(x) = x^3 \text{ and } \psi(x) = -x.$$

Obviously, both functions are explicitly quasiconvex (actually, ψ is even linear), but their sum is not quasiconvex.

Next we present an interesting characterization of convexity for real-valued functions, given in terms of quasiconvexity of their linear perturbations, established by Crouzeix [7, Prop. 9 of Ch. 1].

Proposition 2.6. *For any scalar function $\varphi : D \to \mathbb{R}$, the following assertions are equivalent :*

1° *φ is convex.*

2° *$\varphi + \psi$ is quasiconvex, for every linear functional $\psi : X \to \mathbb{R}$ (restricted to D).*

2.3. **Generalized convex vector functions.** Next we recall three concepts of generalized convexity for vector functions, currently used in vector optimization, along with some useful characterizations (see, e.g., Luc [19] and Popovici [22]).

Definition 2.4. A function $f : D \to Y$ is said to be C-convex if

$$f((1-t)x + tx') \in (1-t)f(x) + tf(x') - C, \quad \forall x, x' \in D, \forall t \in]0,1[.$$

Proposition 2.7. *For any function $f : D \to Y$, the following assertions are equivalent :*

1° *f is C-convex.*

2° *For any $y, y' \in Y$, $x, x' \in D$ and $t \in]0,1[$,*

$$f(x) \in y - C \text{ and } f(x') \in y' - C \implies f((1-t)x + tx') \in (1-t)y + ty' - C.$$

Definition 2.5. A function $f : D \to Y$ is said to be C-quasiconvex if

$$(f(x) + C) \cap (f(x') + C) \subseteq f((1-t)x + tx') + C, \quad \forall x, x' \in D, \forall t \in]0,1[.$$

Proposition 2.8. *For any function $f : D \to Y$, the following statements hold:*

1° *f is C-quasiconvex.*

2° *For any $y \in Y$, $x, x' \in D$ and $t \in]0,1[$,*

$$f(x) \in y - C \text{ and } f(x') \in y - C \implies f((1-t)x + tx') \in y - C.$$

Definition 2.6. A function $f : D \to Y$ is said to be explicitly C-quasiconvex if

$$(f(x) + C) \cap (f(x') + \operatorname{int} C) \subseteq f((1-t)x + tx') + \operatorname{int} C, \quad \forall x, x' \in D, \forall t \in]0,1[.$$

Proposition 2.9. *For any function $f : D \to Y$, the following statements hold:*

1° *f is explicitly C-quasiconvex.*

2° *For any $y \in Y$, $x, x' \in D$ and $t \in]0,1[$,*

$$f(x) \in y - C \text{ and } f(x') \in y - \operatorname{int} C \implies f((1-t)x + tx') \in y - \operatorname{int} C.$$

Example 2.4. Consider the particular framework of the finite-dimensional real Euclidean space $Y = \mathbb{R}^m$, endowed with the standard ordering cone $C = \mathbb{R}^m_+$. In view of Propositions 2.7 and 2.8 it is easily seen that a function $f = (f_1, \ldots, f_m) : D \to \mathbb{R}^m$ is $\mathbb{R}^m_+$-convex ($\mathbb{R}^m_+$-quasiconvex) if and only if f is componentwise convex (quasiconvex), i.e., all scalar functions $f_1 : D \to \mathbb{R}$, ..., $f_m : D \to \mathbb{R}$ are convex (quasiconvex, respectively). Moreover, according to Popovici [22, Th. 3.1], f is explicitly $\mathbb{R}^m_+$-quasiconvex if and only if f is componentwise explicitly quasiconvex. Obviously, for $m = 1$, we recover the classical notions of convexity, quasiconvexity and explicit quasiconvexity.

Remark 2.3. The following relationships between the generalized convexity concepts introduced above hold (see, e.g., Luc [19] and Popovici [22]):

a) C-convex functions are explicitly C-quasiconvex (even if C is not closed).

b) Explicitly C-quasiconvex functions are C-quasiconvex (the closedness assumption on C, stated throughout our paper, being essential).

Remark 2.4. Similarly to the stability properties of convex scalar functions, mentioned in Remark 2.2, the C-convexity of vector functions is preserved under multiplication by nonnegative real numbers as well as under addition. However, neither C-quasiconvexity nor explicitly C-quasiconvexity is stable under addition, in view of Examples 2.3 and 2.4.

Next we present a counterpart of Proposition 2.6 for vector functions, obtained by Kuroiwa, Popovici and Rocca [16].

Proposition 2.10. *For any vector function $f : D \to Y$, the following assertions are equivalent :*

1° *f is C-convex.*

2° *$f + g$ is C-quasiconvex for every linear operator $g : X \to Y$ (restricted to D).*

3. Characterization of generalized convex vector functions by means of nonlinear scalarization functions

We start this section by presenting a characterization of C-quasiconvex vector functions by means of nonlinear scalarization functions, obtained by Luc [19, Prop. 6.3, Ch. 1].

Proposition 3.1. *For any vector function $f : D \to Y$, the following assertions are equivalent :*

1° *f is C-quasiconvex.*

2° *The scalar function $\sigma_a \circ f : D \to \mathbb{R}$ is quasiconvex for every $a \in Y$.*

In what concerns C-convexity we have the following result.

Theorem 3.1. *For any vector function $f : D \to Y$, the following assertions hold :*

1° *If f is C-convex, then $\sigma_a \circ f : D \to \mathbb{R}$ is convex for every $a \in Y$.*

2° *If $\sigma_a \circ f : D \to \mathbb{R}$ is convex for every $a \in Y$, while C is a polyhedral cone in $Y = \mathbb{R}^m$ as defined in Example 2.1, then f is C-convex.*

Proof. In order to prove 1°, assume that f is C-convex and let $a \in Y$. By Proposition 2.2, the function σ_a is convex and C-increasing. Thus, the real-valued function $\sigma_a \circ f$ is C-convex, according to a well-known result of Luc [19, Prop. 6.8, Ch. 1].

Assertion 2° is a particular instance of a result obtained by La Torre, Popovici and Rocca [18, Th. 11] in a more general framework (where Y is assumed to be a normed space and C satisfies a weak polyhedrality type property). □

The following result gives a characterization of C-convex vector functions in terms of quasiconvexity of certain scalar functions, defined by means of nonlinear scalarization functions and linear perturbations.

Theorem 3.2. *For any vector function $f : D \to Y$, the following assertions are equivalent:*

1° *f is C-convex.*

2° *The scalar function $\sigma_a \circ (f+g) : D \to \mathbb{R}$ is quasiconvex, for any $a \in Y$ and any linear operator $g : X \to Y$ (restricted to D).*

Proof. According to Proposition 2.10, f is C-convex if and only if $f+g$ is C-quasiconvex for any linear operator $g : X \to Y$. Applying Proposition 3.1 for $f+g$ in the role of f, we get the desired conclusion. □

A characterization of explicitly C-quasiconvex vector functions, similar to Proposition 3.1, has been obtained by us in [11, Th. 4.4] for $Y = \mathbb{R}^m$ and a polyhedral cone C as in Example 2.1. By using a new approach, we show now that this characterization is still valid in the general framework of this paper.

Theorem 3.3. *For any vector function $f : D \to Y$, the following assertions are equivalent:*

1° *f is explicitly C-quasiconvex.*

2° *For every $a \in Y$, the scalar function $\sigma_a \circ f : D \to \mathbb{R}$ is explicitly quasiconvex.*

Proof. In order to prove the implication $1^\circ \Rightarrow 2^\circ$, assume that f is explicitly C-quasiconvex and consider an arbitrary point $a \in Y$. We will prove that $\sigma_a \circ f$ is explicitly quasiconvex by means of Proposition 2.5. To this aim, consider any $\lambda \in \mathbb{R}$ and any points $x, x' \in D$ such that

$$(\sigma_a \circ f)(x) \leq \lambda \ \text{ and } \ (\sigma_a \circ f)(x') < \lambda.$$

Then, by definition of σ_a, we have $\sigma(f(x) - a) \leq \lambda$ and $\sigma(f(x') - a) < \lambda$, which in view of (2.4) means that

$$\sigma(f(x) - a) - \lambda = \sigma(f(x) - a - \lambda e) \leq 0$$

and

$$\sigma(f(x') - a) - \lambda = \sigma(f(x') - a - \lambda e) < 0.$$

By (2.2) and (2.3), it follows that $f(x) - a - \lambda e \in -C$ and $f(x') - a - \lambda e \in -\operatorname{int} C$, i.e.,

$$f(x) \in (a + \lambda e) - C$$

and

$$f(x') \in (a + \lambda e) - \operatorname{int} C.$$

Since f is explicitly C-quasiconvex, we deduce by Proposition 2.9 that

$$f((1-t)x + tx') \in (a + \lambda e) - \operatorname{int} C, \quad \forall t \in]0,1[.$$

In view of relation (2.3), this actually means that $\sigma(f((1-t)x + tx') - a - \lambda e) < 0$, which can be rewritten as $\sigma(f((1-t)x + tx') - a) - \lambda < 0$ by (2.4). Finally, recalling the definition of σ_a, we conclude that $\sigma_a(f((1-t)x + tx')) < \lambda$, i.e.,

$$(\sigma_a \circ f)((1-t)x + tx')) < \lambda,$$

as desired. Thus $\sigma_a \circ f : D \to \mathbb{R}$ is explicitly quasiconvex.

Now, let us prove the converse implication $2^\circ \Rightarrow 1^\circ$. Assume that $\sigma_a \circ f : D \to \mathbb{R}$ is explicitly quasiconvex for every $a \in Y$. We will show that f is explicitly C-quasiconvex by means of Proposition 2.9. Let $y \in Y$ and let $x, x' \in D$ be such that

$$f(x) \in y - C \text{ and } f(x') \in y - \operatorname{int} C.$$

By (2.2) and (2.3), it follows that $\sigma(f(x) - y) \leq 0$ and $\sigma(f(x') - y) < 0$, hence $(\sigma_y \circ f)(x) \leq 0$ and $(\sigma_y \circ f)(x') < 0$ by definition of σ_y. Since $\sigma_y \circ f$ is explicitly quasiconvex, we infer by Proposition 2.5 that $(\sigma_y \circ f)((1-t)x + tx') < 0$ for any $t \in]0,1[$. Using again the definition of σ_y, we deduce that $\sigma(f((1-t)x + tx') - y) < 0$, which in view of (2.3) yields the desired relation

$$f((1-t)x + tx') \in y - \operatorname{int} C.$$

Thus f is explicitly C-quasiconvex. □

By using the explicit representation of the nonlinear scalarization functions for polyhedral cones, we deduce from Theorem 3.1, Proposition 3.1 and Theorem 3.3 the following result.

Corollary 3.1. *Let C be a polyhedral cone of $Y = \mathbb{R}^m$ as described in Example 2.1 and let $e \in \operatorname{int} C$. Then, for any vector function $f = (f_1, \ldots, f_m) : D \to \mathbb{R}^m$, the following assertions are equivalent :*

1° *f is C-convex (C-quasiconvex, explicitly C-quasiconvex, respectively).*

2° *For any $a \in \mathbb{R}^m$, the real-valued function*

$$\max_{i \in I_p} \frac{\langle f(x) - a, u^i \rangle}{\langle e, u^i \rangle}$$

is convex (quasiconvex, explicitly quasiconvex, respectively).

Furthermore, in the framework of Example 2.4 we can deduce as a particular instance of Corollary 3.1 the following componentwise characterizations of generalized convex vector functions.

Corollary 3.2. *For any vector function $f = (f_1, \ldots, f_m) : D \to \mathbb{R}^m$ the following assertions are equivalent :*

1° *f is componentwise convex (quasiconvex, explicitly quasiconvex, respectively).*

2° *For any numbers $a_1, \ldots, a_m \in \mathbb{R}$, the scalar function*

$$\max_{i \in I_m} (f_i - a_i)$$

is convex (quasiconvex, explicitly quasiconvex, respectively).

4. Conclusion

The nonlinear scalarization functions in the sense of Gerstewitz (Tammer) are shown to be a powerful tool for characterizing different notions of generalized convexity, currently used in vector optimization, namely C-convexity, C-quasiconvexity and explicit C-quasiconvexity with respect to the ordering convex cone C of the outcome space. An interesting topic for further research would be to study whether similar characterizations hold for other notions of generalized convexity known in the literature. Moreover, the nonlinear scalarization functions, σ_a, could be used to introduce new classes of generalized convex vector functions, f, in such a way that $\sigma_a \circ f$ would satisfy a specific generalized convexity property for all $a \in Y$.

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