KARUSH-KUHN-TUCKER OPTIMALITY CONDITIONS AND DUALITY FOR SET OPTIMIZATION PROBLEMS WITH MIXED CONSTRAINTS

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Abstract. In this paper, we consider set optimization problems with mixed constraints. We first investigate necessary and sufficient Karush-Kuhn-Tucker optimality conditions for strict minimal solutions. Then, we formulate types of Mond-Weir and Wolfe dual problems and explore duality relations under convexity assumptions. Some examples are provided to illustrate our results.

Keywords. Set optimization problems with mixed constraints; Strict minimal solutions; Karush-Kuhn-Tucker optimality conditions; Mond-Weir duality; Wolfe duality.

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1. INTRODUCTION

Set-valued optimization problems play an important role in optimization theory because many models in practice can be described via set-valued optimization problems; see the books [4, 12, 22] for more details. There are two main types of approaches to define solution concepts of set-valued optimization problems: the vector approach and the set approach. The former is the usual one, the latter arises naturally in the real-world application in socio-economic; see the papers [16, 17, 23], the survey [9] and the references therein. In the vector approach, many types of generalized derivatives were employed to establish the optimality conditions for set-valued optimization problems in two directions: the dual spaces and the primal spaces. For some recent results in this direction, see the books [12, 22], the papers [2, 3, 7, 10, 13, 19, 20, 24] and the references therein. In the set approach, the optimality conditions for some types of solutions of set optimization problems were established in terms of the directional derivatives of the continuous selections in [1]. The paper [14] investigated the necessary optimality conditions in terms of contingent derivatives and Mordukhovich coderivatives for optimal solutions of set optimization problems with respect to the possibly and lower set less order relation. By using Studniarski derivatives, the necessary and sufficient optimality conditions were derived in [15] for optimal solutions of set optimization problems with respect to the lower set less order relation. To the best of our knowledge, there is no paper dealing with establishing optimality conditions for the

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set optimization problems with mixed constraints. Moreover, the Mond-Weir and Wolfe dual problems of set optimizations with mixed constraints also have not been investigated yet.

Motivated by the above observations, in this paper, we establish Karush-Kuhn-Tucker optimality conditions and discuss the duality problems for the set optimization problems with mixed constraints. The paper is organized as follows. Section 2 recalls basic concepts and some preliminaries. The Karush-Kuhn-Tucker necessary and sufficient optimality conditions for strict minimal solutions are established in Section 3. Section 4 is devoted to exploring Mond-Weir and Wolfe dual problems of the set optimization problems with mixed constraints. Some examples are provided to illustrate our results.

2. PRELIMINARIES

In this paper, let X, Y, Z and W be Banach spaces, $C \subseteq Y$, $D \subseteq Z$ be pointed closed convex cones with nonempty interior. Denote X^* the topological dual space of X. The notion B_X represents the open unit ball of X and $B_X(x, \delta)$ stands for the open ball with center $x \in X$ and radius δ. For a given set $A \subseteq X$, int$A$, cl$A$, ∂A, cone$A$ denote its interior, closure, boundary and the cone $\{\lambda a \mid \lambda \geq 0, a \in A\}$ of A, respectively (resp). A point $\bar{y}$ is said to be a weak efficient point of $B \subset Y$ with respect to (w.r.t.) C, denoted by $\bar{y} \in \text{WMin}_C B$, iff $(B - \bar{y}) \cap (-\text{int}C) = \emptyset$. A subset $B \subset Y$ has the weakly C-minimal property if for all $y \in B$ there exists $\bar{y} \in \text{WMin}_C B$ such that $\bar{y} - y \in -\text{int}C \cup \{0\}$, i.e., $B - \bar{y} \subset \text{int}C \cup \{0\}$. For the cones $C \subseteq Y$ and $D \subseteq Z$,

$$C^+ := \{y^* \in Y^* \mid \langle y^*, c \rangle \geq 0, \forall c \in C\},$$

$$D^+ := \{d^* \in Z^* \mid \langle d^*, z \rangle \geq 0, \forall z \in D\}$$

are positive polar cones of C, D, resp. A convex set $B \subset Y$ is called a base for C if $0 \notin \text{cl}B$ and $C = \{tb \mid t \in \mathbb{R}_+, b \in B\}$.

Definition 2.1. [4, 12] Let A be a nonempty subset of X.

(i) The contingent (or Bouligand) cone of A at $\bar{x} \in \text{cl}A$ is

$$T(A, \bar{x}) := \{x \in X \mid \exists t_n \downarrow 0, \exists x_n \to x, \forall n \in \mathbb{N}, \bar{x} + t_n x_n \in A\}.$$

(ii) The adjacent cone of A at $\bar{x} \in \text{cl}A$ is

$$T^b(A, \bar{x}) := \{x \in X \mid \forall t_n \downarrow 0, \exists x_n \to x, \forall k \in \mathbb{N}, \bar{x} + t_n x_n \in A\}.$$

(iii) The interior tangent cone of A at $\bar{x}$ is

$$IT(A, \bar{x}) := \{x \in X \mid \forall t_n \downarrow 0, \forall x_n \to x, \forall n \text{ large}, \bar{x} + t_n x_n \in A\}.$$

Remark 2.1. The following properties can be checked directly.

(i) $IT(A, \bar{x}) \subset T^b(A, \bar{x}) \subset T(A, \bar{x})$.
(ii) $0 \in T(A, \bar{x})$ and $0 \in IT(A, \bar{x})$ if $\bar{x} \in \text{int}A$.
(iii) [4, 12] If A is a convex set then $T^b(A, \bar{x}) = T(A, \bar{x}) = \text{clcone}(A - \bar{x})$.
(iv) [11, 12] If A is a convex set then $IT(A, \bar{x}) = \text{intcone}(A - \bar{x}) = \text{cone}(\text{int}A - \bar{x})$.

For a given set-valued map $P : X \rightrightarrows Y$, the domain, graph and epigraph of P are defined respectively by

$$\mathrm{dom}P := \{x \in X : P(x) \neq \emptyset\},$$
$$\mathrm{gr}P := \{(x,y) \in X \times Y : y \in P(x)\},$$
$$\mathrm{epi}P := \{(x,y) \in X \times Y : y \in P(x) + C\}.$$

The so-called profile mapping of P is P_+ defined by $P_+(x) = P(x) + C$. P is C-convex if its epigraph is convex, i.e., for all $x, x' \in X$ and $\lambda \in [0,1]$,

$$\lambda P(x) + (1 - \lambda)P(x') \subset P(\lambda x + (1 - \lambda)x') + C.$$

Definition 2.2. [8] Let $P : X \rightrightarrows Y$ be a set-valued map, $(\bar{x}, \bar{y}) \in \mathrm{gr}P$ and $u \in X$. P is said to be directionally metrically subregular at $(\bar{x}, \bar{y})$ in direction u w.r.t. $S \subset X$ iff there are $\alpha \geq 0$ and $\gamma > 0$ such that, for all $t \in]0, \gamma[$ and $v \in B_X(u, \gamma)$ with $\bar{x} + tv \in S$,

$$d(\bar{x} + tv, P^{-1}(\bar{y}) \cap S) \leq \alpha d(\bar{y}, P(\bar{x} + tv)).$$

Definition 2.3. [4] Let $P : X \rightrightarrows Y$ and $(\bar{x}, \bar{y}) \in \mathrm{gr}P$.

(i) The set-valued map P is said to be Aubin at $(\bar{x}, \bar{y})$ if there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$, and $L > 0$ such that

$$P(x) \cap V \subset P(x') + L\|x - x'\| \mathrm{cl}B_Y, \forall x, x' \in U.$$

(ii) The set-valued map P is said to be C-Aubin at $(\bar{x}, \bar{y})$ if there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$, and $L > 0$ such that

$$P(x) \cap V \subset P(x') + C + L\|x - x'\| \mathrm{cl}B_Y, \forall x, x' \in U.$$

Definition 2.4. Let $P : X \rightrightarrows Y$ be a set-valued map and $(\bar{x}, \bar{y}) \in \mathrm{gr}P$.

(i) [4] The contingent derivative $DP(\bar{x}, \bar{y})$ of P at $(\bar{x}, \bar{y})$ is

$$DP(\bar{x}, \bar{y})(u) := \Big\{v \in Y \mid \exists t_n \downarrow 0, \ \exists(u_n, v_n) \to (u, v), \ \forall n \in \mathbb{N}, \bar{y} + t_n v_n \in P(\bar{x} + t_n u_n)\Big\}.$$

(ii) [4] The adjacent derivative $D^b P(\bar{x}, \bar{y})$ of P at $(\bar{x}, \bar{y})$ is

$$D^b P(\bar{x}, \bar{y})(u) := \Big\{v \in Y \mid \forall t_n \downarrow 0, \ \exists(u_n, v_n) \to (u, v), \ \forall n \in \mathbb{N}, \bar{y} + t_n v_n \in P(\bar{x} + t_n u_n)\Big\}.$$

(iii) [7] The lower (Dini) derivative $D^l P(\bar{x}, \bar{y})$ of P at $(\bar{x}, \bar{y})$ is

$$D^l P(\bar{x}, \bar{y})(u) := \Big\{v \in Y \mid \forall t_n \downarrow 0, \ \forall u_n \to u, \ \exists v_n \to v, \ \forall n \in \mathbb{N}, \bar{y} + t_n v_n \in P(\bar{x} + t_n u_n)\Big\}.$$

(iv) [7] The Dubovitskii-Miljutin derivative $D^{DM} P(\bar{x}, \bar{y})$ of P at $(\bar{x}, \bar{y})$ is

$$D^{DM} P(\bar{x}, \bar{y})(u) := \Big\{v \in Y \mid \forall t_n \downarrow 0, \ \forall(u_n, v_n) \to (u, v), \ \forall n \text{ large}, \bar{y} + t_n v_n \in P(\bar{x} + t_n u_n)\Big\}.$$

Remark 2.2. Let $P : X \rightrightarrows Y$ be a set-valued map, $(\bar{x}, \bar{y}) \in \mathrm{gr}P$ and $u \in X$.

(i) $D^{DM} P(\bar{x}, \bar{y})(u) \subset D^l P(\bar{x}, \bar{y})(u) \subset D^b P(\bar{x}, \bar{y})(u) \subset DP(\bar{x}, \bar{y})(u)$.
(ii) [4] If P is convex then $D^b P(\bar{x}, \bar{y})(u) = DP(\bar{x}, \bar{y})(u)$.
(iii) If P is C-convex then $D^b P_+(\bar{x}, \bar{y})(u) = DP_+(\bar{x}, \bar{y})(u)$.
(iv) [7] If P is Aubin at $(\bar{x}, \bar{y})$ then $D^l P(\bar{x}, \bar{y})(u) = D^b P(\bar{x}, \bar{y})(u)$.
(v) If P is C-Aubin at $(\bar{x}, \bar{y})$ then $D^l P_+(\bar{x}, \bar{y})(u) = D^b P_+(\bar{x}, \bar{y})(u)$.

Lemma 2.1. [6] *Let $P : X \rightrightarrows Y$, $(\bar{x}, \bar{y}) \in \mathrm{gr}P$, and P be C-convex. Then, for all $x \in X$,*

$$P(x) - \bar{y} \subseteq DP_+(\bar{x}, \bar{y})(x - \bar{x}).$$

Let A and B be nonempty subsets of Y, denoted by $A, B \in \mathscr{P}(Y)$. Recall some set relations [16, 17] based on cone ordering C as follows:

(i) $A \preceq_C^l B \Leftrightarrow A + C \supseteq B$,
(ii) $A \prec_C^l B \Leftrightarrow A + \mathrm{int}C \supseteq B$.

3. KARUSH-KUHN-TUCKER OPTIMALITY CONDITIONS

In the sequel, let $F : X \rightrightarrows Y$, $G : X \rightrightarrows Z$, $H : X \rightrightarrows W$ be three set-valued maps. We consider the following set optimization problem with mixed constraints:

$$(P) : \preceq_C^l - \min_{x \in X}\{F(x) \mid G(x) \cap (-D) \neq \emptyset, 0 \in H(x)\}.$$

Denote $\Omega := \{x \in X \mid G(x) \cap (-D) \neq \emptyset, 0 \in H(x)\}$ the set of feasible points of (P). A point $\bar{x} \in \Omega$ is a strict minimal solution [1] with respect to $\prec_C^l$ of (P), denoted by $\bar{x} \in \mathrm{Min}(P, \prec_C^l)$, if for any $x \in \Omega$, one has $F(x) \not\prec_C^l F(\bar{x})$.

Lemma 3.1. [18] *Let $\bar{x} \in X$ and $\bar{y} \in F(\bar{x})$. Suppose that $(F - \bar{y}, G, H)(X) + (\mathrm{int}C \times D \times \{0\})$ is a convex set and $\mathrm{int}H(X) \neq \emptyset$, where $(F - \bar{y})(x) := F(x) - \bar{y}$. If the system*

$$(F(x) - \bar{y}) \cap -\mathrm{int}C \neq \emptyset, G(x) \cap -D \neq \emptyset, 0 \in H(x)$$

admits no solution $x \in X$, then there exists $(c^*, d^*, w^*) \in (C^+ \times D^+ \times W^*) \backslash \{(0,0,0)\}$ such that

$$\langle c^*, y - \bar{y} \rangle + \langle d^*, z \rangle + \langle w^*, w \rangle \geq 0,$$

for all $x \in X$ and for all $(y, z, w) \in (F, G, H)(x)$.

For the sake of convenience, we always assume that $\mathrm{dom}D(F_+, G_+, H)(x, y, z, w) = X$ and $\mathrm{dom}DF_+(x, y) \cap \mathrm{dom}DG_+(x, z) \cap \mathrm{dom}DH(x, w) = X$ for all $x \in X, y \in F(x), z \in G(x) \cap -D$ and $w \in H(x)$.

3.1. Necessary Optimality Conditions.

Proposition 3.1. *Let $\bar{x} \in X, \bar{y} \in F(\bar{x}), \bar{z} \in G(\bar{x}) \cap -D, 0 \in H(\bar{x})$ and $\bar{x}$ be a strict minimal solution w.r.t. $\prec_C^l$ of (P). Suppose that the following conditions hold:*

(i) $\mathrm{WMin}_C F(\bar{x}) = \{\bar{y}\}$, $F(\bar{x})$ *has the weakly C-minimal property,*
(ii) $(F - \bar{y}, G, H)(X) + (\mathrm{int}C \times D \times \{0\})$ *is a convex set and $\mathrm{int}H(X) \neq \emptyset$.*

Then, there exists $(c^*, d^*, w^*) \in (C^+ \times D^+ \times W^*) \backslash \{(0,0,0)\}$ such that

$$\langle c^*, y \rangle + \langle d^*, z \rangle + \langle w^*, w \rangle \geq 0 \tag{3.1}$$

and

$$\langle d^*, \bar{z} \rangle = 0, \tag{3.2}$$

for all $x \in X$ and for all $(y, z, w) \in D(F_+, G_+, H)(\bar{x}, \bar{y}, \bar{z}, 0)(x)$.
Moreover, $c^* \in C^+ \backslash \{0\}$ if the following constraint qualification holds

(SCQ) : $0 \in \mathrm{int}H(X)$ *and there exists $\hat{x} \in X$ such that $G(\hat{x}) \cap -\mathrm{int}D \neq \emptyset$ and $0 \in H(\hat{x})$.*

Proof. We first prove that $(F(x) - \bar{y}) \cap -\text{int}C = \emptyset$ for any $x \in \Omega$. Suppose to the contrary that there exists $\hat{x} \in \Omega$ such that $(F(\hat{x}) - \bar{y}) \cap -\text{int}C \neq \emptyset$. Then, there exists $\hat{y} \in F(\hat{x})$ satisfying $\hat{y} - \bar{y} \in -\text{int}C$, or equivalently, $\bar{y} \in \hat{y} + \text{int}C$. Since $\text{WMin}_C F(\bar{x}) = \{\bar{y}\}$ and $F(\bar{x})$ has weakly C-minimal property, one has

$$F(\bar{x}) \subseteq \bar{y} + \text{int}C \cup \{0\} \subseteq F(\hat{x}) + C + \text{int}C \cup \{0\} \subseteq F(\hat{x}) + \text{int}C,$$

contradicting with the fact that $\bar{x}$ is a strict minimal solution w.r.t. $\prec_C^l$ of (P). Hence, for any $x \in \Omega$, $(F(x) - \bar{y}) \cap -\text{int}C = \emptyset$. This leads that the system

$$(F(x) - \bar{y}) \cap -\text{int}C \neq \emptyset, G(x) \cap -D \neq \emptyset, 0 \in H(x)$$

admits no solution $x \in X$. Then, we deduce from (ii) and Lemma 3.1 the existence of $(c^*, d^*, w^*) \in C^+ \times D^+ \times W^* \setminus \{(0,0,0)\}$ such that, for all $x \in X$ and for all $(y, z, w) \in (F, G, H)(x)$,

$$\langle c^*, y - \bar{y} \rangle + \langle d^*, z \rangle + \langle w^*, w \rangle \geq 0. \tag{3.3}$$

We justify that $\langle d^*, \bar{z} \rangle = 0$. Indeed, by taking $x = \bar{x}, y = \bar{y} \in F(\bar{x}), z = \bar{z} \in G(\bar{x})$ and $w = 0 \in H(\bar{x})$ in (3.3), we obtain that $\langle d^*, \bar{z} \rangle \geq 0$. On the other hand, $d^* \in D^+$ and $\bar{z} \in -D$ give that $\langle d^*, \bar{z} \rangle \leq 0$. Thus, $\langle d^*, \bar{z} \rangle = 0$. Since $c^* \in C^+$ and $d^* \in D^+$, we derive from (3.3) that, for all $x \in X$ and for all $(y, z, w) \in (F, G, H)(x) + (C \times D \times \{0\})$,

$$\langle c^*, y - \bar{y} \rangle + \langle d^*, z - \bar{z} \rangle + \langle w^*, w \rangle \geq 0. \tag{3.4}$$

Finally, let $x \in X$ and $(y, z, w) \in D(F_+, G_+, H)(\bar{x}, \bar{y}, \bar{z}, 0)(x)$. Then, there exist $t_n \downarrow 0, x_n \in X$ and $(y_n, z_n, w_n) \in (F, G, H)(x_n) + (C \times D \times \{0\})$ such that

$$\left(\frac{x_n - \bar{x}}{t_n}, \frac{y_n - \bar{y}}{t_n}, \frac{z_n - \bar{z}}{t_n}, \frac{w_n - 0}{t_n} \right) \to (x, y, z, w).$$

From (3.4), one yields

$$\left\langle c^*, \frac{y_n - \bar{y}}{t_n} \right\rangle + \left\langle d^*, \frac{z_n - \bar{z}}{t_n} \right\rangle + \left\langle w^*, \frac{w_n - 0}{t_n} \right\rangle \geq 0.$$

Letting $n \to +\infty$ in the above inequality, we get

$$\langle c^*, y \rangle + \langle d^*, z \rangle + \langle w^*, w \rangle \geq 0.$$

Now, let (SCQ) hold. We will verify that $c^* \neq 0$ by contraposition. Suppose that $c^* = 0$. Then, from (3.3), for all $(z, w) \in (G, H)(X) + D \times \{0\}$,

$$\langle d^*, z \rangle + \langle w^*, w \rangle \geq 0. \tag{3.5}$$

Employing (SCQ) leads us to the existence of $\hat{z} \in G(\hat{x})$ with $-\hat{z} \in \text{int}D$ and $0 \in H(\hat{x})$. Since $d^* \in D^+$ and $\hat{z} \in -\text{int}D$, one gets

$$\langle d^*, \hat{z} \rangle < 0. \tag{3.6}$$

If $d^* \neq 0$, then we deduce from $(\hat{z}, 0) \in (G, H)(\bar{x}) + D \times \{0\} \subset (G, H)(X) + D \times \{0\}$ and (3.5) that

$$\langle d^*, \hat{z} \rangle = \langle d^*, \hat{z} \rangle + \langle w^*, 0 \rangle \geq 0,$$

contradicting with (3.6), so $d^* = 0$. Since $0 \in \text{int}H(X)$, it follows from (3.5) that there is $\delta > 0$ satisfying $\langle w^*, w \rangle \geq 0$ for all $w \in B_W(0, \delta) \subseteq H(X)$, i.e., $w^* = 0$. This is unreasonable because $(c^*, d^*, w^*) \neq (0,0,0)$. Hence, $c^* \neq 0$. $\qquad \square$

In the following we give a result, similar to Proposition 3.1, for the case when the derivatives of objective and constraint functions are considered separately.

Proposition 3.2. *Let $\bar{x} \in X$, $\bar{y} \in F(\bar{x})$, $\bar{z} \in G(\bar{x}) \cap (-D)$, $0 \in H(\bar{x})$ and $\bar{x}$ be a strict minimal solution w.r.t. $\prec_C^l$ of (P). Suppose that the following provisos hold:*

(i) $\mathrm{WMin}_C F(\bar{x}) = \{\bar{y}\}$, $F(\bar{x})$ has the weakly C-minimal property,

(ii) F, G, H are C-convex, D-convex, convex, respectively,

(iii) F_+ is C-Aubin at $(\bar{x}, \bar{y})$, G_+ is D-Aubin at $(\bar{x}, \bar{z})$,

(iv) H is directionally metrically subregular at $(\bar{x}, 0)$ w.r.t. X.

Then there exists $(c^*, d^*, w^*) \in C^+ \times D^+ \times W^* \setminus \{(0,0,0)\}$ such that $\langle d^*, \bar{z} \rangle = 0$ and

$$\langle c^*, y \rangle + \langle d^*, z \rangle + \langle w^*, w \rangle \geq 0,$$

for every $x \in X$, $y \in DF_+(\bar{x}, \bar{y})(x)$, $z \in DG_+(\bar{x}, \bar{z})(x)$ and $w \in DH(\bar{x}, 0)(x)$. Moreover, $c^* \neq 0$ if the following constraint qualification holds

$$(\text{KRZCQ}): (DG_+(\bar{x}, \bar{z}), DH(\bar{x}, 0))(X) + \mathrm{cone}(D + \bar{z}) \times \{0\} = Z \times W.$$

Proof. By similar arguments as in Proposition 3.1, we infer from (i) that

$$(F(x) - \bar{y}) \cap -\mathrm{int}C = \emptyset, \forall x \in \Omega. \tag{3.7}$$

Moreover, we derive from (ii), (iii) and Remark 2.2 that $D^l F_+(\bar{x}, \bar{y})(x) = DF_+(\bar{x}, \bar{y})(x)$ and $D^l G_+(\bar{x}, \bar{z})(x) = DG_+(\bar{x}, \bar{z})(x)$.

Firstly, we verify that, for all $x \in X$,

$$\left(D^l F_+(\bar{x}, \bar{y})(x), D^l G_+(\bar{x}, \bar{z})(x), DH(\bar{x}, 0)(x) \right) \cap ((-\mathrm{int}C) \times (-\mathrm{int}D - \bar{z}) \times \{0\}) = \emptyset. \tag{3.8}$$

Arguing by contradiction, suppose that $x \in X$ and $(y, z) \in (-\mathrm{int}C) \times (-\mathrm{int}D - \bar{z})$ such that

$$(y, z, 0) \in (D^l F_+(\bar{x}, \bar{y})(x), D^l G_+(\bar{x}, \bar{z})(x), DH(\bar{x}, 0)(x)).$$

As $0 \in DH(\bar{x}, 0)(x)$, there exist $t_n \downarrow 0$, $(x_n, w_n) \to (x, 0)$ satisfying $t_n w_n \in H(\bar{x} + t_n x_n)$. We deduce from the directional metrical subregularity of H the existence of $\gamma > 0$ such that, for n large enough,

$$d(\bar{x} + t_n x_n, H^{-1}(0)) \leq \gamma d(0, H(\bar{x} + t_n x_n)).$$

Thus, there is $x_n' \in H^{-1}(0)$ such that

$$\|\bar{x} + t_n x_n - x_n'\| \leq \gamma \|t_n w_n\|.$$

Setting $\bar{x}_n := \frac{1}{t_n}(x_n' - \bar{x})$, we have $\bar{x} + t_n \bar{x}_n = x_n' \in H^{-1}(0)$, $\|x_n - \bar{x}_n\| \leq \gamma \|w_n\|$, and hence, $\bar{x}_n \to x$. As $z \in D^l G_+(\bar{x}, \bar{z})(x) \cap (-\mathrm{int}D - \bar{z})$, for the above t_n and x_n, there exist $z_n \to z$ fulfilling $\bar{z} + t_n z_n \in G(\bar{x} + t_n x_n) + D$. Moreover, since $z \in -\mathrm{int}D - \bar{z}$, $z_n \to z$ and $-\mathrm{int}D - \bar{z}$ is open, one has $\bar{z} + t_n z_n \in -\mathrm{int}D$ for n large. Therefore, $(G(\bar{x} + t_n x_n) + D) \cap (-\mathrm{int}D) \neq \emptyset$ leads to $G(\bar{x} + t_n x_n) \cap (-D) \neq \emptyset$. This implies that $\bar{x} + t_n x_n \in \Omega$ for large n.

Since $y \in D^l F_+(\bar{x}, \bar{y})(x)$, for the above t_n and x_n, there exist $y_n \to y$ and $c_n \in C$ such that $\bar{y} + t_n y_n - c_n \in F(\bar{x} + t_n x_n)$ for all $n \in \mathbb{N}$. As $y \in -\mathrm{int}C$, $y_n \to y$ and $-\mathrm{int}C$ is open, one has $t_n y_n \in -\mathrm{int}C$ for n large enough. Hence, $t_n y_n - c_n \in -\mathrm{int}C - C = -\mathrm{int}C$. Thus, there exists $\bar{x} + t_n x_n \in \Omega$ such that

$$t_n y_n - c_n \in (F(\bar{x} + t_n x_n) - \bar{y}) \cap (-\mathrm{int}C),$$

contradicting with (3.7). This confirms (3.8).

Moreover, we derive from (ii) that $(DF_+(\bar{x},\bar{y})(x), DG_+(\bar{x},\bar{z})(x), DH(\bar{x},0)(x))$ is a convex set. Hence, it follows from (3.8) and the standard separation theorem that there exists $(c^*, d^*, w^*) \in Y^* \times Z^* \times W^* \setminus (0,0,0)$ such that

$$\langle c^*, y \rangle + \langle d^*, z \rangle + \langle w^*, w \rangle \geq \langle c^*, c' \rangle + \langle d^*, d' \rangle + \langle d^*, -\bar{z} \rangle,$$

for all $x \in X$, $(y,z,w) \in (DF_+(\bar{x},\bar{y})(x), DG_+(\bar{x},\bar{z})(x), DH(\bar{x},0)(x))$, $c' \in -\text{int}C$ and $d' \in -\text{int}D$. We assert by the assumptions $-\text{int}C$ and $-\text{int}D$ are cones that

$$\langle c^*, y \rangle + \langle d^*, z \rangle + \langle w^*, w \rangle + \langle d^*, \bar{z} \rangle \geq \langle c^*, \varepsilon c' \rangle + \langle d^*, \eta d' \rangle, \tag{3.9}$$

for all $x \in X$, $(y,z,w) \in (DF_+(\bar{x},\bar{y})(x), DG_+(\bar{x},\bar{z})(x), DH(\bar{x},0)(x))$, $c' \in -\text{int}C$, $d' \in -\text{int}D$, $\varepsilon > 0$ and $\eta > 0$. We claim that $\langle c^*, c \rangle \geq 0$ for all $c \in \text{int}C$. If otherwise, there exists $0 \neq \bar{c} \in \text{int}C$ such that $\langle c^*, \bar{c} \rangle < 0$. Then, $0 \neq \bar{c}' := -\bar{c} \in -\text{int}C$ and $\langle c^*, \bar{c}' \rangle > 0$. Hence, we deduce from (3.9) that

$$\langle c^*, y \rangle + \langle d^*, z \rangle + \langle w^*, w \rangle + \langle d^*, \bar{z} \rangle \geq \langle c^*, \varepsilon \bar{c}' \rangle + \langle d^*, \eta d' \rangle,$$

which leads to a contradiction when passing $\varepsilon \to +\infty$. Thus, $\langle c^*, c \rangle \geq 0$ for all $c \in \text{int}C$, and consequently,

$$\langle c^*, c \rangle \geq 0, \ \forall c \in C = \text{cl}C = \text{cl}(\text{int}C),$$

i.e., $c^* \in C^+$. By similar arguments, one gets $d^* \in D^+$.

Now, we will prove that $\langle d^*, \bar{z} \rangle = 0$. Letting $\varepsilon \to 0^+$ and $\eta \to 0^+$ in (3.9), we ensure that

$$\langle c^*, y \rangle + \langle d^*, z \rangle + \langle w^*, w \rangle + \langle d^*, \bar{z} \rangle \geq 0, \tag{3.10}$$

for all $x \in X$, $(y,z,w) \in (DF_+(\bar{x},\bar{y})(x), DG_+(\bar{x},\bar{z})(x), DH(\bar{x},0)(x))$. It follows from the strictly positive homogeneousness of contingent derivatives that, for any $t > 0$,

$$tDF_+(\bar{x},\bar{y})(0) = DF_+(\bar{x},\bar{y})(t.0) = DF_+(\bar{x},\bar{y})(0),$$
$$tDG_+(\bar{x},\bar{z})(0) = DG_+(\bar{x},\bar{z})(t.0) = DG_+(\bar{x},\bar{z})(0),$$
$$tDH(\bar{x},0)(0) = DH(\bar{x},0)(t.0) = DH(\bar{x},0)(0).$$

This, taking to account (3.10) with $x = 0$, entails that

$$t(\langle c^*, y \rangle + \langle d^*, z \rangle + \langle w^*, w \rangle) + \langle d^*, \bar{z} \rangle \geq 0,$$

for all $t > 0$. Passing to limit as $t \to 0^+$, one has $\langle d^*, \bar{z} \rangle \geq 0$. On the other hand, since $\bar{z} \in -D$, $\langle d^*, \bar{z} \rangle \leq 0$, which in turn implies the equality $\langle d^*, \bar{z} \rangle = 0$. Since $c^* \in C^+$, $d^* \in D^+$ and $\langle d^*, \bar{z} \rangle = 0$, we use (3.10) to deduce that

$$\langle c^*, y \rangle + \langle d^*, z \rangle + \langle w^*, w \rangle \geq 0, \tag{3.11}$$

for all $x \in X$, $y \in DF_+(\bar{x},\bar{y})(x)$, $z \in DG_+(\bar{x},\bar{z})(x)$ and $w \in DH(\bar{x},0)(x)$.

Finally, let (KRZCQ) hold. We will prove that $c^* \neq 0$. For every $(z,w) \in Z \times W$, there exist $\beta \geq 0$, $x \in X$, $\tilde{z} \in DG_+(\bar{x},\bar{z})(x)$, $\tilde{w} \in DH(\bar{x},0)(x)$ and $d \in D$ such that

$$(z,w) = (\tilde{z},\tilde{w}) + (\beta(d+\bar{z}),0).$$

Reasoning by contraposition, assume that $c^* = 0$. Then, using (3.11) tells us the inequality $\langle d^*, \tilde{z} \rangle + \langle w^*, \tilde{w} \rangle \geq 0$. From $d \in D$, $d^* \in D^+$ and $\langle d^*, \bar{z} \rangle = 0$, one gets

$$\langle d^*, z \rangle + \langle w^*, w \rangle = \langle d^*, \tilde{z} \rangle + \langle w^*, \tilde{w} \rangle + \beta \langle d^*, d \rangle + \beta \langle d^*, \bar{z} \rangle \geq 0.$$

Since $(z,w) \in Z \times W$ is arbitrary, we arrive at $(d^*, w^*) = (0,0)$. This is a contradiction to $(c^*, d^*, w^*) \neq (0,0,0)$. $\qquad\square$

3.2. **Sufficient Optimality Conditions.**

Lemma 3.2. *Let $\bar{x} \in \Omega$. If, for each $x_t \in \Omega$, there exists $\bar{y}_t \in F(\bar{x})$ such that*

$$(F(x_t) - \bar{y}_t) \cap -\text{int}C = \emptyset, \tag{3.12}$$

then $\bar{x}$ is a strict minimal solution w.r.t. $\prec^l_C$ of (P).

Proof. Suppose by contradiction that $\bar{x} \in \Omega$ is not a strict minimal solution w.r.t. $\prec^l_C$ of (P). Then, there exists $\hat{x} \in \Omega$ such that $F(\hat{x}) \prec^l_C F(\bar{x})$, i.e., $F(\hat{x}) + \text{int}C \supset F(\bar{x})$. Consequently, for all $\bar{y}_t \in F(\bar{x})$, there exists $\hat{y}_t \in F(\hat{x})$ such that $\bar{y}_t \in \hat{y}_t + \text{int}C$, or equivalently, $\hat{y}_t - \bar{y}_t \in -\text{int}C$. Hence, $(F(\hat{x}) - \bar{y}_t) \cap -\text{int}C \neq \emptyset$, contradicting to (3.12). $\qquad\square$

Proposition 3.3. *Let $\bar{x} \in X$, $\bar{z} \in G(\bar{x}) \cap -D$, $\bar{w} = 0 \in H(\bar{x})$ and F, G, H be C-convex, D-convex, convex, resp. Suppose that there exist $c^* \in C^+ \setminus \{0\}$ and $(d^*, w^*) \in D^+ \times W^*$ such that, for all $x \in X$, for all $\bar{y} \in F(\bar{x})$ and for all $(y, z, w) \in (DF_+(\bar{x}, \bar{y})(x), DG_+(\bar{x}, \bar{z})(x), DH(\bar{x}, 0)(x)) \setminus \{(0,0,0)\}$,*

$$\langle c^*, y \rangle + \langle d^*, z \rangle + \langle w^*, w \rangle \geq 0, \tag{3.13}$$

and $\langle d^*, \bar{z} \rangle = 0$. Then, $\bar{x}$ is a strict minimal solution w.r.t. $\prec^l_C$ of (P).

Proof. Reasoning ad absurdum, assume that $\bar{x}$ is not a strict minimal solution w.r.t. $\prec^l_C$ of (P). Invoking from Lemma 3.2, there is $x_t \in \Omega$ such that, for all $\bar{y}_t \in F(\bar{x})$,

$$(F(x_t) - \bar{y}_t) \cap -\text{int}C \neq \emptyset.$$

This implies the existence of $x_t \in \Omega$ such that, for all $\bar{y}_t \in F(\bar{x})$, there exists $y_t \in F(x_t)$ satisfying $y_t - \bar{y}_t \in -\text{int}C$. This is nothing else but there are $y_t \in F(x_t), z_t \in G(x_t) \cap -D$, and $0 \in H(x_t)$ such that $y_t - \bar{y}_t \in -\text{int}C$, and thus,

$$\langle c^*, y_t - \bar{y}_t \rangle < 0. \tag{3.14}$$

Applying Lemma 2.1 gives us that

$$(y_t - \bar{y}_t, z_t - \bar{z}, 0 - \bar{w}) \in (DF_+(\bar{x}, \bar{y})(x_t - \bar{x}), DG_+(\bar{x}, \bar{z})(x_t - \bar{x}), DH(\bar{x}, 0)(x_t - \bar{x})).$$

Therefore, (3.13) amounts to

$$\langle c^*, y_t - \bar{y}_t \rangle + \langle d^*, z_t - \bar{z} \rangle = \langle c^*, y_t - \bar{y}_t \rangle + \langle d^*, z_t - \bar{z} \rangle + \langle w^*, 0 \rangle \geq 0.$$

Because $d^* \in D^+$ and $z_t \in G(x_t) \cap -D$, we have $\langle d^*, z_t \rangle \leq 0$. This together with $\langle d^*, \bar{z} \rangle = 0$ leads that

$$\langle c^*, y_t - \bar{y}_t \rangle \geq -\langle d^*, z_t \rangle \geq 0,$$

contradicting to (3.14). $\qquad\square$

Example 3.1. Let $X = \mathbb{R}^2, Y = Z = W = \mathbb{R}$ and $C = D = \mathbb{R}_+$, $F(x_1, x_2) = \{y \in \mathbb{R} \mid y \geq x_1^2\}$, $G(x_1, x_2) = \{z \in \mathbb{R} \mid z \geq -2x_2\}$ and $H(x_1, x_2) = \{w \in \mathbb{R} \mid w = -x_2\}$. Then, $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = 0\}$. By taking $\bar{x} = (0,0)$, $\bar{y} = 0$ and $\bar{z} = 0$, one has $F(x) + \text{int}C \not\supset F(\bar{x}), \forall x \in \Omega$, i.e., $F(x) \not\prec^l_C F(\bar{x}), \forall x \in \Omega$, in other words, $\bar{x}$ is a strict minimal solution w.r.t. $\prec^l_C$ of (P). We can check that F is C-convex, G is D-convex, H is convex and H is directionally metrically subregular at $(\bar{x}, 0)$ w.r.t X. Since $F_+(x) = F(x)$, $G_+(x) = G(x)$, it is easy to justify that F_+ is C-Aubin at $(\bar{x}, \bar{y})$, G_+ is D-Aubin at $(\bar{x}, \bar{z})$. By some calculations, we get that, for all $x \in X$,

$$DF_+(\bar{x}, \bar{y})(x) = \{y \in \mathbb{R} \mid y \geq 0\}, DG_+(\bar{x}, \bar{z})(x) = \{z \in \mathbb{R} \mid z \geq -2x_2\},$$

$$DH(\bar{x},0)(x) = \{w \in \mathbb{R} \mid w = -x_2\}.$$

Hence,

$$(DG_+(\bar{x},\bar{z}), DH(\bar{x},0))(X) + \mathrm{cone}(D + \bar{z}) \times \{0\} = Z \times W,$$

i.e., (KRZCQ) holds, and thus, all assumptions of Proposition 3.2 are fulfilled. By choosing $(c^*,d^*,w^*) = (1,0,0)$, one gets that $(c^*,d^*,w^*) \in (\mathbb{R}_+ \setminus \{0\}) \times \mathbb{R}_+ \times \mathbb{R}$ satisfying $\langle d^*, \bar{z} \rangle = 0$ and

$$\langle c^*, y \rangle + \langle d^*, z \rangle + \langle w^*, w \rangle \geq 0, \tag{3.15}$$

for every $x \in X$, $y \in DF_+(\bar{x},\bar{y})(x)$, $z \in DG_+(\bar{x},\bar{z})(x)$ and $w \in DH(\bar{x},0)(x)$. In addition, the hypotheses of Proposition 3.3 are also satisfied. So, we can derive from (3.15) that $\bar{x}$ is a strict minimal solution w.r.t. $\prec_C^l$ of (P).

4. DUALITY

In this section, we formulate a Mond-Weir [21] type dual problem and a Wolfe [25] type dual problem of (P) and explore weak and strong duality relations.

4.1. Mond-Weir type duality. For $x' \in X$, $y' \in F(x')$, $z' \in G(x') \cap (-D), w' \in H(x')$, $c^* \in C^+ \setminus \{0\}, d^* \in D^+$ and $w^* \in W^*$, let us denote

$$L(x',y',z',w',c^*,d^*,w^*) = F(x').$$

We consider the Mond-Weir type dual problem of (P) as follows

$$(D_{MW}): \begin{cases} \preceq_C^l - \max L(x',y',z',w',c^*,d^*,w^*) = F(x'), \\ s.t. \ \langle c^*, y \rangle + \langle d^*, z \rangle + \langle w^*, w \rangle \geq 0, \\ \forall (y,z,w) \in \bigcup_{x \in X} (DF_+(x',y')(x), DG_+(x',z')(x), DH(x',w')(x)), \\ \langle d^*, z' \rangle + \langle w^*, w' \rangle \geq 0, \\ (c^*,d^*,w^*) \in (C^+ \setminus \{0\}) \times D^+ \times W^*. \end{cases}$$

The feasible set of problem (D_{MW}) is

$$\Omega_{MW} := \Big\{ (x',y',z',w',c^*,d^*,w^*) \in X \times Y \times Z \times W \times (C^+ \setminus \{0\}) \times D^+ \times W^*, \ y' \in F(x'),$$

$$z' \in G(x') \cap (-D), w' \in H(x'), \langle c^*, y \rangle + \langle d^*, z \rangle + \langle w^*, w \rangle \geq 0,$$

$$\forall (y,z,w) \in \bigcup_{x \in X} (DF_+(x',y')(x), DG_+(x',z')(x), DH(x',w')(x)), \langle d^*, z' \rangle + \langle w^*, w' \rangle \geq 0 \Big\}.$$

A point $(\bar{x}',\bar{y}',\bar{z}',\bar{w}',\bar{c}^*,\bar{d}^*,\bar{w}^*) \in \Omega_{MW}$ is a strict maximal solution w.r.t. $\prec_C^l$ of (D_{MW}) if, for all $(x', y', z', w', c^*, d^*, w^*) \in \Omega_{MW}$,

$$L(\bar{x}',\bar{y}',\bar{z}',\bar{w}',\bar{c}^*,\bar{d}^*,\bar{w}^*) \not\prec_C^l L(x',y',z',w',c^*,d^*,w^*).$$

Proposition 4.1. *(Weak duality) Suppose that F,G,H be C-convex, D-convex, convex, resp. If $x \in \Omega$ and $(x',y',z',w',c^*,d^*,w^*) \in \Omega_{MW}$, then $F(x) \not\prec_C^l L(x',y',z',w',c^*,d^*,w^*)$.*

Proof. We assume to the contrary that $F(x) \prec_C^l L(x',y',z',w',c^*,d^*,w^*) = F(x')$. Then, one gets that $F(x) + \text{int}C \supseteq F(x')$. This ensures the existence of $y \in F(x)$ and $y' \in F(x')$ such that $y - y' \in -\text{int}C$. Since $c^* \in C^+ \setminus \{0_{Y^*}\}$, one has

$$\langle c^*, y - y' \rangle < 0. \tag{4.1}$$

The fact on $(x',y',z',w',c^*,d^*,w^*) \in \Omega_{MW}$ ensures that $y' \in F(x')$, $z' \in G(x') \cap (-D)$, $w' \in H(x')$, $(c^*, d^*, w^*) \in (C^+ \setminus \{0\}) \times D^+ \times W^*$,

$$\langle d^*, z' \rangle + \langle w^*, w' \rangle \geq 0, \tag{4.2}$$

and

$$\langle c^*, y \rangle + \langle d^*, z \rangle + \langle w^*, w \rangle \geq 0, \forall (y,z,w) \in \bigcup_{x \in X} (DF_+(x',y')(x), DG_+(x',z')(x), DH(x',w')(x)). \tag{4.3}$$

It follows from the convexity assumptions and Lemma 2.1 that, $\forall x \in X$,

$$(F(x) - y', G(x) - z', H(x) - w') \subset (DF_+(x',y')(x-x'), DG_+(x',z')(x-x'), DH(x',w')(x-x')).$$

Since $x \in \Omega$, there exists $z \in G(x) \cap (-D)$ and $0 \in H(x)$. This implies that

$$\langle d^*, z \rangle \leq 0. \tag{4.4}$$

Furthermore, for any $y \in F(x)$, one has

$$\begin{aligned}(y - y', z - z', 0 - w') &\in (F(x) - y', G(x) - z', H(x) - w') \\ &\subset (DF_+(x',y')(x-x'), DG_+(x',z')(x-x'), DH(x',w')(x-x')),\end{aligned}$$

and hence, utilizing (4.3) tells us that

$$\langle c^*, y - y' \rangle + \langle d^*, z \rangle - (\langle d^*, z' \rangle + \langle w^*, w' \rangle) = \langle c^*, y - y' \rangle + \langle d^*, z - z' \rangle + \langle w^*, 0 - w' \rangle \geq 0.$$

Combining this with (4.2) and (4.4), we arrive at

$$\langle c^*, y - y' \rangle \geq -\langle d^*, z \rangle + (\langle d^*, z' \rangle + \langle w^*, w' \rangle) \geq 0,$$

which contradicts with (4.1). This completes the proof. $\square$

Proposition 4.2. *(Strong duality) Let $\bar{x} \in X$, $\bar{y} \in F(\bar{x})$, $\bar{z} \in G(\bar{x}) \cap (-D)$, $\bar{w} = 0 \in H(\bar{x})$. Suppose that the following conditions hold:*

(i) $\text{WMin}_C F(\bar{x}) = \{\bar{y}\}$, $F(\bar{x})$ *has the weakly C-minimal property,*
(ii) *F, G, H are C-convex, D-convex, convex, respectively,*
(iii) F_+ *is C-Aubin at $(\bar{x},\bar{y})$, G_+ is D-Aubin at $(\bar{x},\bar{z})$,*
(iv) *H is directionally metrically subregular at $(\bar{x},0)$ w.r.t. X,*
(v) $(DG_+(\bar{x},\bar{z}), DH(\bar{x},0))(X) + \text{cone}(D + \bar{z}) \times \{0\} = Z \times W$.

If $\bar{x} \in \Omega$ is a strict minimal solution w.r.t. $\prec_C^l$ of (P), then there exists $\bar{c}^* \in C^+ \setminus \{0\}, \bar{d}^* \in D^+$ and $\bar{w}^* \in W^*$ such that $(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{c}^*, \bar{d}^*, \bar{w}^*)$ is a strict maximal solution w.r.t. $\prec_C^l$ of (D_{MW}).

Proof. We derive from Proposition 3.2 the existence of $(\bar{c}^*, \bar{d}^*, \bar{w}^*) \in C^+ \setminus \{0\} \times D^+ \times W^*$ such that, for all $x \in X$ and for all $(y,z,w) \in (DF_+(\bar{x},\bar{y})(x), DG_+(\bar{x},\bar{z})(x), DH(\bar{x},0)(x)) \setminus \{(0,0,0)\}$,

$$\langle \bar{c}^*, y \rangle + \langle \bar{d}^*, z \rangle + \langle \bar{w}^*, w \rangle \geq 0 \text{ and } \langle \bar{d}^*, \bar{z} \rangle + \langle \bar{w}^*, \bar{w} \rangle = \langle \bar{d}^*, \bar{z} \rangle = 0.$$

Hence, $(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{c}^*, \bar{d}^*, \bar{w}^*) \in \Omega_{MW}$ and $L(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{c}^*, \bar{d}^*, \bar{w}^*) = F(\bar{x})$. Now, suppose to the contrary that $(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{c}^*, \bar{d}^*, \bar{w}^*)$ is not a strict minimal solution w.r.t. $\prec_C^l$ of (D_{MW}). Then, there exists $(x', y', z', w', c^*, d^*, w^*) \in \Omega_{MW}$ such that

$$F(\bar{x}) = L(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{c}^*, \bar{d}^*, \bar{w}^*) \prec_C^l L(x', y', z', w', c^*, d^*, w^*),$$

contradicting with Proposition 4.1. Hence, $(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{c}^*, \bar{d}^*, \bar{w}^*)$ is a strict maximal solution w.r.t. $\prec_C^l$ of (D_{MW}). $\square$

Remark 4.1. Let $X = \mathbb{R}^n, Y = \mathbb{R}^p, Z = \mathbb{R}^q, W = \mathbb{R}^r, C = \mathbb{R}_+^p, D = \mathbb{R}_+^q$, $F := (f_1, ..., f_p)$, $G := (g_1, ..., g_q), H := (h_1, ..., h_r)$ be single-valued functions, where $f_i, i = 1, ..., p$, $g_j, j = 1, ..., q$, and $h_k, k = 1, ..., r$, are continuously differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$. Then, for $\bar{x} \in \mathbb{R}^n$,

$$DF_+(\bar{x}, f(\bar{x}))(x) = \nabla f(\bar{x})(x) + \mathbb{R}_+^p, DG_+(\bar{x}, g(\bar{x}))(x) = \nabla g(\bar{x})(x) + \mathbb{R}_+^q,$$

$$DH(\bar{x}, h(\bar{x}))(x) = \{\nabla h(\bar{x})(x)\}, \forall x \in X.$$

The Mond-Weir dual problem of (P) is

$$(D_{MW}): \begin{cases} \mathbb{R}_+^p - \max L(x', f(x'), g(x'), h(x'), c^*, d^*, w^*) = f(x'), \\ \quad s.t. \ \nabla f(x')^T c^* + \nabla g(x')^T d^* + \nabla h(x')^T w^* = 0, \\ \quad \langle d^*, g(x') \rangle + \langle w^*, h(x') \rangle \geq 0, \\ \quad (c^*, d^*, w^*) \in (\mathbb{R}_+^p \setminus \{0\}) \times \mathbb{R}_+^q \times \mathbb{R}^r, \end{cases}$$

since

$$\langle \nabla f(x')^T c^* + \nabla g(x')^T d^* + \nabla h(x')^T w^*, x \rangle$$
$$= \langle c^*, \nabla f(x')(x) \rangle + \langle d^*, \nabla g(x')(x) \rangle + \langle w^*, \nabla h(x')(x) \rangle$$
$$\geq 0$$

for all $x \in \mathbb{R}^n$ if and only if

$$\nabla f(x')^T c^* + \nabla g(x')^T d^* + \nabla h(x')^T w^* = 0.$$

When $p = 1$, this problem is exactly the Mond-Weir dual problem (see [5, 21]).

Example 4.1. Let $X = \mathbb{R}^2, Y = Z = W = \mathbb{R}$ and $C = D = \mathbb{R}_+$, $F(x_1, x_2) = \{y \in \mathbb{R} \mid y \geq x_1^2\}$, $G(x_1, x_2) = \{z \in \mathbb{R} \mid z \geq -2x_2\}$ and $H(x_1, x_2) = \{w \in \mathbb{R} \mid w = -x_2\}$. Consider the following set optimization problem with mixed constraints:

$$(P): \ \preceq_C^l - \min_{x \in X} \{F(x) \mid G(x) \cap (-D) \neq \emptyset, 0 \in H(x)\}.$$

Then, $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = 0\}$. For $x' \in X, y' \in F(x'), z' \in G(x') \cap (-D), w' \in H(x')$, one gets that, for $x \in \mathbb{R}^2$,

$$DF_+(x', y')(x) = \begin{cases} \{y \in \mathbb{R} \mid y \geq 2x_1'x_1\}, & \text{if } y' = x_1'^2, \\ \mathbb{R}, & \text{if } y' > x_1'^2, \end{cases}$$

$$DG_+(x', z')(x) = \begin{cases} \{z \in \mathbb{R} \mid z \geq -2x_2\}, & \text{if } z' = -2x_2', \\ \mathbb{R}, & \text{if } z' > -2x_2', \end{cases}$$

$$DH(x', w')(x) = \{w \in \mathbb{R} \mid w = -x_2\}.$$

The Mond-Weir type dual problem of (P) is

$$(D_{MW}): \begin{cases} \preceq_C^l -\max L(x',y',z',w',c^*,d^*,w^*) = F(x'), \\ s.t. \ c^*y + d^*z + w^*w \geq 0, \\ \forall (y,z,w) \in \bigcup_{x \in \mathbb{R}^2} (DF_+(x',y')(x), DG_+(x',z')(x), DH(x',w')(x)), \\ d^*z' + w^*w' \geq 0, \\ (c^*,d^*,w^*) \in (\mathbb{R}_+ \setminus \{0\}) \times \mathbb{R}_+ \times \mathbb{R}. \end{cases}$$

By picking $\bar{x} = (0,0)$, $\bar{y} = 0$ and $\bar{z} = 0$, we obtain $F(\bar{x}) = \{y \in \mathbb{R} \mid y \geq 0\}$, $G(\bar{x}) = \{z \in \mathbb{R} \mid z \geq 0\}$, $H(\bar{x}) = \{0\}$, and hence, $\bar{y} \in F(\bar{x})$, $\bar{z} \in G(\bar{x}) \cap (-D)$, $\bar{w} = 0 \in H(\bar{x})$. By some straightforward calculations show that all the assumptions in Proposition 4.2 hold. Taking $(c^*,d^*,w^*) = (1,0,0)$, we yield the existence of $(c^*,d^*,w^*) \in (\mathbb{R}_+ \setminus \{0\}) \times \mathbb{R}_+ \times \mathbb{R}$ such that $d^*\bar{z} = 0$ and

$$c^*y + d^*z + w^*w \geq 0,$$

for every $x \in \mathbb{R}^2$, $y \in DF_+(\bar{x},\bar{y})(x)$, $z \in DG_+(\bar{x},\bar{z})(x)$ and $w \in DH(\bar{x},\bar{w})(x)$. Hence, we derive from Proposition 4.2 that $(\bar{x},\bar{y},\bar{z},\bar{w},\bar{c}^*,\bar{d}^*,\bar{w}^*)$ is a strict maximal solution w.r.t. $\prec_C^l$ of (D_{MW}).

We can check directly that $(\bar{x},\bar{y},\bar{z},\bar{w},\bar{c}^*,\bar{d}^*,\bar{w}^*)$ is a strict maximal solution w.r.t. $\prec_C^l$ of (D_{MW}) as follows. Firstly, we will prove that if $(x',y',z',w',c^*,d^*,w^*) \in \Omega_{MW}$ then $x'_1 = 0$. Reasoning by contraposition, assume that there exists $(x',y',z',w',c^*,d^*,w^*) \in \Omega_{MW}$ and $x'_1 \neq 0$. Since

$$c^*y + d^*z + w^*w \geq 0,$$

for all $(y,z,w) \in \bigcup_{x \in \mathbb{R}^2} (DF_+(x',y')(x), DG_+(x',z')(x), DH(x',w')(x))$, $0 \in DG_+(x',z')((x_1,0))$ and $0 \in DH(x',w')((x_1,0))$, one has

$$c^*y \geq 0, \forall y \in \bigcup_{x \in X_1} DF_+(x',y')(x), \tag{4.5}$$

where $X_1 = \{x \in \mathbb{R}^2 \mid x_2 = 0\}$. Then, there are only two cases as follows.

∗ *Case 1.* If $y' > x_1'^2$, then $DF_+(x',y')(x) = \mathbb{R}$ for all $x \in X_1$. Hence, we deduce from (4.5) that $c^* = 0$, contradicting with $(x',y',z',w',c^*,d^*,w^*) \in \Omega_{MW}$.

∗ *Case 2.* If $y' = x_1'^2 > 0$, then $DF_+(x',y')(x) = \{y \in \mathbb{R} \mid y \geq 2x_1'x_1\}$ for all $x \in X_1$. Hence, by taking $x = (1,0) \in X_1$, $y = 2x_1' \in DF_+(x',y')((1,0))$, $x = (-1,0) \in X_1$, $y = -2x_1' \in DF_+(x',y')((-1,0))$, we deduce from (4.5) that $c^* = 0$, contradicting with $(x',y',z',w',c^*,d^*,w^*) \in \Omega_{MW}$.

Thus, if $(x',y',z',w',c^*,d^*,w^*) \in \Omega_{MW}$ then $x'_1 = 0$. Hence, for all $(x',y',z',w',c^*,d^*,w^*) \in \Omega_{MW}$, one has

$$F(\bar{x}) + \mathrm{int}C \not\supset F(x'), \text{ i.e., } F(\bar{x}) \not\prec_C^l F(x').$$

Equivalently, for all $(x',y',z',w',c^*,d^*,w^*) \in \Omega_{MW}$,

$$L(\bar{x},\bar{y},\bar{z},\bar{w},\bar{c}^*,\bar{d}^*,\bar{w}^*) \not\prec_C^l L(x',y',z',w',c^*,d^*,w^*),$$

i.e., $(\bar{x},\bar{y},\bar{z},\bar{w},\bar{c}^*,\bar{d}^*,\bar{w}^*)$ is a strict maximal solution w.r.t. $\prec_C^l$ of (D_{MW}).

4.2. **Wolfe type duality.** For $x' \in X$, $y' \in F(x')$, $z' \in G(x') \cap (-D)$, $w' \in H(x')$, $c^* \in C^+ \setminus \{0\}$, $d^* \in D^+$, $w^* \in W^*$, and a fixed point $\tilde{c} \in C \setminus \{0\}$, denote

$$\widetilde{L}(x',y',z',w',c^*,d^*,w^*) = F(x') + \langle d^*,z' \rangle \tilde{c} + \langle w^*,w' \rangle \tilde{c}.$$

We define the Wolfe type dual problem of (P) as follows:

$$(D_W): \begin{cases} \preceq^l_C - \max \widetilde{L}(x',y',z',w',c^*,d^*,w^*) = F(x') + \langle d^*,z' \rangle \tilde{c} + \langle w^*,w' \rangle \tilde{c}, \\ \quad s.t. \ \langle c^*,y \rangle + \langle d^*,z \rangle + \langle w^*,w \rangle \geq 0, \\ \quad \forall (y,z,w) \in \bigcup_{x \in X} (DF_+(x',y')(x), DG_+(x',z')(x), DH(x',w')(x)), \\ \quad \langle c^*,\tilde{c} \rangle = 1, \\ \quad (c^*,d^*,w^*) \in (C^+ \setminus \{0\}) \times D^+ \times W^*. \end{cases}$$

The set of feasible points of (D_W) is

$$\Omega_W := \Big\{ (x',y',z',w',c^*,d^*,w^*) \in X \times Y \times Z \times W \times (C^+ \setminus \{0\}) \times D^+ \times W^*, \ y' \in F(x'),$$

$$z' \in G(x') \cap (-D), 0 \in H(x'), \langle c^*,y \rangle + \langle d^*,z \rangle + \langle w^*,w \rangle \geq 0,$$

$$\forall (y,z,w) \in \bigcup_{x \in X} (DF_+(x',y')(x), DG_+(x',z')(x), DH(x',w')(x)), \ \langle c^*,\tilde{c} \rangle = 1 \Big\}.$$

A point $(\bar{x}',\bar{y}',\bar{z}',\bar{w}',\bar{c}^*,\bar{d}^*,\bar{w}^*) \in \Omega_W$ is said to be a strict maximal solution w.r.t. $\prec^l_C$ of (D_W) if, for all $(x',y',z',w',c^*,d^*,w^*) \in \Omega_W$,

$$\widetilde{L}(\bar{x}',\bar{y}',\bar{z}',\bar{w}',\bar{c}^*,\bar{d}^*,\bar{w}^*) \nprec^l_C \widetilde{L}(x',y',z',w',c^*,d^*,w^*).$$

Proposition 4.3. *(Weak duality) Suppose that F,G,H be C-convex, D-convex, convex, resp. If $x \in \Omega$ and $(x',y',z',w',c^*,d^*,w^*) \in \Omega_W$, then $F(x) \nprec^l_C \widetilde{L}(x',y',z',w',c^*,d^*,w^*)$.*

Proof. We assume to contrary that

$$F(x) \prec^l_C \widetilde{L}(x',y',z',w',c^*,d^*,w^*) = F(x') + \langle d^*,z' \rangle \tilde{c} + \langle w^*,w' \rangle \tilde{c}.$$

Then, we get that

$$F(x) + \text{int}C \supseteq F(x') + \langle d^*,z' \rangle \tilde{c} + \langle w^*,w' \rangle \tilde{c}.$$

This implies the existence of $y \in F(x)$ and $y' \in F(x')$ such that

$$y - (y' + \langle d^*,z' \rangle \tilde{c} + \langle w^*,w' \rangle \tilde{c}) \in -\text{int}C.$$

Since $c^* \in C^+ \setminus \{0\}$ and $\langle c^*,\tilde{c} \rangle = 1$, we derive from the above inequality that

$$\langle c^*,y-y' \rangle - (\langle d^*,z' \rangle + \langle w^*,w' \rangle) = \langle c^*,y-y' \rangle - (\langle d^*,z' \rangle + \langle w^*,w' \rangle) \langle c^*,\tilde{c} \rangle < 0. \tag{4.6}$$

The fact $(x',y',z',w',c^*,d^*,w^*) \in \Omega_W$ justifies that $y' \in F(x')$, $z' \in G(x') \cap (-D)$, $w' \in H(x')$, $(c^*,d^*,w^*) \in (C^+ \setminus \{0\}) \times D^+ \times W^*$ and

$$\langle c^*,y \rangle + \langle d^*,z \rangle + \langle w^*,w \rangle \geq 0, \tag{4.7}$$

for all $(y,z,w) \in \bigcup_{x \in X} (DF_+(x',y')(x), DG_+(x',z')(x), DH(x',w')(x))$. It follows from the convexity assumptions and Lemma 2.1 that, $\forall x \in X$,

$$(F(x)-y', G(x)-z', H(x)-w') \subset (DF_+(x',y')(x-x'), DG_+(x',z')(x-x'), DH(x',w')(x-x')).$$

Since $x \in \Omega$, there exists $z \in G(x) \cap (-D)$ and $0 \in H(x)$. This leads that

$$\langle d^*, z \rangle \leq 0. \tag{4.8}$$

Thus, for any $y \in F(x)$, one has

$$\begin{aligned}(y - y', z - z', 0 - w') &\in (F(x) - y', G(x) - z', H(x) - w') \\ &\subset (DF_+(x', y')(x - x'), DG_+(x', z')(x - x'), DH(x', w')(x - x')).\end{aligned}$$

Therefore, we deduce from (4.7) that

$$\langle c^*, y - y' \rangle + \langle d^*, z \rangle - (\langle d^*, z' \rangle + \langle w^*, w' \rangle) = \langle c^*, y - y' \rangle + \langle d^*, z - z' \rangle + \langle w^*, 0 - w' \rangle \geq 0.$$

This along with (4.8) guarantees that

$$\langle c^*, y - y' \rangle - (\langle d^*, z' \rangle + \langle w^*, w' \rangle) \geq -\langle d^*, z \rangle \geq 0,$$

which contradicts with (4.6). $\qquad \square$

Proposition 4.4. *(Strong duality) Let $\bar{x} \in X$, $\bar{y} \in F(\bar{x})$, $\bar{z} \in G(\bar{x}) \cap (-D)$, $\bar{w} = 0 \in H(\bar{x})$. Suppose that the following conditions hold:*

(i) $\mathrm{WMin}_C F(\bar{x}) = \{\bar{y}\}$, $F(\bar{x})$ has the weakly C-minimal property,
(ii) F, G, H are C-convex, D-convex, convex, respectively,
(iii) F_+ is C-Aubin at $(\bar{x}, \bar{y})$, G_+ is D-Aubin at $(\bar{x}, \bar{z})$,
(iv) H is directionally metrically subregular at $(\bar{x}, 0)$ w.r.t. X,
(v) $(DG_+(\bar{x}, \bar{z}), DH(\bar{x}, 0))(X) + \mathrm{cone}(D + \bar{z}) \times \{0\} = Z \times W$.

If $\bar{x} \in \Omega$ is a strict minimal solution w.r.t. $\prec_C^l$ of (P), then there exists $\bar{c}^* \in C^+ \setminus \{0\}, \bar{d}^* \in D^+$ and $\bar{w}^* \in W^*$ such that $(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{c}^*, \bar{d}^*, \bar{w}^*)$ is a strict maximal solution w.r.t. $\prec_C^l$ of (D_W).

Proof. According to Proposition 3.2, there exists $(\tilde{c}^*, \tilde{d}^*, \tilde{w}^*) \in C^+ \setminus \{0\} \times D^+ \times W^*$ such that, for all $x \in X$ and for all $(y, z, w) \in (DF_+(\bar{x}, \bar{y})(x), DG_+(\bar{x}, \bar{z})(x), DH(\bar{x}, 0)(x)) \setminus \{(0, 0, 0)\}$,

$$\langle \tilde{c}^*, y \rangle + \langle \tilde{d}^*, z \rangle + \langle \tilde{w}^*, w \rangle \geq 0 \text{ and } \langle \tilde{d}^*, \bar{z} \rangle + \langle \tilde{w}^*, \bar{w} \rangle = \langle \tilde{d}^*, \bar{z} \rangle = 0.$$

Since $\tilde{c} \in C \setminus \{0\}$, $\tilde{c}^* \in C^+ \setminus \{0\}$, one has $\alpha := \langle \tilde{c}^*, \tilde{c} \rangle > 0$. Setting $(\bar{c}^*, \bar{d}^*, \bar{w}^*) := \frac{1}{\alpha}(\tilde{c}^*, \tilde{d}^*, \tilde{w}^*)$, we have

$$(\bar{c}^*, \bar{d}^*, \bar{w}^*) \in C^+ \setminus \{0\} \times D^+ \times W^* \text{ and } \langle \bar{c}^*, \tilde{c} \rangle = 1.$$

Moreover, for all $x \in X$ and for all $(y, z, w) \in (DF_+(\bar{x}, \bar{y})(x), DG_+(\bar{x}, \bar{z})(x), DH(\bar{x}, 0)(x)) \setminus \{(0, 0, 0)\}$,

$$\langle \bar{c}^*, y \rangle + \langle \bar{d}^*, z \rangle + \langle \bar{w}^*, w \rangle = \frac{1}{\alpha}(\langle \tilde{c}^*, y \rangle + \langle \tilde{d}^*, z \rangle + \langle \tilde{w}^*, w \rangle) \geq 0.$$

Hence, $(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{c}^*, \bar{d}^*, \bar{w}^*) \in \Omega_W$. Now, suppose to the contrary that $(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{c}^*, \bar{d}^*, \bar{w}^*)$ is not a strict maximal solution w.r.t. $\prec_C^l$ of (D_W). Then, there exists $(x', y', z', w', c^*, d^*, w^*) \in \Omega_W$ such that $\widetilde{L}(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{c}^*, \bar{d}^*, \bar{w}^*) \prec_C^l \widetilde{L}(x', y', z', w', c^*, d^*, w^*)$. Since $\langle \bar{w}^*, \bar{w} \rangle = \langle \bar{w}^*, 0 \rangle = 0$ and $\langle \bar{d}^*, \bar{z} \rangle = \frac{1}{\alpha}\langle \tilde{d}^*, \bar{z} \rangle = 0$, one gets

$$F(\bar{x}) = \widetilde{L}(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{c}^*, \bar{d}^*, \bar{w}^*) \prec_C^l \widetilde{L}(x', y', z', w', c^*, d^*, w^*),$$

contradicting with Proposition 4.3. So, $(\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{c}^*, \bar{d}^*, \bar{w}^*)$ is a strict maximal solution w.r.t. $\prec_C^l$ of (D_W). $\qquad \square$

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