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SEMICONTINUITY OF THE COMPOSITION OF SET-VALUED MAP AND SCALARIZATION FUNCTION FOR SETS

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Abstract. In this paper, we deal with the inheritation of the semicontinuity of set-valued maps via general scalarization for sets, which is regarded as the framework of generalizations of results by Kuwano, Tanaka, and Yamada in 2010. Since the unified scalarization functions for sets satisfy certain desired semicontinuity, our main theorems can be reduced to the results in earlier study.

Keywords. Set optimization; Sublinear scalarization; Set relation; Set-valued map; Semicontinuity.

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1. Introduction

Continuity is one of the most beneficial concepts in mathematical analysis. For example, it is well known that every continuous real-valued function on a compact set always has a minimum and maximum. Consequently, we can make reasonable models for several social systems and propose approximate solutions of optimization problems by iterative method. For set-valued maps, there are many different kinds of notions on semicontinuity and important results which are not only in theoretical studies but also in many algorithmic analyses related to mathematical programming problems. Accordingly there are many interesting researches in set-valued analysis and various continuity notions for set-valued maps and their properties (see, e.g., [2, 7]).

On the other hand, scalarization plays a key role in vector optimization and set optimization because optimization problems with set-valued functions become easy to handle by converting vectors or sets into real numbers. Based on Tammer's sublinear scalarization function for vectors [1, 2], some authors ([3, 6, 8, 14, 15]) investigated scalarization functions for sets related to set relations [9]. Especially, Nishizawa, Tanaka, and Georgiev [13] suggested interesting results on inherited properties of set-valued maps. Besides, Hamel and Löhne [4] introduced several useful nonlinear scalarization functions for sets with respect to set relations. It provided some motivation for Kuwano, Tanaka, and Yamada to reframe those researches into unification of nonlinear scalarization. In [11], they discussed certain inherited properties on continuity of set-valued maps via the unified scalarizations introduced in [10]. Furthermore, Sonda, Kuwano, and Tanaka [16] extended that work to the case of cone-continuity.

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The aim of this paper is to show certain results on the inheritation of the semicontinuity for set-valued maps via general scalarization for sets, which are regarded as generalizations of results in [11]. In fact, since the unified scalarization functions introduced in [10] satisfy certain desired semicontinuity, our main theorems can be reduced into special cases which are results in [11].

The organization of the paper is as follows. In Section 2, we recall some basic concepts on set relations in set optimization and the scalarization scheme for sets in a real vector space such that each scalarization function has order-monotone property for set relation. In Section 3, we introduce a new concept of the invariant property for set-valued maps with respect to a binary relationship on a family of sets, which is regarded as some kind of continuity from the view point of order-monotonicity. Then, we show certain inheritation of the semicontinuity of set-valued maps via general scalarization for sets. The last section, Section 4, ends this paper.

2. Set relations and scalarization functions for sets

Throughout the paper, let Y be a real topological vector space unless otherwise specified. Let θ_Y be the zero vector in Y and $\mathscr{P}(Y)$ denote the set of all nonempty subsets of Y. The topological interior, topological closure, convex hull, and complement of a set $A \in \mathscr{P}(Y)$ are denoted by $\mathrm{int}A$, $\mathrm{cl}A$, $\mathrm{co}A$, and A^{c}, respectively.

For given $A, B \in \mathscr{P}(Y)$ and $t \in \mathbb{R}$, the algebraic sum $A+B$ and the scalar multiplication tA are defined as follows:

$$A+B := \{a+b \mid a \in A,\ b \in B\}, \quad tA := \{ta \mid a \in A\}.$$

In particular, we denote $A+\{y\}$ by $A+y$ (or $y+A$) and $(-1)A$ by $-A$ for $A \in \mathscr{P}(Y)$ and $y \in Y$. We define $Ty := \{ty \mid t \in T\}$ for $y \in Y$ and $T \subset \mathbb{R}$.

Let X be a nonempty set and $\preccurlyeq$ a binary relation on X. The relation $\preccurlyeq$ is said to be

(1) reflexive if for all $x \in X$, $x \preccurlyeq x$;
(2) irreflexive if for all $x \in X$, $x \not\preccurlyeq x$;
(3) transitive if for all $x, y, z \in X$, $x \preccurlyeq y$ and $y \preccurlyeq z$ imply $x \preccurlyeq z$;
(4) antisymmetric if for all $x, y \in X$, $x \preccurlyeq y$ and $y \preccurlyeq x$ imply $x = y$;
(5) complete if for all $x, y \in X$, $x \preccurlyeq y$ or $y \preccurlyeq x$.

The relation $\preccurlyeq$ is called

(1) a preorder if it is reflexive and transitive;
(2) a strict order if it is irreflexive and transitive;
(3) a partial order if it is reflexive, transitive, and antisymmetric;
(4) a total order if it is reflexive, transitive, antisymmetric, and complete.

A set $C \in \mathscr{P}(Y)$ is called a cone if $ty \in C$ for every $y \in C$ and $t > 0$. Let us remark that a cone considered in this paper does not necessarily contain the zero vector θ_Y. Let C be a convex cone in Y. Then, $C+C = C$ holds, and $\mathrm{int}C$ and $\mathrm{cl}C$ are also convex cones. If $\theta_Y \in C$, we define a preoder $\leq_C$ on Y induced by C as follows:

$$\text{for } y_1, y_2 \in Y, y_1 \leq_C y_2 \stackrel{\text{def}}{\Longleftrightarrow} y_2 - y_1 \in C.$$

This preorder is compatible with the linear structure:

$$\text{for all } y_1, y_2, y_3 \in Y, \quad y_1 \leq_C y_2 \implies y_1 + y_3 \leq_C y_2 + y_3; \tag{2.1}$$

$$\text{for all } y_1, y_2 \in Y \text{ and } t > 0, \quad y_1 \leq_C y_2 \implies ty_1 \leq_C ty_2. \tag{2.2}$$

If $\theta_Y \notin C$, then $\leq_C$ is irreflexive and hence a strict order. In addition, assuming that C is pointed (i.e., $C \cap (-C) = \{\theta_Y\}$), one can check that $\leq_C$ is antisymmetric and becomes a partial order.

Proposition 2.1 ([5])**.** *Let C, C' be convex cones in Y and $d \in Y$. Assume that $C + (0, +\infty)d \subset C'$. Then, for any $y_1, y_2 \in Y$ and $t, t' \in \mathbb{R}$ with $t > t'$,*

$$y_1 + td \leq_C y_2 \implies y_1 + t'd \leq_{C'} y_2.$$

As generalizations of partial orderings for vectors, we give a definition of certain binary relations between sets in Y, called set relations.

Definition 2.1 (set relations, [9])**.** Let C be a convex cone in Y. The six types of set relations are defined by

(1) $A \leq_C^{(1)} B \overset{\text{def}}{\iff} \forall a \in A,\ \forall b \in B,\ a \leq_C b \iff A \subset \bigcap_{b \in B}(b - C)$;
(2) $A \leq_C^{(2L)} B \overset{\text{def}}{\iff} \exists a \in A \text{ s.t. } \forall b \in B,\ a \leq_C b \iff A \cap (\bigcap_{b \in B}(b - C)) \neq \emptyset$;
(3) $A \leq_C^{(3L)} B \overset{\text{def}}{\iff} \forall b \in B,\ \exists a \in A \text{ s.t. } a \leq_C b \iff B \subset A + C$;
(4) $A \leq_C^{(2U)} B \overset{\text{def}}{\iff} \exists b \in B \text{ s.t. } \forall a \in A,\ a \leq_C b \iff (\bigcap_{a \in A}(a + C)) \cap B \neq \emptyset$;
(5) $A \leq_C^{(3U)} B \overset{\text{def}}{\iff} \forall a \in A,\ \exists b \in B \text{ s.t. } a \leq_C b \iff A \subset B - C$;
(6) $A \leq_C^{(4)} B \overset{\text{def}}{\iff} \exists a \in A,\ \exists b \in B \text{ s.t. } a \leq_C b \iff A \cap (B - C) \neq \emptyset$

for A, $B \in \mathscr{P}(Y)$.

Here, the letters L and U stand for "lower" and "upper," respectively.

In general, the relation $\leq_C^{(j)}$ is transitive for $j = 1, 2L, 2U, 3L, 3U$ and not transitive for $j = 4$. If $\theta_Y \in C$, $\leq_C^{(j)}$ is reflexive for $j = 3L, 3U, 4$ and hence a preorder for $j = 3L, 3U$. If $\theta_Y \notin C$, $\leq_C^{(j)}$ is irreflexive and hence a strict order for $j = 1, 2L, 2U$. For each $j = 1, 2L, 2U, 3L, 3U, 4$, the relation $\leq_C^{(j)}$ satisfies the following properties for all $A, B \in \mathscr{P}(Y)$:

(1) $A \leq_C^{(j)} B \implies A + y \leq_C^{(j)} B + y$ for $y \in Y$;
(2) $A \leq_C^{(j)} B \implies tA \leq_C^{(j)} tB$ for $t > 0$.

From the definition, we easily obtain the following implications:

$$\begin{cases} A \leq_C^{(1)} B \implies A \leq_C^{(2L)} B \implies A \leq_C^{(3L)} B \implies A \leq_C^{(4)} B; \\ A \leq_C^{(1)} B \implies A \leq_C^{(2U)} B \implies A \leq_C^{(3U)} B \implies A \leq_C^{(4)} B \end{cases} \tag{2.3}$$

for $A, B \in \mathscr{P}(Y)$.

Also, we have the following equivalences.

Proposition 2.2. *For A, $B \in \mathscr{P}(Y)$, the following statements hold:*

(1) $A \leq_C^{(1)} B \iff -B \leq_C^{(1)} -A$;
(2) $A \leq_C^{(2L)} B \iff -B \leq_C^{(2U)} -A$;
(3) $A \leq_C^{(3L)} B \iff -B \leq_C^{(3U)} -A$;
(4) $A \leq_C^{(2U)} B \iff -B \leq_C^{(2L)} -A$;
(5) $A \leq_C^{(3U)} B \iff -B \leq_C^{(3L)} -A$;

(6) $A \leq_C^{(4)} B \iff -B \leq_C^{(4)} -A$.

Proof. The conclusion follows immediately from Definition 2.1. □

Proposition 2.3. *Let C, C' be convex cones in Y and $d \in Y$. Assume that $C + (0, +\infty)d \subset C'$. Then, for each $j = 1, 2L, 2U, 3L, 3U, 4$, any $A, B \in \mathscr{P}(Y)$, $s, s' \in \mathbb{R}$ with $s < s'$ and $t, t' \in \mathbb{R}$ with $t > t'$,*

$$A \leq_C^{(j)} B + sd \implies A \leq_{C'}^{(j)} B + s'd$$

and

$$A + td \leq_C^{(j)} B \implies A + t'd \leq_{C'}^{(j)} B.$$

Proof. The conclusion follows immediately from Definition 2.1 and Proposition 2.1. □

Based on the set relations, we introduce the following scalarization functions for sets in Y, which are certain generalization as unification of several nonlinear scalarizations proposed in [13].

Definition 2.2 ([5, 10])**.** For each $j = 1, 2L, 2U, 3L, 3U, 4$, we define

$$I_C^{(j)}(A; V, d) := \inf \left\{ t \in \mathbb{R} \,\middle|\, A \leq_C^{(j)} (V + td) \right\}, \tag{2.4}$$

$$S_C^{(j)}(A; V, d) := \sup \left\{ t \in \mathbb{R} \,\middle|\, (V + td) \leq_C^{(j)} A \right\} \tag{2.5}$$

for $A, V \in \mathscr{P}(Y)$ and $d \in Y$; V and d are index parameters for scalarization and play key roles as a reference set and a direction, respectively, as one kind of sublinear-like scalarization for a given set A.

Proposition 2.4. *Let $A, V \in \mathscr{P}(Y)$ and $d \in Y$. Then, the following statements hold:*

$$\begin{aligned}
-I_C^{(1)}(-A; -V, d) &= S_C^{(1)}(A; V, d); \\
-I_C^{(2L)}(-A; -V, d) &= S_C^{(2U)}(A; V, d); \\
-I_C^{(3L)}(-A; -V, d) &= S_C^{(3U)}(A; V, d); \\
-I_C^{(2U)}(-A; -V, d) &= S_C^{(2L)}(A; V, d); \\
-I_C^{(3U)}(-A; -V, d) &= S_C^{(3L)}(A; V, d); \\
-I_C^{(4)}(-A; -V, d) &= S_C^{(4)}(A; V, d).
\end{aligned}$$

Proof. The conclusion follows immediately from Proposition 2.2. □

Lemma 2.1. *Let $V, W \in \mathscr{P}(Y)$, $d \in C$ and $\alpha \in \mathbb{R}$. Then*

$$W \cap \left(\bigcap_{v \in V} (v - C) + \alpha d \right) = \emptyset \implies \alpha \leq I_C^{(1)}(A; V, d), \ \forall A \in \mathscr{P}(Y) \text{ with } A \cap W \neq \emptyset.$$

Proof. We assume that $\alpha > I_C^{(1)}(A; V, d)$ for some $A \in \mathscr{P}(Y)$ with $A \cap W \neq \emptyset$. By Proposition 2.3, we obtain

$$A \subset \left(\bigcap_{v \in V} (v - C) + \alpha d \right).$$

Since $A \cap W \neq \emptyset$, we have

$$W \cap \left(\bigcap_{v \in V} (v - C) + \alpha d \right) \neq \emptyset.$$

This is a contradiction, and therefore the conclusion follows. $\square$

Lemma 2.2. *Let $V, W \in \mathscr{P}(Y)$, $d \in C$ and $\alpha \in \mathbb{R}$. Then*

$$W \cap ((V - C) + \alpha d) = \emptyset \implies \alpha \leq I_C^{(3U)}(A; V, d),\ \forall A \in \mathscr{P}(Y) \text{ with } A \cap W \neq \emptyset.$$

Proof. The proof is similar to that of Lemma 2.1. $\square$

Lemma 2.3. *Let $V, W \in \mathscr{P}(Y)$, $d \in C$ and $\alpha \in \mathbb{R}$. Then*

$$W \subset \left(\bigcap_{v \in V} (v - C) + \alpha d \right) \implies \alpha \geq I_C^{(2L)}(A; V, d),\ \forall A \in \mathscr{P}(Y) \text{ with } A \cap W \neq \emptyset.$$

Proof. The proof is similar to that of Lemma 2.1. $\square$

Lemma 2.4. *Let $V, W \in \mathscr{P}(Y)$, $d \in C$ and $\alpha \in \mathbb{R}$. Then*

$$W \subset ((V - C) + \alpha d) \implies \alpha \geq I_C^{(4)}(A; V, d),\ \forall A \in \mathscr{P}(Y) \text{ with } A \cap W \neq \emptyset.$$

Proof. The proof is similar to that of Lemma 2.1. $\square$

Lemma 2.5. *Let $V, W, Q \in \mathscr{P}(Y)$ and $d \in C$. Then,*

$$W \subset Q \implies I_C^{(j)}(W; V, d) \leq I_C^{(j)}(Q; V, d)$$

for each $j = 1, 2U, 3U$.

Proof. When $j = 1$, we assume that there exists $t' \in \mathbb{R}$ such that

$$I_C^{(1)}(W; V, d) > t' > I_C^{(1)}(Q; V, d).$$

By Proposition 2.3, we have

$$W \not\subset \bigcap_{v \in V} (v - C) + t'd,$$

and

$$Q \subset \bigcap_{v \in V} (v - C) + t'd.$$

This contradicts $W \subset Q$. Hence,

$$I_C^{(1)}(W; V, d) \leq I_C^{(1)}(Q; V, d).$$

When $j = 2U, 3U$, both proofs are similar to the case of $j = 1$. $\square$

Lemma 2.6. *Let $V, W, Q \in \mathscr{P}(Y)$ and $d \in C$. Then,*

$$W \subset Q \implies I_C^{(j)}(W; V, d) \geq I_C^{(j)}(Q; V, d)$$

for each $j = 2L, 3L, 4$.

Proof. The proofs are similar to those of Lemma 2.5. $\square$

3. Inheritation of semicontinuity of set-valued maps via scalarization

We consider set-valued maps $F : X \to \mathscr{P}(Y)$, where X is a topological space and we introduce a new concept of invariant property for those maps with respect to a binary relationship on a family of sets in Y, which is regarded as some kind of continuity from the view point of order-monotonicity. Let $\mathscr{N}(x)$ and $\preccurlyeq$ be a neighborhood system of a point $x \in X$ and a binary relation on $\mathscr{P}(Y)$, respectively.

Definition 3.1. We define the following two binary relations on $\mathscr{P}(Y)$.

(1) $W \preccurlyeq_1 A \overset{\text{def}}{\Longleftrightarrow} \operatorname{int} W \cap A \neq \emptyset$,

(2) $W \preccurlyeq_2 A \overset{\text{def}}{\Longleftrightarrow} A \subset \operatorname{int} W$.

Definition 3.2. Let $F : X \to \mathscr{P}(Y)$ be a set-valued map, $x_0 \in X$ and $\preccurlyeq$ a binary relation on $\mathscr{P}(Y)$. We say that F is $\preccurlyeq$-continuous at x_0 if, $\forall W \subset Y$ with $W \preccurlyeq F(x_0)$, there exists $V \in \mathscr{N}(x_0)$ such that $W \preccurlyeq F(x), \forall x \in V$.

For $\preccurlyeq_1$ and $\preccurlyeq_2$ as special cases, we can easily check that the corresponding notions coincide with usual "lower (semi)continuity" and "upper (semi)continuity" for set-valued maps, respectively.

(1) F is lower continuous ($\preccurlyeq_1$-continuous) at x_0 if

$$\forall W \subset Y \text{ with } W \preccurlyeq_1 F(x_0), \exists V \in \mathscr{N}(x_0) \text{ s.t. } W \preccurlyeq_1 F(x), \forall x \in V.$$

(2) F is upper continuous ($\preccurlyeq_2$-continuous) at x_0 if

$$\forall W \subset Y \text{ with } W \preccurlyeq_2 F(x_0), \exists V \in \mathscr{N}(x_0) \text{ s.t. } W \preccurlyeq_2 F(x), \forall x \in V.$$

Definition 3.3. Let $\varphi : \mathscr{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$ be a scalarization function, $A_0 \in \mathscr{P}(Y)$, and $\preccurlyeq$ a binary relation on $\mathscr{P}(Y)$. Then,

(1) we say that φ is $\preccurlyeq$-lower semicontinuous at A_0 if

$$\forall r < \varphi(A_0),\ \exists W \in \mathscr{P}(Y) \text{ such that } W \preccurlyeq A_0 \text{ and } r < \varphi(A), \forall A \in U(W, \preccurlyeq),$$

(2) we say that φ is $\preccurlyeq$-upper semicontinuous at A_0 if

$$\forall r > \varphi(A_0),\ \exists W \in \mathscr{P}(Y) \text{ such that } W \preccurlyeq A_0 \text{ and } r > \varphi(A), \forall A \in U(W, \preccurlyeq),$$

where $U(W, \preccurlyeq) := \{A \in \mathscr{P}(Y) \mid W \preccurlyeq A\}$.

Theorem 3.1. *Let $F : X \to \mathscr{P}(Y)$, $\varphi : \mathscr{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $x_0 \in X$, and $\preccurlyeq$ a binary relation on $\mathscr{P}(Y)$. If F is $\preccurlyeq$-continuous at x_0 and φ is $\preccurlyeq$-lower semicontinuous at $F(x_0)$, then $\varphi \circ F$ is lower semicontinuous at x_0 where $\varphi \circ F(x) := \varphi(F(x))$ for each $x \in X$.*

Proof. For any $r < \varphi(F(x_0))$, since φ is $\preccurlyeq$-lower semicontinuous at $F(x_0)$, there exists $W_0 \in \mathscr{P}(Y)$ such that $W_0 \preccurlyeq F(x_0)$ and $r < \varphi(A)$ for any $A \in U(W_0, \preccurlyeq)$. Then, since F is $\preccurlyeq$-continuous at x_0, there exists $V \in \mathscr{N}(x_0)$ such that $W_0 \preccurlyeq F(x)$ for any $x \in V$. It is easy to check that $F(x) \in U(W_0, \preccurlyeq)$ and thus we obtain $r < \varphi(F(x))$ for any $x \in V$. This means that $\varphi \circ F$ is lower semicontinuous at x_0.

□

Theorem 3.2. *Let $F : X \to \mathscr{P}(Y)$, $\varphi : \mathscr{P}(Y) \to \mathbb{R} \cup \{\pm\infty\}$, $x_0 \in X$, and $\preccurlyeq$ a binary relation on $\mathscr{P}(Y)$. If F is $\preccurlyeq$-continuous at x_0 and φ is $\preccurlyeq$-upper semicontinuous at $F(x_0)$, then $\varphi \circ F$ is upper semicontinuous at x_0.*

Proof. The proof is similar to that of Theorem 3.1. $\square$

Proposition 3.1. *Let $A_0, V \in \mathscr{P}(Y)$ and $d \in \operatorname{int} C$. The following statements hold:*

(1) $I_C^{(j)}(\cdot;V,d)$ is $\preccurlyeq_1$-lower semicontinuous at A_0 for $j = 1, 3U$;

(2) $I_C^{(j)}(\cdot;V,d)$ is $\preccurlyeq_1$-upper semicontinuous at A_0 for $j = 2L, 4$.

Proof. First, we prove (i). Let $r < I_C^{(j)}(A_0;V,d)$. It is sufficient to show that there exists $W \in \mathscr{P}(Y)$ such that

$$W \preccurlyeq_1 A_0 \tag{3.1}$$

and

$$r < I_C^{(j)}(A;V,d), \forall A \in U(W, \preccurlyeq_1). \tag{3.2}$$

Take α, $\beta \in \mathbb{R}$ satisfying $r < \beta < \alpha < I_C^{(j)}(A_0;V,d)$.

When $j = 1$, we have

$$A_0 \not\subset \bigcap_{v \in V}(v - C) + \alpha d$$

and

$$\bigcap_{v \in V}(v - C) + \alpha d \supset \operatorname{cl}\left(\bigcap_{v \in V}(v - C) + \beta d\right).$$

It follows that there exists $y \in Y$ such that $y \in (\operatorname{cl}(\bigcap_{v \in V}(v - C) + \beta d))^{\mathrm{c}} \cap A_0$. Thus we can take $W \in \mathscr{P}(Y)$ with

$$y \in \operatorname{int} W \tag{3.3}$$

and

$$W \cap \left(\bigcap_{v \in V}(v - C) + \beta d\right) = \emptyset. \tag{3.4}$$

From (3.3), W satisfies (3.1). By Lemma 2.1 and (3.4), $\beta \leq I_C^{(1)}(A_0;V,d)$ for all $A \in \mathscr{P}(Y)$ with $A \cap W \neq \emptyset$. This implies that W satisfies (3.2). Therefore $I_C^{(1)}(\cdot;V,d)$ is $\preccurlyeq_1$-lower semicontinuous at A_0. When $j = 3U$, we can similarly prove the statement by using Lemma 2.2 instead of Lemma 2.1.

Next, we prove (ii). Let $r > I_C^{(j)}(A_0;V,d)$. It is sufficient to show that there exists $W \in \mathscr{P}(Y)$ such that

$$W \preccurlyeq_1 A_0 \tag{3.5}$$

and

$$r > I_C^{(j)}(A;V,d), \ \forall A \in U(W, \preccurlyeq_1). \tag{3.6}$$

Take α, $\beta \in \mathbb{R}$ satisfying $r > \beta > \alpha > I_C^{(j)}(A_0;V,d)$. When $j = 2L$, we have

$$A_0 \cap \left(\bigcap_{v \in V}(v - C) + \alpha d\right) \neq \emptyset$$

and

$$\bigcap_{v \in V}(v - C) + \alpha d \subset \operatorname{int}\left(\bigcap_{v \in V}(v - C) + \beta d\right).$$

It follows that there exists $y \in Y$ such that $y \in A_0 \cap \operatorname{cl}\left(\bigcap_{v\in V}(v-C)+\beta d\right)$. Thus we can take $W \in \mathscr{P}(Y)$ with

$$y \in \operatorname{int} W \tag{3.7}$$

and

$$W \subset \bigcap_{v\in V}(v-C)+\beta d. \tag{3.8}$$

From (3.7), we have that W satisfies (3.5). By Lemma 2.3 and (3.8), $\beta \geq I_C^{(2L)}(A_0;V,d)$ for all $A \in \mathscr{P}(Y)$ with $A \cap W \neq \emptyset$. This implies that W satisfies (3.6), and therefore $I_C^{(2L)}(\cdot;V,d)$ is $\preccurlyeq_1$-lower semicontinuous at A_0. When $j=4$, we can similarly prove the statement by using Lemma 2.4 instead of Lemma 2.3. □

Proposition 3.2. *Let $A_0, V \in \mathscr{P}(Y)$ and $d \in \operatorname{int} C$. The following statements hold:*

(1) $S_C^{(j)}(\cdot;V,d)$ is $\preccurlyeq_1$-lower semicontinuous at A_0 for $j = 2U, 4$;

(2) $S_C^{(j)}(\cdot;V,d)$ is $\preccurlyeq_1$-upper semicontinuous at A_0 for $j = 1, 3L$.

Proof. The conclusion follows immediately from Propositions 2.4 and 3.1. □

Proposition 3.3. *Let $A_0, V \in \mathscr{P}(Y)$ and $d \in \operatorname{int} C$. The following statements hold:*

(1) $I_C^{(j)}(\cdot;V,d)$ is $\preccurlyeq_2$-upper semicontinuous at A_0 for $j = 1, 3U$;

(2) $I_C^{(j)}(\cdot;V,d)$ is $\preccurlyeq_2$-lower semicontinuous at A_0 for $j = 2L, 4$.

Proof. First, we prove (i). Let $r > I_C^{(j)}(A_0;V,d)$. It is sufficient to show that there exists $W \in \mathscr{P}(Y)$ such that

$$W \preccurlyeq_2 A_0 \tag{3.9}$$

and

$$r > I_C^{(j)}(A;V,d), \forall A \in U(W, \preccurlyeq_2). \tag{3.10}$$

Take $\alpha, \beta \in \mathbb{R}$ satisfying $r > \beta > \alpha > I_C^{(j)}(A_0;V,d)$. When $j=1$, we have

$$A_0 \subset \bigcap_{v\in V}(v-C)+\alpha d \subset \operatorname{int}\left(\bigcap_{v\in V}(v-C)+\beta d\right).$$

Putting $W := \bigcap_{v\in V}(v-C)+\beta d$, we obtain $A_0 \subset \operatorname{int} W$, that is, (3.9). It holds that $I_C^{(1)}(W;V,d) \leq \beta < r$ because $W \leq_C^{(1)} V + \beta d$. For any $A \in U(W, \preccurlyeq_2)$, by $A \subset W$ and Lemma 2.5, $I_C^{(1)}(A;V,d) \leq I_C^{(1)}(W;V,d)$ and thus $I_C^{(1)}(A;V,d) < r$. Hence (3.10) holds, and therefore $I_C^{(1)}(\cdot;V,d)$ is $\preccurlyeq_2$-upper semicontinuous at A_0. When $j = 3U$, we can similarly prove the statement.

Next we prove (ii). Let $r < I_C^{(j)}(A_0;V,d)$. It is sufficient to show that there exists $W \in \mathscr{P}(Y)$ such that

$$W \preccurlyeq_2 A_0 \tag{3.11}$$

and

$$r < I_C^{(j)}(A;V,d), \forall A \in U(W, \preccurlyeq_2). \tag{3.12}$$

Take $\alpha, \beta \in \mathbb{R}$ satisfying $r < \beta < \alpha < I_C^{(j)}(A_0;V,d)$. When $j = 2L$, we have

$$A_0 \subset \left(\bigcap_{v\in V}(v-C)+\alpha d\right)^{\mathrm{c}} \subset \operatorname{int}\left(\bigcap_{v\in V}(v-C)+\beta d\right)^{\mathrm{c}}.$$

Putting $W := (\bigcap_{v\in V}(v-C)+\beta d)^{\mathrm{c}}$, we obtain $A_0 \subset \operatorname{int} W$, that is, (3.11). It holds that $I_C^{(2L)}(W;V,d) \geq \beta > r$ because $W \not\leq_C^{(2L)} V+\beta d$. For any $A \in U(W,\preccurlyeq_2)$, by $A \subset W$ and Lemma 2.6, we have

$$I_C^{(2L)}(A;V,d) \geq I_C^{(2L)}(W;V,d).$$

Thus, $I_C^{(2L)}(A;V,d) > r$. Hence (3.12) holds. It follows that $I_C^{(2L)}(\cdot;V,d)$ is $\preccurlyeq_2$-lower semicontinuous at A_0. When $j=4$, we can similarly prove the statement. □

Proposition 3.4. *Let $A_0, V \in \mathscr{P}(Y)$ and $d \in \operatorname{int} C$. The following statements hold:*

(1) $S_C^{(j)}(\cdot;V,d)$ is $\preccurlyeq_2$-upper semicontinuous at A_0 for $j = 2U, 4$;

(2) $S_C^{(j)}(\cdot;V,d)$ is $\preccurlyeq_2$-lower semicontinuous at A_0 for $j = 1, 3L$.

Proof. The conclusion follows immediately from Propositions 2.4 and 3.3. □

From Propositions 3.1, 3.2, 3.3, and 3.4, we have the following results as corollaries of Theorems 3.1 and 3.2; see [11, 12].

Theorem 3.3. *Let $F : X \to \mathscr{P}(Y)$. For each $V \in \mathscr{P}(Y)$ and $d \in \operatorname{int} C$, the following statements hold:*

(1) For each $j = 1, 3U$,

(a) if F is $\preccurlyeq_1$-continuous (l.s.c.) at x_0, then $I_C^{(j)}(F(\cdot);V,d)$ is lower semicontinuous at x_0,

(b) if F is $\preccurlyeq_2$-continuous (u.s.c.) at x_0, then $I_C^{(j)}(F(\cdot);V,d)$ is upper semicontinuous at x_0;

(2) For each $j = 2L, 4$,

(c) if F is $\preccurlyeq_1$-continuous (l.s.c.) at x_0, then $I_C^{(j)}(F(\cdot);V,d)$ is upper semicontinuous at x_0,

(d) if F is $\preccurlyeq_2$-continuous (u.s.c.) at x_0, then $I_C^{(j)}(F(\cdot);V,d)$ is lower semicontinuous at x_0.

Theorem 3.4. *Let $F : X \to \mathscr{P}(Y)$. For each $V \in \mathscr{P}(Y)$ and $d \in \operatorname{int} C$, the following statements hold:*

(1) For each $j = 2U, 4$,

(a) if F is $\preccurlyeq_1$-continuous (l.s.c.) at x_0, then $S_C^{(j)}(F(\cdot);V,d)$ is lower semicontinuous at x_0,

(b) if F is $\preccurlyeq_2$-continuous (u.s.c.) at x_0, then $S_C^{(j)}(F(\cdot);V,d)$ is upper semicontinuous at x_0;

(2) For each $j = 1, 3L$,

(c) if F is $\preccurlyeq_1$-continuous (l.s.c.) at x_0, then $S_C^{(j)}(F(\cdot);V,d)$ is upper semicontinuous at x_0,

(d) if F is $\preccurlyeq_2$-continuous (u.s.c.) at x_0, then $S_C^{(j)}(F(\cdot);V,d)$ is lower semicontinuous at x_0.

4. Conclusion

In the paper, we find out mathematical background and mechanism for the inheritation of the semicontinuity of set-valued maps by using general scalarization for sets, which are regarded as the framework of generalizations of results in earlier study. Since the unified scalarization functions for sets introduced in [10] satisfy certain desired semicontinuity, our main results can be reduced to results in [11, 12] as special cases.

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