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SOLVING BILEVEL PROBLEMS WITH POLYHEDRAL CONSTRAINT SET

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Abstract. In this paper, we study the relationship between bilevel programmes and polyhedral projection problems. Extending a well-known result by Fülöp, we show that solving a bilevel problem with polyhedral constraints is equivalent to optimise the upper level objective over certain facets of an associated polyhedral projection problem. Utilising this result, we show how solutions to such bilevel problems can be computed.

Keywords. Bilevel programming; Multiple Objective Linear Programming; Polyhedral Projection.

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1. Introduction

Let a function $g\colon \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}$, matrices $A \in \mathbb{R}^{m\times d}$, $B \in \mathbb{R}^{m\times q}$ and vectors $b \in \mathbb{R}^m$, $c \in \left(\mathbb{R}^d\right)^\star$ be given. We consider the following *bilevel optimisation problem*:

$$\min_{\bar{x},y} g(\bar{x},y), \tag{1.1a}$$

where $\bar{x}$ solves

$$\begin{aligned} \min_{x}\ & cx \\ \text{s.t. } & Ax + By = b \\ & x \geq 0. \end{aligned} \tag{1.1b}$$

For fixed $y \in \mathbb{R}^q$ problem (1.1b) is called *inner problem* (also referred to as 'lower' or 'follower' problem). The inner problem (1.1b) controls the *inner variable* $x \in \mathbb{R}^d$, while the *outer problem* (1.1a) optimises over $y \in \mathbb{R}^q$ and is restricted by solutions (We understand *solution* to be a feasible point at which the optimal value is attained. This is sometimes referred to as 'optimal solution'.) to (1.1b). The ambiguity arising from the possibility of multiple solutions of the inner problem (1.1b) is resolved by assuming that the choice of a solution at the inner level can be influenced in favour of the outer level. This is known as *optimistic* formulation of the bilevel problem, which is widely studied in the literature (compare [12]). The optimistic approach explains why minimisation in (1.1a) is performed additionally with respect to $\bar{x}$.

In this note, we discuss how (1.1) can be solved in practice. Even in the case of g being linear (in this case (1.1) becomes a *bilevel linear programme* (BLP)), problem (1.1) is not convex

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in general and solving it poses a hard task [1, 4, 13, 23]. A common approach for solving general bilevel problems is to replace the inner optimisation problem—in case it is convex and a constraint qualification is satisfied—with its KKT conditions (see [11, 14]). Even in case of a linear inner problem like (1.1b) this step results in a non-convex problem. Handling the resulting non-linear complementary slackness condition is done by using a branch-and-bound approach [2, 3] or by splitting the complementary condition into two linear constraints involving binary variables [15]. The latter approach involves 'big-M' constraints, and it is a challenging task to find appropriate constants a priori (compare [23]). Alternatively, non-linear solvers could be used to directly solve the KKT-reformulated problem. Numerical issues arising due to non-regularity of this problem [25] can be handled by using a regularisation approach [26]. However, local optima of the regularised KKT-reformulation acquired by this method are not local optima of the original bilevel problem in general [10]. In [23], a local optimum of the KKT-reformulation obtained by the regularisation approach is used for finding appropriate parameters (big-M constant and initial values for binary variables) of the mixed integer reformulation from [15].

Another approach utilises simplex-like methods as follows: A solution to (1.1) is computed without taking the inner optimisation problem into account. It is then tested whether the solution obtained is also a solution to (1.1); and, if this is not the case, the process is iterated with neighboring vertices with gradually inferior objective function values (see [6, 7, 27]). This kind of approach is considered computationally too expensive, as it is essentially a vertex enumeration process of the high-dimensional feasible region of (1.1).

In contrast to the aforementioned method we implement a result from [16] which involves a polyhedron of much smaller dimension. In his seminal paper [16], Fülöp examines the relationship between bilevel linear programmes (BLP) and multiple objective linear programmes (MOLP). He shows that solving (BLP) is equivalent to optimising a linear function over the Pareto-Front of a corresponding (MOLP). If (1.1) has q upper level variables, Fülöp's associated (MOLP) has $q+2$ objectives.

A similar distinction between solution approaches—focusing on feasible region vs. image space based methods—can be found in the literature concerning (MOLP)s: The classical approach is based on decision space analysis, where the goal is to enumerate *all* efficient points (see, for example, [28]). Usually being numerically challenging, this decision space based enumeration quickly becomes intractable when a high number of variables is involved.

An alternative approach is based on objective space analysis. Among the early references are [5, 9] and a theoretical framework has been developed, for instance, in [17, 18, 19]. Using such image space based methods makes it possible to handle (MOLP)s with a high number of decision variables. Accordingly, applying those ideas to (BLP) allows to compute solutions to (1.1) with a high number of lower level variables.

In this note, we discuss how linear bilevel optimisation problems can be solved in practice using a polyhedral calculus toolbox like `bensolve tools` [8]. The core of the polyhedral calculus toolbox is the ability to compute projections of polyhedra. Note that the current version of `bensolve tools` utilises the (MOLP) solver `bensolve` [21] and the equivalence between multiple objective linear programming and polyhedral projection as pointed out in [20].

It turns out that (BLP) with q upper level variables can be solved by computing a $q+1$-dimensional projected polyhedron and by solving linear programmes over certain facets of this polyhedron. Internally, the current version of `bensolve tools` solves an associated (MOLP) with $q+2$ objectives. This means that the toolbox already utilises one of Fülöp's ideas.

In §2, we shortly introduce the optimal value reformulation of (1.1) and define several terms we need for the subsequent discussion. In §3 we show how Fülöp's approach can be improved further by considering a polyhedral projection problem instead of an (MOLP). §4 contains a numerical example which serves as a proof of concept.

2. Optimal Value Reformulation

Consider the bilevel optimisation problem (1.1). When the upper level has chosen $y \in \mathbb{R}^q$, the feasible set for the inner problem becomes

$$P_y := \{ x \in \mathbb{R}^d \mid x \geq 0, Ax = b - By \} . \tag{2.1}$$

The follower will then choose $x \in \mathbb{R}^d$ among the set of *rational reactions*:

$$\Phi(y) := \{ x \in P_y \mid cx \leq \varphi(y) \} , \tag{2.2}$$

where φ is the *optimal value function* of (1.1b), i.e.

$$\varphi \colon y \mapsto \inf \{ cx \mid x \in P_y \} . \tag{2.3}$$

Using (2.2), the bilevel problem (1.1) can be reformulated as single level optimisation problem:

$$\min_{\bar{x},y} g(\bar{x}, y) \text{ s.t. } \bar{x} \in \Phi(y). \tag{2.4}$$

Problem (2.4) is the *optimal value reformulation* of (1.1) (see [11, 22]).

We denote the *characteristic cone* (also *recession cone*) of a convex polyhedron by cc·. For example the characteristic cone of the inner feasible set (2.1) is

$$\operatorname{cc} P_y = \{ x \in \mathbb{R}^d \mid x \geq 0, Ax = 0 \} . \tag{2.5}$$

The k-th unitvector (i.e. a vector with k-th component equal to one and zero elsewhere) is denoted by e_k.

3. Reformulation as Polyhedral Projection Problem

We consider the epigraph of the optimal value function (2.3):

$$\operatorname{epi} \varphi = \left\{ \begin{pmatrix} y \\ r \end{pmatrix} \middle| \exists x \in \mathbb{R}^d \colon x \geq 0, Ax + By = b, r \geq cx \right\} . \tag{3.1}$$

(Here we use the fact that a feasible, bounded linear programme has a solution.) Note that (3.1) is a *P-representation* of a convex polyhedron (compare [8]), which shows that φ is a polyhedral convex function.

In the following we will show that the constraint set of (2.4) is closely related to certain facets of (3.1) (also compare ([27], Theorem 3.2) and ([13], Theorem 2.1)).

An affine halfspace $H \subseteq \mathbb{R}^{q+1}$ with normal vector $(h,t) \in \left(\mathbb{R}^{q+1}\right)^\star$,

$$H = \left\{ \begin{pmatrix} \hat{y} \\ \hat{r} \end{pmatrix} \in \mathbb{R}^{q+1} \middle| h\hat{y} + t\hat{r} \leq \alpha \right\} , \tag{3.2a}$$

is said to *support* $\operatorname{epi}\varphi$ in a point $\begin{pmatrix} y \\ r \end{pmatrix} \in \operatorname{epi}\varphi$, if $H \supseteq \operatorname{epi}\varphi$ and $hy + tr = \alpha$ hold. A face F of the polyhedron $\operatorname{epi}\varphi$ is the intersection of $\operatorname{epi}\varphi$ with the boundary of a supporting halfspace:

$$F = \left\{ \begin{pmatrix} \hat{y} \\ \hat{r} \end{pmatrix} \in \operatorname{epi}\varphi \;\middle|\; h\hat{y} + t\hat{r} = \alpha \right\}. \tag{3.2b}$$

We call a face of $\operatorname{epi}\varphi$ *non-vertical* whenever $t < 0$ in (3.2b). (Note that because $e_{q+1} \in \operatorname{cc}\operatorname{epi}\varphi$, $t \leq 0$ holds for any halfspace supporting $\operatorname{epi}\varphi$.) A face F of $\operatorname{epi}\varphi$ is called *proper* if $F \neq \operatorname{epi}\varphi$. A *facet* is a proper face of maximal dimension. We denote the set of non-vertical faces of $\operatorname{epi}\varphi$ by $\mathscr{F}$.

Proposition 3.1. [24, Theorem 23.10] *Let φ be a polyhedral convex function and let y be a point such that $\varphi(y)$ is finite. Then $\partial\varphi(y) \neq \emptyset$.*

Theorem 3.1. *The following equivalence holds:*

$$\bar{x} \in \Phi(y) \iff \bar{x} \in P_y \wedge \exists F \in \mathscr{F} : \begin{pmatrix} y \\ c\bar{x} \end{pmatrix} \in F. \tag{3.3}$$

Proof. $\bar{x} \in \Phi(y)$ implies $\bar{x} \in P_y$ and that $\varphi(y) = c\bar{x}$ is finite. φ is polyhedral convex (see (3.1)), hence there exists a subgradient $u \in \partial\varphi(y)$ by Proposition 3.1. The *subgradient inequality*,

$$z \in \mathbb{R}^q \implies \varphi(z) \geq \varphi(y) + u(z - y),$$

shows that the normal vector $(u, -1)$ generates a supporting halfspace of $\operatorname{epi}\varphi$ in $\begin{pmatrix} y \\ \varphi(y) \end{pmatrix}$. This supporting halfspace defines a non-vertical face of $\operatorname{epi}\varphi$. Thus, the '$\implies$'-direction of (3.3) is shown.

For the reverse direction, we take a point $\bar{x} \in P_y$ and a face $F \in \mathscr{F}$ according to (3.2) with $\binom{y}{c\bar{x}} \in F$. As $\bar{x} \in P_y$, we have

$$\varphi(y) \leq c\bar{x}.$$

F is supposed to be non-vertical, therefore $t < 0$ holds. Hence,

$$tc\bar{x} \leq t\varphi(y).$$

Adding the scalar product hy on both sides gives

$$hy + tc\bar{x} \leq hy + t\varphi(y),$$

which in turn shows

$$\alpha = hy + tc\bar{x} \leq hy + t\varphi(y) \leq \alpha,$$

where the first equality is due to $\binom{y}{c\bar{x}} \in F$ (3.2b), and the last inequality due to $\begin{pmatrix} y \\ \varphi(y) \end{pmatrix} \in \operatorname{epi}\varphi$ (3.2a). We get $c\bar{x} = \varphi(y)$. Taking into account $\bar{x} \in P_y$, we derive $\bar{x} \in \Phi(y)$. □

Remark 3.1. Theorem 3.1 characterises the conditions under which the set of rational reactions $\Phi(y)$ is empty: The feasible set P_y (2.1) is empty or $\operatorname{epi}\varphi$ has no non-vertical faces, i.e. $\mathscr{F} = \emptyset$. The latter condition corresponds to unboundedness of the inner problem at y. The remarkable fact here is that unboundedness of the inner problem for a single $y \in \mathbb{R}^q$ implies unboundedness for *all* $y \in \operatorname{dom}\varphi$. From the perspective of computational geometry, this fact is not surprising:

While P_y depends on y, this dependency vanishes at transition to the recession cone, see (2.5). This means we do not have to make further assumptions to the feasible set or the set of rational reactions, which are usually assumed to be compact or bounded (compare [6, 13, 23, 27]).

Algorithm 1: An algorithm for solving (1.1)

Data: Problem (1.1)
Result: Either a solution $\bar{x}^\star, y^\star$ to (1.1) or determine that (1.1) has no solution (infeasible or unbounded)

1 `compute` an irredundant H-representation of epi φ:

$$\operatorname{epi}\varphi = \bigcap_{0\le j<m} \{ \left(\begin{smallmatrix} y \\ r \end{smallmatrix}\right) \in \mathbb{R}^{q+1} \mid h^j y + t^j r \le \alpha_j \}$$

2 **if** epi $\varphi = \emptyset$ **then**
3   **return** `flag_empty`
4 $J \leftarrow \{ j \mid 0 \le j < m,\ t^j \ne 0 \}$
5 **if** $J = \emptyset$ **then**
6   **return** `flag_unbounded`
7 $g^\star \leftarrow +\infty$
8 **for** $j \in J$ **do**
9     $\left(x^j, y^j\right) \leftarrow \arg\min \{ g(x,y) \mid x \ge 0, Ax + By = b, h^j y + t^j cx = \alpha_j \}$
10   **if** $g^\star > g(x^j, y^j)$ **then**
11         $\bar{x}^\star \leftarrow x^j$
12         $y^\star \leftarrow y^j$
13         $g^\star \leftarrow g(\bar{x}^\star, y^\star)$
14 **return** $\bar{x}^\star, y^\star$

Theorem 3.1 shows that in order to find a solution to the bilevel problem (1.1), we can restrict the search to the finitely many facets of epi φ. In Algorithm 1 we show pseudocode of an algorithm exploiting this result. The first step is to represent epi φ as intersection of finitely many affine halfspaces (line 1). This type of representation is known as *H-representation*; such an H-representation is said to be *irredundant* if no halfspace can be left out in the intersection without altering the resulting polyhedron. An irredundant H-representation is known to contain all affine halfspaces which generate the facets of epi φ. The halfspaces—meaning their normal vectors $\left(h^j, t^j\right)$ and right hand sides α_j—can be computed from the P-representation (3.1) of epi φ using, for example, `bensolve tools`. In this step one may detect that epi φ is empty, which means no feasible point for (1.1) exists (line 3). If epi φ is not empty, we collect the indices j such that corresponding facets are non-vertical (line 4). In case there does not exist a non-vertical facet (line 6) we conclude that the inner problem is unbounded for every $y \in \operatorname{dom}\varphi$ (see Remark 3.1). In the remainder of Algorithm 1, the objective function g has to be minimised over each of the finitely many non-verticeal facets (line 9). Of those minimal function values, the smallest one is determined (lines 10–13). A corresponding pair of variables is returned as solution to (1.1) (line 14).

4. The Numerical Example

We present an example with a transportation problem on the inner level. For fixed outer level variable $y \in \mathbb{R}^q$ it can be presented as

$$\begin{aligned} \min \; & \sum_{i=1}^{n}\sum_{j=1}^{\ell} c_{ij}x_{ij} \\ \text{s.t.} \; & \sum_{j=1}^{\ell} x_{ij} \le a_i + (B_1 y)_i, \; i = 1,\dots,n, \\ & \sum_{i=1}^{n} x_{ij} \ge b_j + (B_2 y)_j, \; j = 1,\dots,\ell, \\ & x \ge 0, \end{aligned} \tag{4.1}$$

where $B_1 \in \mathbb{R}^{n\times q}$, $B_2 \in \mathbb{R}^{\ell\times q}$ and n and ℓ are the number of supply and demand depots, respectively. For each $i = 1,\dots,n$ the value $a_i + (B_1 y)_i$ is the amount of goods available at supply depot i and for $j = 1\dots,\ell$, $b_j + (B_2 y)_j$ is the amount to be transported to demand depot j. The objective function components $c_{ij} \in \mathbb{R}$ resemble the cost of transportation of one unit from supply depot i to demand depot j. In order to guarantee feasibility of (4.1) we impose

$$\sum_{i=1}^{n} a_i + (B_1 y)_i \ge \sum_{j=1}^{\ell} b_j + (B_2 y)_j \tag{4.2}$$

on the problem data for all possible values of the outer level variable. Altogether we consider the following bilevel problem

$$\begin{aligned} \min \; & \sum_{k=1}^{q} h_k y_k + \sum_{i=1}^{n}\sum_{j=1}^{\ell} \bar{c}_{ij}x_{ij} \\ \text{s.t.} \; & \bar{x} \text{ solves } (4.1), \\ & \sum_{i=1}^{q} y_i = 1, \\ & y \ge 0 \end{aligned} \tag{4.3}$$

with a linear objective function $g = (h,\bar{c}) \in \left(\mathbb{R}^{q+n\ell}\right)^\star$ for the leader. We have conducted numerical experiments for different sizes of the transportation problem and values of q. In the course of this all entries of $c = (c_{ij}) \in \left(\mathbb{R}^{n\ell}\right)^\star$, $g \in \left(\mathbb{R}^{q+n\ell}\right)^\star$, $a = (a_i) \in \mathbb{R}^n$ and $b = (b_j) \in \mathbb{R}^\ell$ are drawn from a uniform distribution on the interval $[5,10]$ independently of each other. Thereafter, the components of B_1 and B_2 are chosen independently of each other according to a uniform distribution such that (4.2) holds. The computations were performed on a machine with CPU Intel Core i5 2.4GHz and 8GB RAM. The polyhedral projection problem in Algorithm 1, line 1 is solved with the polyhedral calculus toolbox `bensolve tools` and for the linear programmes in line 9 `glpk` is used. For every $q \in \{3,4,5\}$ and every pair

$$(n,\ell) \in \{(5,1),(5,2),(10,5),(10,10),(50,10),(50,20)\,(100,50),(100,100)\}$$

we solve 10 instances of (4.3). The average computation time and number of non-vertical facets of $\operatorname{epi}\varphi$ are depicted in Figure 1. Clearly, the combinatorial complexity of $\operatorname{epi}\varphi$ grows quickly

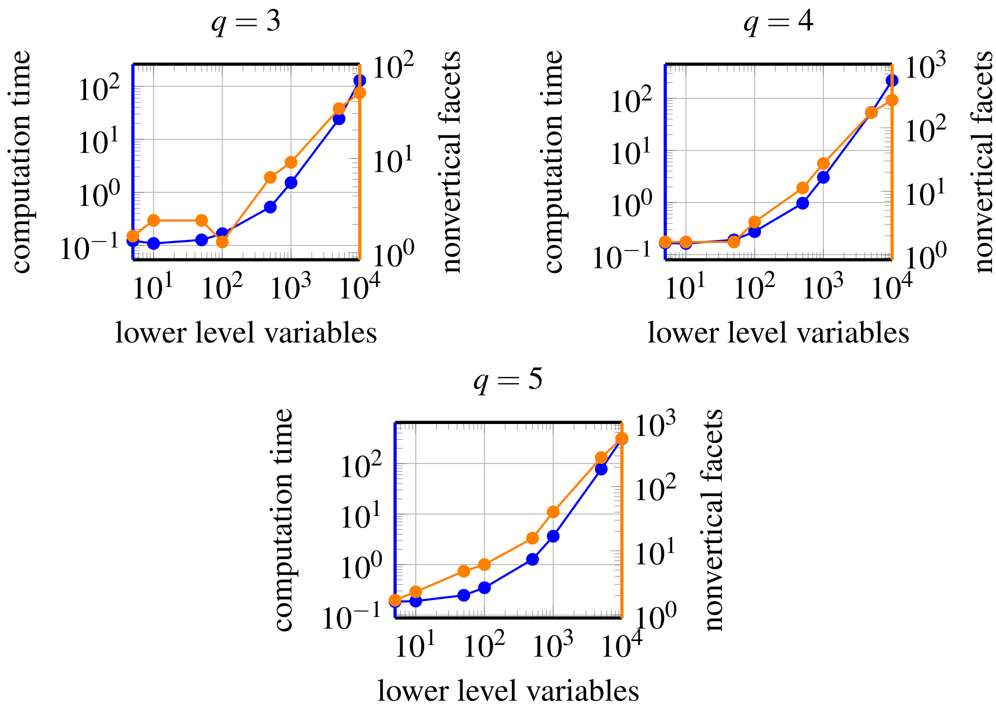

FIGURE 1. **Blue:** The average computation time over 10 random instances of (4.3) for different values of q, n and ℓ. **Orange:** The average number of non-vertical facets of epi φ.

with the number of lower level variables as can be seen by the number of non-vertical facets. It is to be expected that computation time grows exponentially in the dimension of epi φ as a significant part of it is required for solving the polyhedral projection problem which is difficult due to its combinatorial nature.

We would like to draw attention to the fact that solving the outer problems in Algorithm 1 (line 9 of the algorithm) may lead to numerical issues. This is due to the reason that only facets of epi φ are considered. Therefore the feasible regions of the outer problems are not full-dimensional. One possibility to bypass this complication is to shift the halfspaces defining the facets of epi φ by a small margin in the direction of r, i.e. in line 9 of the algorithm we replace $h^j y + t^j cx = \alpha_j$ by

$$h^j y + t^j cx \geq \alpha_j + t^j \varepsilon \tag{4.4}$$

for some small $\varepsilon > 0$. Thereby we do not require x to be an optimal solution of the inner problem but only an ε-approximate solution. Of course, this modified programme does not yield a solution to the considered bilevel problem but has a smaller optimal objective function value. However, if ε is chosen reasonably small the difference between the optimal objective function values will be small as well. Clearly, for $\varepsilon = 0$ we obtain the original problem. This idea is exemplarily illustrated for problem (4.3) with $q = 3$, $n = 100$ and $\ell = 100$. Figure 2 shows the difference between the optimal values of the disturbed and the original problem as well as the relative computation time for different values of ε. Evidently, the performance of the algorithm does not significantly suffer under the modification and is invariant under the value of ε. We will investigate this approach in more detail as part of future research.

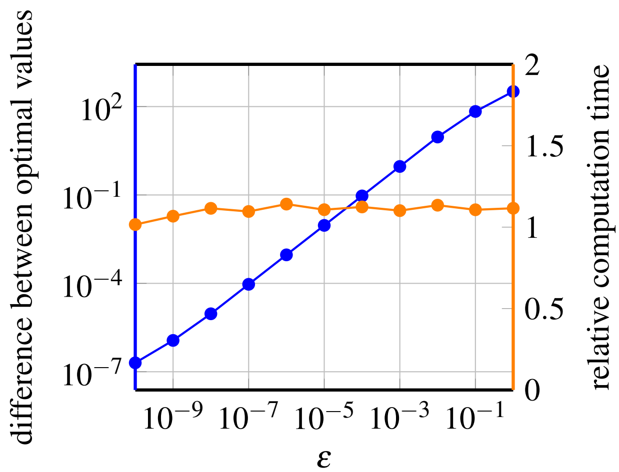

FIGURE 2. **Blue:** The difference between the optimal values of the disturbed problem where Inequality (4.4) was used and the original problem. **Orange:** The relative computation time of the disturbed problem compared to the original one.

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