J. Appl. Numer. Optim. 1 (2019), No. 3, pp. 335-345
Available online at http://jano.biemdas.com
https://doi.org/10.23952/jano.1.2019.3.10

THE SUBGRADIENT EXTRAGRADIENT METHOD FOR SOLVING MIXED EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS IN HILBERT SPACES

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Abstract. In this paper, we introduce an iterative method based on hybrid methods and hybrid extragradient methods for finding a common solution of mixed equilibrium problems and fixed point problems of nonexpansive mappings in a real Hilbert space. We define the notion of generalized skew-symmetric bifunctions which is a natural extension of a skew-symmetric bifunctions. Further, we prove that the sequences generated by the proposed iterative scheme converge strongly to a common solution of these systems. The results presented in this paper are the supplements, extensions and generalizations of the previously known results in this area.

Keywords. System of mixed equilibrium problems; Fixed-point problem; Hybrid-extragradient method; Generalized skew-symmetric bi-function; Nonexpansive mapping.

2010 Mathematics Subject Classification. 49J40, 90C52.

1. Introduction

In this paper, we are concerned with the problem of finding a common solution of mixed equilibrium problems and fixed point problems in real Hilbert spaces. Lets us recall the involved problems.

The *mixed equilibrium problem* (in short, MEP): Find $x \in C$ such that

$$F(x,y) + \langle Ax, y-x \rangle + \phi(y,x) - \phi(x,x) \geq 0, \ \forall y \in C, \tag{1.1}$$

where C is a nonempty convex and closed set in a real Hilbert space H, $A : C \to H$ is a nonlinear mapping and $F : C \times C \to \mathbb{R}$, $\phi : C \times C \to \mathbb{R}$, are bi-functions. The solution set of MEP (1.1) is denoted by Sol(MEP(1.1)).

This problem is quite general and it includes several known problems as special cases. For example, if $\phi \equiv 0$ and A $\equiv 0$, then (1.1) is reduced to the *equilibrium problem* (in short, EP): Find $x \in C$ such that

$$F(x,y) \geq 0, \ \forall y \in C, \tag{1.2}$$

which was studied by Blum and Oettli [1]. The solution set of EP (1.2) is denoted by Sol(EP(1.2)).

In the development of various fields of science and engineering, the equilibrium problem provides a framework for many problems, such as, variational inclusion problem, variational inequality problem, saddle point problem, complementary problem, Nash equilibrium problem

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Received May 23, 2019; Accepted September 16, 2019.

in noncooperative games, minimization problem, minimax inequality problem, and fixed point problem, see [1, 8, 10, 12, 16, 17, 20].

If $\phi \equiv 0$ and F $\equiv 0$, (1.1) becomes the following variational inequality problem (in short, VIP): Find $x \in C$ such that

$$\langle Ax, y - x \rangle \geq 0, \ \forall y \in C, \tag{1.3}$$

which was studied by Hartmann and Stampacchia [9]. The solution set of (1.3) is denoted by Sol(VIP(1.3)).

If $\phi \equiv 0$, then (1.1) is reduced to the following generalized equilibrium problem (in short, GEP): Find $x \in C$ such that

$$F(x,y) + \langle Ax, y - x \rangle \geq 0, \ \forall y \in C. \tag{1.4}$$

The Solution set of (1.4) is denoted by Sol(GEP(1.4)).

In 1976, Korpelevich [11] introduced the following iterative method in Hilbert space H, which is known as extragradient iterative method for (1.3):

$$\left.\begin{array}{l} x_0 \in C \subseteq H, \\ u_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A u_n), \end{array}\right\} \tag{1.5}$$

where $\lambda > 0$, A is a monotone and Lipschitz continuous mapping and P_C is the metric projection of H onto C. He proved that if Sol(VIP(1.3)) is nonempty, then, under some suitable conditions, the sequence generated by (1.5) converges to a solution of VIP(1.3).

In 2006, with the help of the extragradient iterative method for (1.3) given in [11], Nadezkhina and Takahashi [13] introduced and studied the following extragradient method and proved a strong convergence theorem:

$$\left.\begin{array}{l} x_0 \in C \subseteq H, \\ u_n = P_C(x_n - r_n A x_n), \\ y_n = \alpha_n x_n + (1-\alpha_n) T P_C(x_n - r_n A u_n), \\ C_n = \{z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{array}\right\} \tag{1.6}$$

For further generalizations of iterative method (1.6), we refer to [3, 4, 5, 18].

Another related an important problem is the *fixed point problem* (FPP). As above, $C \subseteq H$ is a nonempty, convex and closed subset of a real Hilbert space H. Given a mapping $T : C \to C$, the fixed point problem is formulated as follows:

$$\text{Find } x \in C \text{ such that } x = Tx. \tag{1.7}$$

The above problems have been extensively studied due to its theoretical and practical importance. Our concern in this paper is to study the solution set of (1.1) and (1.7) and propose an iterative scheme which is motivated by several related methods in this area ([2, 13, 15, 18]). The paper is organized as follows. We first recall some basic definitions and results in Section 2. Our algorithm is presented and analyzed in Section 3. Conclusions are given in 4.

2. PRELIMINARIES

Throughout the paper, let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let C be a nonempty convex closed set in H. The weak convergence of $\{x_n\}$ to x is denoted by $x^n \rightharpoonup x$ as $n\to\infty$, while the strong convergence of $\{x^n\}$ to x is written as $x^n \to x$ as $n\to\infty$. The identity operator on H is denoted by I.

For every point $x\in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x-P_C x\| \le \|x-y\|,\ \forall y\in C. \tag{2.1}$$

The mapping P_C is called the *metric projection* of H onto C. It is well known that P_C is *nonexpansive*, that is, $\|P_C x - P_C y\| \le \|x-y\|$, $\forall x,y\in H$ and furthermore, *firmly nonexpansive*,

$$\langle x-y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2,\ \forall x,y\in H. \tag{2.2}$$

Moreover, $P_C x$ is characterized by the fact $P_C x\in C$ and $\langle x-P_C x, y-P_C x\rangle \le 0$, $\forall y\in C$. This implies that

$$\|x-y\|^2 \ge \|x-P_C x\|^2 + \|y-P_C x\|^2,\ \forall x\in H,\ \forall y\in C.$$

Further, it is well known that every nonexpansive operator $T: H\to H$ satisfies, for all $(x,y)\in H\times H$, the inequality

$$\langle (x-T(x))-(y-T(y)), T(y)-T(x)\rangle \le \frac{1}{2}\|(T(x)-x)-(T(y)-y)\|^2.$$

For all $(x,y)\in H\times \mathrm{Fix}(T)$, we have

$$\langle x-T(x), y-T(x)\rangle \le \frac{1}{2}\|T(x)-x\|^2,$$

see, e.g., [6, Theorem 3.1] and [7, Theorem 2.1].

In a real Hilbert space H, we have

$$\|\lambda x+(1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$$

for all $x,y\in H$ and $\lambda\in[0,1]$. It is also known that every Hilbert space satisfies the *Opial's Condition* [14], i.e., for any sequence $\{x^n\}$ with $x^n \rightharpoonup x$ the inequality

$$\liminf_{n\to\infty}\|x^n - x\| < \liminf_{n\to\infty}\|x^n - y\|$$

holds for every $y\in H$ with $y\ne x$.

Definition 2.1. Given a mapping $A: H\to H$, recall that

(i) A is said to be *monotone* if

$$\langle Ax-Ay, x-y\rangle \ge 0,\ \forall x,y\in H;$$

(ii) A is said to be α-*inverse strongly monotone* (or, α-*ism*) if there exists a constant $\alpha>0$ such that

$$\langle Ax-Ay, x-y\rangle \ge \alpha\|Ax-Ay\|^2,\ \forall x,y\in H;$$

(iii) A is said to be *firmly nonexpansive* if it is 1-*ism*.

(iv) A is said to be *L-Lipschitz continuous* if there exists a constant $L>0$ such that

$$\|Ax-Ay\| \le L\|x-y\|,\ \forall x,y\in H.$$

Remark 2.1. If A is an α-ism mapping, then A is monotone and Lipschitz continuous. The converse is not true in general.

Lemma 2.1. [21]. *Let H be a real Hilbert space and let D be a nonempty, closed and convex subset of H. Let the sequence $\{x^n\} \subset H$ be Fejér-monotone with respect to D, i.e., for every $u \in D$,*

$$\|x^{n+1} - u\| \leq \|x^n - u\|, \ \forall n \geq 0.$$

Then $\{P_D(x^n)\}_{n=0}^{\infty}$ converges strongly to some $z \in D$.

Lemma 2.2. [13]. *Let H be a real Hilbert space, $\{\alpha_n\}_{n=0}^{\infty}$ be a real sequence satisfying $a \geq \alpha_n \leq b < 1$ for all $n > 0$, and let $\{v^n\}_{n=0}^{\infty}$ and $\{w^n\}_{n=0}^{\infty}$ be two sequences in H such that for some $\sigma \geq 0$,*

$$\limsup_{n\to\infty} \|v^n\| \leq \sigma, \ \limsup_{n\to\infty} \|w^n\| \leq \sigma,$$

and

$$\lim_{n\to\infty} \|\alpha_n v^n + (1-\alpha_n) w^n\| = \sigma.$$

Then

$$\lim_{n\to\infty} \|v^n - w^n\| = 0.$$

Condition 2.1. Let F and ϕ satisfy the following conditions:

(1) $F(x,x) = 0, \ \forall x \in C$;

(2) F is monotone, i.e.,

$$F(x,y) + F(y,x) \leq 0, \ \forall x,y \in C;$$

(3) For each $y \in C$, $x \to F(x,y)$ is *hemi-upper semicontinuous*, i.e.,

$$\text{for each } x,y,z \in C, \ \limsup_{t\to 0^+} F(tz + (1-t)x, y) \leq F(x,y);$$

(4) For each $x \in C$, $y \to F(x,y)$ is convex and lower semicontinuous;

(5) $\phi(\cdot,\cdot)$ is weakly continuous and $\phi(.,y)$ is convex, i.e.,

$$t\phi(x_1,y) + (1-t)\phi(x_2,y) \leq \phi(tx_1 + (1-t)x_2, y), \ \forall x_1,x_2 \in C, \ \forall t \in [0,1];$$

(6) ϕ is *generalized skew symmetric*, i.e.,

$$\phi(x,x) - \phi(x,y) + \phi(y,y) - \phi(y,z) + \phi(z,z) - \phi(z,x) \geq 0, \ \forall x,y,z \in C.$$

We also need the following technical result.

Lemma 2.3. [19] *Let H be a real Hilbert space and let C be a nonempty, closed, and convex subset of H. Let $F, \phi : C \times C \to \mathbb{R}$ be nonlinear mappings satisfying Condition 2.1. For each $x \in C$, there exist a bounded subset $D_x \subseteq C$ and $z_x \in C$ such that, for any $y \in C \setminus D_x$,*

$$F(y,z_x) + \phi(z_x,y) - \phi(y,y) + \frac{1}{r}\langle z_x - y, y - z\rangle < 0. \tag{2.3}$$

Define $T_r : H \to C$ as follows:

$$T_r z = \left\{ x \in C : F(x,y) + \phi(y,x) - \phi(x,x) + \frac{1}{r}\langle y - x, x - z\rangle \geq 0, \ \forall y \in C \right\}, \tag{2.4}$$

where r is a positive real number. Then the following statements hold:

(i) *$T_r z$ is nonempty for each $z \in H$;*

(ii) *T_r is single-valued firmly nonexpansive;*

(iii) $\mathrm{Fix}(T_r)$= *Sol(MEP(1.1));*

(iv) *Sol(MEP(1.1)) is convex and closed.*

Remark 2.2. It follows from Lemma 2.3 (i)-(ii) that

$$rF(T_rx,y)+r\phi(y,T_rx)-r\phi(T_rx,T_rx)+\langle T_rx-x,y-T_rx\rangle \geq 0,\ \forall y\in C,\ \forall x\in H. \tag{2.5}$$

Further, Lemma 2.3 (ii) implies that T_r is nonexpansive and furthermore, (2.1) implies that, for all $x,y\in H$, the following inequality holds.

$$\|T_rx-y\|^2 \leq \|x-y\|^2-\|T_rx-x\|^2+2rF(T_rx,y)-2r[\phi(y,T_rx)-\phi(T_rx,T_rx)]. \tag{2.6}$$

The function $F: C\times C\to\mathbb{R}$ is said to be 2-monotone if

$$F(x,y)+F(y,z)+F(z,x)\leq 0,\ \forall x,y,z\in C.$$

By taking $y=z$, it is clear that every 2-monotone bi-function is a monotone bi-function. For example, if $F(x,y)=x(y-x)$, then F is a 2-monotone bi-function.

3. MAIN RESULTS

Let C be a nonempty, closed, and convex subset of H. Let $F: C\times C\to\mathbb{R}$ and $\phi: C\times C\to\mathbb{R}$ be bi-functions, and $A: C\to H$ be a monotone operator. Our algorithm is presented below.

Algorithm 3.1.

Initialization: Choose an arbitrary starting point $x^0\in H$.

Iterative Steps: Given the current iterate x^n, compute:
Step 1. Compute

$$y^n = T_{r_n}(x^n-r_nAx^n),$$

construct the set

$$Q_n=\{w\in H \mid \langle (x^n-r_nAx^n)-y^n, w-y^n\rangle\leq 0\}$$

and calculate the next iterate x^{n+1} as follows

$$x^{n+1}=T_{r_nQ_n}(x^n-r_nAy^n),$$

where $T_{r_nQ_n}$ is defined by (2.4) with $C=Q_n$.
Set $n:=n+1$ and go to **Step 1**.

Theorem 3.1. *Let H be a real Hilbert space and let C be a nonempty, closed, and convex subset of H. Let $F: C\times C\to\mathbb{R}$ be 2-monotone bi-function. Let $\phi: C\times C\to\mathbb{R}$ be generalized skew-symmetric bi-function satisfying Condition 2.1 and $A: C\to H$ be an inverse-strongly monotone mapping. Let the sequences $\{x^n\}$ and $\{y^n\}$ be generated by Algorithm 3.1 and $u\in$ Sol(MEP(1.1)). Then*

$$\|x^{n+1}-u\|^2\leq\|x^n-u\|^2-(1-(r_nL)^2)\|y^n-x^n\|^2,\ \forall n\geq 0. \tag{3.1}$$

Proof. Denote $t^n = T_{r_nQ_n}(x^n - r_nAy^n),\ \forall n \geq 0$ and let $u \in \mathrm{Sol}(\mathrm{MEP}(1.1))$. Using (2.5), (2.6) and the definition of A, we estimate

$$\begin{aligned}
\|t^n - u\|^2 &\leq \|x^n - r_nAy^n - u\|^2 - \|x^n - r_nAy^n - t^n\|^2 \\
&\quad +2r_nF(t^n,u) + 2r_n[\phi(u,t^n) - \phi(t^n,t^n)] \\
&= \|x^n - u\|^2 - \|x^n - t^n\|^2 + 2r_n\langle Ay^n - Au, u - y^n\rangle + 2r_n\langle Au, u - y^n\rangle \\
&\quad +2r_n\langle Ay^n, t^n - y^n\rangle + 2r_nF(t^n,u) + 2r_n[\phi(u,t^n) - \phi(t^n,t^n)] \\
&\leq \|x^n - u\|^2 - \|x^n - t^n\|^2 + 2r_n\langle Au, u - y^n\rangle + 2r_n\langle Ay^n, y^n - t^n\rangle \\
&\quad +2r_nF(t^n,u) + 2r_n[\phi(u,t^n) - \phi(t^n,t^n)] \\
&\leq \|x^n - u\|^2 - \|x^n - t^n\|^2 + 2r_n\langle Ay^n, y^n - t^n\rangle \\
&\quad +2r_n[F(t^n,u) + F(u,y^n)] + 2r_n[\phi(u,t^n) - \phi(t^n,t^n) + \phi(y^n,u) - \phi(u,u)] \\
&\leq \|x^n - u\|^2 - \|x^n - y^n\|^2 - \|y^n - t^n\|^2 - 2\langle x^n - y^n, y^n - t^n\rangle \\
&\quad +2r_n\langle Ay^n, y^n - t^n\rangle + 2r_n[F(u,y^n) + F(t^n,u)] \\
&\quad +2r_n[\phi(u,t^n) - \phi(t^n,t^n) + \phi(y^n,u) - \phi(u,u)] \\
&\leq \|x^n - u\|^2 - \|x^n - y^n\|^2 - \|y^n - t^n\|^2 - 2\langle y^n - (x^n - r_nAx^n), t^n - y^n\rangle \\
&\quad +2r_n\langle Ax^n - Ay^n, t^n - y^n\rangle + 2r_n[F(u,y^n) + F(t^n,u)] \\
&\quad +2r_n[\phi(u,t^n) - \phi(t^n,t^n) + \phi(y^n,u) - \phi(u,u)] \\
&\leq \|x^n - u\|^2 - \|x^n - y^n\|^2 - \|y^n - t^n\|^2 + 2r_n\langle Ax^n - Ay^n, t^n - y^n\rangle \\
&\quad +2r_n[F(u,y^n) + F(y^n,t^n) + F(t^n,u)] \\
&\quad +2r_n[\phi(u,t^n) - \phi(t^n,t^n) + \phi(y^n,u) - \phi(u,u) + \phi(t^n,y^n) - \phi(y^n,y^n)] \\
&\leq \|x^n - u\|^2 - \|x^n - y^n\|^2 - \|y^n - t^n\|^2 + 2r_n\langle Ax^n - Ay^n, t^n - y^n\rangle \\
&\leq \|x^n - u\|^2 - \|x^n - y^n\|^2 - \|y^n - t^n\|^2 + 2r_n\|Ax^n - Ay^n\|\|t^n - y^n\|,
\end{aligned}$$

where the last inequality follows the 2-monotonicity of F and the generalized skew symmetric of ϕ. Next, we can continue the evaluation and get

$$\begin{aligned}
\|t^n - u\|^2 &\leq \|x^n - u\|^2 - \|x^n - y^n\|^2 - \|y^n - t^n\|^2 + \|y^n - t^n\|^2 + (r_nL)^2\|x^n - y^n\|^2 \\
&\leq \|x^n - u\|^2 - (1 - (r_nL)^2)\|x^n - y^n\|^2.
\end{aligned}$$

□

Theorem 3.2. *Let H be a real Hilbert space and let C be a nonempty, closed, and convex subset of H. Let $F : C \times C \to \mathbb{R}$ be 2-monotone bi-function. Let $\phi : C \times C \to \mathbb{R}$ be generalized skew-symmetric bi-function satisfying Condition 2.1. Let $A : C \to H$ be monotone and Lipschitz continuous on C with constant $L > 0$ and let $r_n < 1/L$. Then any sequences $\{x^n\}$ and $\{y^n\}$ generated by Algorithm 3.1 weakly converge to the same solution $u^* \in \mathrm{Sol}(\mathrm{MEP}(1.1))$ and furthermore, $u^* = \lim_{n\to\infty} P_\Gamma(x^n)$, where $\Gamma = \mathrm{Sol}(\mathrm{MEP}(1.1))$.*

Proof. Let $u \in \Gamma$ and define $\rho = 1 - (r_nL)^2$. From $r_n < L, \rho \in (0,1)$, and (3.1), we have

$$\rho\|y^n - x^n\|^2 \leq \|x^n - u\|^2.$$

Using (3.1) with $n \leftarrow (n-1)$, we have

$$\rho\|y^n - x^n\|^2 + \rho\|y^{n-1} - x^{n-1}\|^2 \leq \|x^{n-1} - u\|^2.$$

Continuing this , we get for all integers $n \geq 0$, $\rho \sum_{n=0}^{N} \|y^n - x^n\|^2 \leq \|x^0 - u\|^2$. Therefore

$$\rho \sum_{n=0}^{\infty} \|y^n - x^n\|^2 \leq \|x^0 - u\|^2.$$

It follows that

$$\lim_{n\to\infty} \|y^n - x^n\| = 0. \tag{3.2}$$

By (3.1), we see that sequence $\{x^n\}$ is bounded. So, there exists a weakly convergent subsequence $\{x^{n_k}\}$ of $\{x^n\}$, say $x^{n_k} \rightharpoonup \hat{x}$. Tit follows from (3.2) that there exists a weakly convergent subsequence $\{y^{n_k}\}$ of $\{y^n\}$ such that $y^{n_k} \rightharpoonup \hat{x}$.

First, let us show that $\hat{x} \in \Gamma$. The relation $y^n = T_{r_n}(x^n - r_n A x^n)$ implies that

$$F(y^n, y) + \phi(y, y^n) - \phi(y^n, y^n) + \frac{1}{r_n}\langle y - y^n, y^n - (x^n - r_n A x^n)\rangle \geq 0, \ \forall y \in C,$$

and

$$\phi(y, y^n) - \phi(y^n, y^n) + \frac{1}{r_n}\langle y - y^n, y^n - x^n\rangle \geq F(y, y^n) + \langle Ax^n, y^n - y\rangle, \ \forall y \in C,$$

by the monotonicity of F. It follows that

$$\phi(y, y^{n_k}) - \phi(y^{n_k}, y^{n_k}) + \langle y - y^{n_k}, \frac{y^{n_k} - x^{n_k}}{r_{n_k}}\rangle \geq F(y, y^{n_k}) + \langle Ax^{n_k}, y^{n_k} - y\rangle, \ \forall y \in C. \tag{3.3}$$

For t, with $0 < t \leq 1$, let $y_t = ty + (1-t)\hat{x} \in C$. Then from (3.3), we have

$$\begin{aligned} \langle Ay_t, y_t - y^{n_k}\rangle \ \geq \ & \langle Ay_t - Ay^{n_k}, y_t - y^{n_k}\rangle - \phi(y_t, y^{n_k}) + \phi(y^{n_k}, y^{n_k}) \\ & - \langle y - y^{n_k}, \tfrac{y^{n_k} - x^{n_k}}{r_{n_k}}\rangle + F(y_t, y^{n_k}) + \langle Ax^{n_k} - Ay^{n_k}, y_t - y^{n_k}\rangle. \end{aligned}$$

Since A is monotone and Lipschitz continuous, $\|Ay^{n_k} - Ax^{n_k}\| \to 0$ as $\|y^{n_k} - x^{n_k}\| \to 0$. Further, since F is weak lower semicontinuous in second argument and ϕ is weak continuous therefore from the above inequality, we have

$$\phi(y_t, \hat{x}) - \phi(\hat{x}, \hat{x}) \geq F(y_t, \hat{x}) - \langle Ay_t, y_t - \hat{x}\rangle.$$

Now,

$$\begin{aligned} 0 &= F(y_t, y_t) \\ &\leq tF(y_t, y) + (1-t)F(y_t, \hat{x}) \\ &\leq tF(y_t, y) + (1-t)[\phi(y_t, \hat{x}) - \phi(\hat{x}, \hat{x}) + \langle Ay_t, y_t - \hat{x}\rangle \\ &\leq tF(y_t, y) + (1-t)t[\phi(y, \hat{x}) - \phi(\hat{x}, \hat{x})] + (1-t)\langle Ay_t, ty + (1-t)\hat{x} - \hat{x}\rangle \\ &= F(y_t, y) + (1-t)[\phi(y, \hat{x}) - \phi(\hat{x}, \hat{x}) + \langle Ay_t, y - \hat{x}\rangle. \end{aligned}$$

Letting $t \to 0$, we have

$$F(\hat{x}, y) + \phi(y, \hat{x}) - \phi(\hat{x}, \hat{x}) + \langle A\hat{x}, y - \hat{x}\rangle \geq 0, \ \forall y \in C,$$

which implies that $\hat{x} \in \Gamma$.

In order to show that the entire sequence weakly converges to $\hat{x}$, we assume that there is another subsequence $\{x^{n_j}\}$ of $\{x^n\}$ that weakly converges to some $\hat{x}' \neq \hat{x}$ and $\hat{x}' \in \Gamma$. From

Theorem 3.1, it follows that the sequence $\{\|x^n - \hat{x}\|\}$ is decreasing for each $u \in \Gamma$. By the Opial's Condition, we have

$$\begin{aligned} \lim_{n\to\infty} \|x^n - \hat{x}\| &= \liminf_{j\to\infty} \|x^{n_j} - \hat{x}\| < \lim_{n\to\infty} \|x^n - \hat{x}'\| \\ &= \lim_{n\to\infty} \|x^n - \hat{x}'\| = \liminf_{j\to\infty} \|x^{n_j} - \hat{x}'\| \\ &< \liminf_{j\to\infty} \|x^{n_j} - \hat{x}\| \\ &= \lim_{n\to\infty} \|x^n - \hat{x}\|. \end{aligned}$$

This is a contradiction. Thus $\hat{x}' = \hat{x}$. This implies that $\{x^n\}$ and $\{y^n\}$ converges weakly to the same point $\hat{x} \in \Gamma$. Put $u^n = P_\Gamma(x^n)$. By (2.2) and Lemma 2.1, we see that $\{u^n\}$ converges strongly to some $\hat{x}' \in \Gamma$. We also have $\langle \hat{x} - \hat{x}', \hat{x}' - \hat{x} \rangle \geq 0$. Hence $\hat{x} = \hat{x}'$, which completes the proof. □

Next, we give a modification of Algorithm 3.1 to find a common solution to Sol(MEP(1.1)) and Fix(S) (1.7) for a nonexpansive mapping $S : C \to C$.

Algorithm 3.2.

Initialization: Choose an arbitrary starting point $x^0 \in H$.

Iterative Steps: Given the current iterate x^n, compute:
Step 1. Compute

$$y^n = T_{r_n}(x^n - r_n A x^n),$$

construct the set

$$Q_n = \{w \in H \mid \langle (x^n - r_n A x^n) - y^n, w - y^n \rangle \leq 0\}$$

and calculate the next iterate x^{n+1} as follows

$$x^{n+1} = \alpha_n x^n + (1 - \alpha_n) S T_{r_n Q_n}(x^n - r_n A y^n).$$

Set $n := n+1$ and go to **Step 1**.

Theorem 3.3. *Let H be a real Hilbert space and let C be a nonempty, closed, and convex subset of H. Let $F : C \times C \to \mathbb{R}$ be 2-monotone bi-function and let $\phi : C \times C \to \mathbb{R}$ be generalized skew-symmetric bi-function satisfying Condition 2.1. Let $A : C \to H$ be monotone and Lipschitz continuous on C with constant $L > 0$ and let $r_n < \frac{1}{L}$. Moreover, let $S : C \to C$ be a nonexpansive mapping and $\alpha_n \subseteq (0,1)$. Let $\{x^n\}$ and $\{y^n\}$ be any two sequences generated by Algorithm 3.2. Then both the sequences weakly converge to the same solution $u^* \in \Gamma$,* where $\Gamma = \mathrm{Fix(S)}(1.7) \cap$ Sol(MEP(1.1)) *and furthermore,*

$$u^* = \lim_{n\to\infty} P_\Gamma(x^n).$$

Proof. Denote $t^n = T_{r_n Q^n}(x^n - r_n A(y^n))$, $\forall n \geq 0$ and let $u \in \Gamma$. As estimated in Theorem 3.1, we have

$$\|t^n - u\|^2 \leq \|x^n - u\|^2 - (1 - (r_n L)^2)\|x^n - y^n\|^2.$$

Using (2.2), we have

$$\begin{aligned}\|x^{n+1}-u\|^2 &= \alpha_n\|x^n-u\|^2+(1-\alpha_n)\|St^n-u\|^2-\alpha_n(1-\alpha_n)\|(x^n-u)-(St^n-u)\|^2\\ &\leq \alpha_n\|x^n-u\|^2+(1-\alpha_n)\|St^n-u\|^2\\ &\leq \alpha_n\|x^n-u\|^2+(1-\alpha_n)\|t^n-u\|^2\\ &\leq \|x^n-u\|^2+(1-\alpha_n)((r_nL)^2-1)\|x^n-y^n\|^2.\end{aligned} \tag{3.4}$$

Hence, $\|x^{n+1}-u\|^2 \leq \|x^n-u\|^2$. It follows that

$$\lim_{n\to\infty}\|x^n-u\| = \sigma. \tag{3.5}$$

So, $\{x^n\}$ and $\{t^n\}$ are bounded. From (3.4), we have

$$(1-\alpha_n)(1-(r_nL)^2)\|x^n-y^n\|^2 \leq \|x^n-u\|^2-\|x^{n+1}-u\|^2,$$

and

$$\|x^n-y^n\|^2 \leq \frac{\|x^n-u\|^2-\|x^{n+1}-u\|^2}{(1-\alpha_n)(1-(r_nL)^2)}.$$

Hence,

$$\lim_{n\to\infty}\|x^n-y^n\| = 0. \tag{3.6}$$

Note that

$$\|y^n-t^n\|^2 \leq \|r_nAy^n-r_nAx^n\|^2 \leq (r_nL)^2\|y^n-x^n\|^2.$$

It follows from (3.6) that

$$\lim_{n\to\infty}\|y^n-t^n\| = 0. \tag{3.7}$$

Using (3.6) and (3.7), we have

$$\lim_{n\to\infty}\|x^n-t^n\| = 0. \tag{3.8}$$

Since $\{x^n\}$ is bounded, we may assume that there exists a subsequence $\{x^{n_j}\}_0^\infty$ which weakly converges to some $\hat{x}\in H$. We next show that $\hat{x}\in\Gamma$. By using the same approach d in the proof of Theorem 3.2, we get $\hat{x}\in \mathrm{Sol}(\mathrm{MEP}(1.1))$. It is now left to show that $\hat{x}\in \mathrm{Fix}(S)(1.7)$. Now, $\|St^n-u\| \leq \|t^n-u\| \leq \|x^n-u\|$. Using (3.5), we have $\limsup_{n\to\infty}\|St^n-u\| \leq \sigma$. Furthermore,

$$\lim_{n\to\infty}\|x^{n+1}-u\| = \lim_{n\to\infty}\|\alpha_n(x^n-u)+(1-\alpha_n)(St^n-u)\| = \sigma. \tag{3.9}$$

Now, applying Lemma 2.2, we have $\lim_{n\to\infty}\|St^n-x^n\| = 0$. Since

$$\|Sx^n-x^n\| \leq \|Sx^n-St^n\|+\|St^n-x^n\| \leq \|x^n-t^n\|+\|St^n-x^n\|.$$

Using (3.8) and (3.9), we have $\lim_{n\to\infty}\|Sx^n-x^n\| = 0$. Since S is nonexpansive on H, $x^{n_j}\rightharpoonup\hat{x}$ and $\lim_{j\to\infty}\|(I-S)(x^{n_j})\| = 0$ we get from the Demiclosedness Principle that $(I-S)\hat{x} = 0$, which means that $\hat{x}\in \mathrm{Fix(S)}(1.7)$. By using similar arguments to those used in the proof of Theorem 3.2, we find that the entire sequence weakly converges to $\hat{x}$. Therefore the sequences $\{x^n\}$ and $\{y^n\}$ weakly converge to $\hat{x}\in\Gamma$. Finally, we put $u^n = P_\Gamma(x^n)$. By (2.2) and Lemma 2.1, we see that $\{u^n\}$ converges strongly to some $\hat{x}'\in\Gamma$. We also have $\langle\hat{x}-\hat{x}',\hat{x}'-\hat{x}\rangle \geq 0$. Hence $\hat{x} = \hat{x}'$, which completes the proof. □

Finally, we have the following results from Theorems 3.1 and 3.2.

Corollary 3.1. *Let H be a real Hilbert space and let C be a nonempty, closed, and convex subset of H. Let $F : C \times C \to \mathbb{R}$ be 2-monotone bi-function satisfying Condition 2.1 and $A : C \to H$ be an inverse-strongly monotone mapping. Let the sequences $\{x^n\}$ and $\{y^n\}$ be generated by Algorithm 3.1 and $u \in \mathrm{Sol}(\mathrm{GEP}(1.4))$. Then*

$$\|x^{n+1} - u\|^2 \leq \|x^n - u\|^2 - (1 - (r_n L)^2)\|y^n - x^n\|^2, \ \forall n \geq 0.$$

Corollary 3.2. *Let H be a real Hilbert space and let C be a nonempty, closed, and convex subset of H. Let $F : C \times C \to \mathbb{R}$ be 2-monotone bi-function satisfying Condition 2.1. Let $A : C \to H$ be monotone and Lipschitz continuous on C with constant $L > 0$ and let $r_n < 1/L$. Then any sequences $\{x^n\}$ and $\{y^n\}$ generated by Algorithm 3.1 weakly converge to the same solution $u^* \in \mathrm{Sol}(\mathrm{GEP}(1.4))$ and furthermore, $u^* = \lim_{n\to\infty} P_\Gamma(x^n)$, where $\Gamma = \mathrm{Sol}(\mathrm{GEP}(1.4))$.*

4. Conclusions

In this paper, We introduced the new extension of skew-symmetric bi-functions, called generalized skew-symmetric bi-functions. We also proposed two subgradient extragradient modifications for finding a common solution to mixed equilibrium problems and fixed point problems in real Hilbert spaces. Strong convergence theorems of common solutions are established in Hilbert spaces.

Acknowledgments
The author is very grateful to the referees for their critical and helpful comments. The author is also thankful to Professor K.R. Kazmi and Professor A. Gibali for their helpful discussions and suggestions regarding this manuscript.

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