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A DESCENT-LIKE METHOD FOR FIXED POINTS AND SPLIT CONCLUSION PROBLEMS

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Abstract. In this paper, a descent-like method is introduced for solving a fixed point problem of a strict pseudocontraction and a split variational inclusion problem. A strong convergence theorem of common solutions is established in the framework of Hilbert spaces without any compact assumptions on any mapping.

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1. Introduction-Preliminaries

Let H_1 and H_2 be two real Hilbert spaces endowed with inner products and induced norms denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively, while H refers to as any of these spaces.

Let $M : H \to 2^H$ be a monotone mapping. M is said to be monotone iff, for all $x, y \in H_1$, $u \in Mx$ and $v \in My$

$$\langle x - y, u - v \rangle \geq 0.$$

It is said to be maximal iff the $Graph(M)$ is not properly contained in the graph of any other monotone mapping. It is known that the graph of maximal monotone operators is weakly-strongly closed [1]. It is also known that a monotone mapping M is maximal iff for $(x,u) \in H \times H$, $\langle x - y, u - v \rangle \geq 0$, for every $(y, v) \in Graph(M)$ implies that $u \in M(x)$. Let r be some positive real number. The resolvent mapping, $Res_r^M : H \to H$ associated with mapping M and number r, is defined by

$$Res_r^M x = (I + rM)^{-1}(x), \forall x \in H,$$

where I stands for identity operator on H. If M is the subdifferential of proper convex lower semi-continuous functions, then the resolvent operator is the called the proximity operator. It is known that $Fix(Res_r^M) = M^{-1}(0)$, where $Fix(Res_r^M)$ stand for the fixed-point set of Res_r^M. The resolve operator plays an important role in many convex optimization problem and many authors studied zero points of maximal monotone operators via resolvent techniques; see, e.g., [2, 3, 4, 5, 6, 7] and the references therein.

Let $A : H_1 \to H_2$ be a bounded linear operator. Let M and N be two maximal monotone operators on H_1 and H_2, respectively. In this paper, we study the following split variational inclusion problem: Find $x^* \in H_1$ such that

$$0 \in M(x^*), \tag{1.1}$$

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and

$$y^* = Ax^* \in H_2 \quad solves \quad 0 \in N(y^*). \tag{1.2}$$

In this paper, we use $SFP(M,N)$ to denote the solution set of the split variational inclusion problem. In 1994, Censor and Elfving [8] first introduced a split feasibility problem in finite-dimensional Hilbert spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. It has been found that the split variational inclusion problem can be used in various disciplines such as image restoration, computer tomograph and radiation therapy treatment planning [9, 10, 11]. Recently, the above split variational inclusion problem has been studied based on Mann-like methods by many authors; see [12, 13, 14, 15] and the references therein.

Let T be a mapping on H. Recall that T is said to be nonexpansive iff

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in H.$$

T is said to be quasi-nonexpansive iff $Fix(T) \neq \emptyset$ and

$$\|x - Ty\| \le \|x - y\|, \quad \forall x \in Fix(T), y \in H.$$

It is known that every nonexpansive mapping satisfies the following properties

$$\langle Tx - Ty, (y - Ty) - (x - Tx)\rangle \le \frac{1}{2}\|(x - Tx) - (y - Ty)\|^2, \quad \forall x, y \in H.$$

In particular, every quasi-nonexpansive mapping satisfies the following properties

$$\langle x - Ty, (y - Ty)\rangle \le \frac{1}{2}\|y - Ty\|^2, \quad \forall x \in Fix(T), y \in H. \tag{1.3}$$

Recall that T is said to be firmly nonexpansive iff

$$\|Tx - Ty\|^2 \le \langle Tx - Ty, x - y\rangle, \quad \forall x, y \in H.$$

It is known that the resolvents of maximal monotone operators is firmly nonexpansive. T is said to be firmly quasi-nonexpansive iff $Fix(T) \neq \emptyset$ and

$$\|x - Ty\|^2 \le \langle x - Ty, x - y\rangle, \quad \forall x \in Fix(T), y \in H.$$

Recall that T is said to be strictly pseudocontractive iff there is a real number $\kappa \in [0,1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\| + \kappa\|(I - T)x - (I - T)\|^2, \quad \forall x, y \in D.$$

The class of strictly pseudocontractive mappings was first introduced and studied by Browder and Petryshy [16] in the framework of Hilbert spaces. Since then, many authors have studied fixed points of strictly pseudocontractive mappings via different methods and techniques; see [17, 18, 19] and the references therein.

In this paper, we investigate the split variational inclusion problem which involving a strict pseudocontractive mapping via a fixed-point method in an infinite dimensional Hilbert spaces. Strong convergence theorems are established without any compact assumptions on mappings.

The following lemmas are essential for our main results.

Lemma 1.1. [16] *Let H be a real Hilbert space. Let T be a κ-strict pseudocontraction with fixed points and let $\{\beta_n\}$ be a sequence in $(0,1)$. Define a mapping S by*

$$T_n x = (1 - \beta_n)Tx + \beta_n x,$$

$\forall x \in H$. If $\beta \in [\kappa, 1)$, then T_n is nonexpansive and $Fix(T_n) = Fix(T)$ for each $n \geq 1$. If $\{x_n\}$ converges weakly to some point x^* and $x_n - Tx_n \to 0$ as $n \to \infty$, then $x^* \in Fix(T)$, that is, $I - T$, where I is the identity mapping, is demiclosed at zero.

Lemma 1.2. [20] *Let H be a Hilbert space and let F be an η-strongly monotone, $\mathscr{L}$-Lipschitz continuous mapping on H. Define a mapping $T^\alpha : H \to H$ by $T^\alpha x = (I - \mu\alpha F)x$, $\forall x \in H$, where α is a real number in $(0,1)$. If $0 < \mu \in (0, \frac{2\eta}{\mathscr{L}^2})$, then T^α is a contraction, that is,*

$$\|T^\alpha x - T^\alpha y\| \leq (1 - \alpha\tau)\|x - y\|,$$

$\forall x, y \in H$, where $\tau = 1 - \sqrt{1 - \mu(2\eta - \mu\mathscr{L}^2)} \in (0,1]$.

Lemma 1.3. [21] *Let H be a real Hilbert space and let M be a maximal operator. For $\lambda > 0$ and $\mu > 0$, we have*

$$(I + \mu M)^{-1}\Big(\frac{\mu}{\lambda}x + \Big(1 - \frac{\mu}{\lambda}\Big)(I + \lambda M)^{-1}x\Big) = (I + \lambda M)^{-1}x, \forall x \in H.$$

Lemma 1.4. [22] *Let H be a real Hilbert space. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in H. Let $\{\sigma_n\}$ be a sequence in $(0,1)$ such that*

$$0 < \liminf_{n\to\infty} \sigma_n \leq \limsup_{n\to\infty} \sigma_n < 1.$$

Assume that $x_{n+1} = \sigma_n y_n + (1 - \sigma_n)x_n$ and

$$\limsup_{n\to\infty} \big(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|\big) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

Lemma 1.5. [23] *Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be sequences of real numbers such that $\alpha_n \in [0,1]$, $\sum_{n=1}^\infty \alpha_n = \infty$, $\limsup_{n\to\infty} \beta_n \leq 0$ and $\sum_{n=1}^\infty \gamma_n < \infty$ Let $\{\lambda_n\}$ be a sequence of nonnegative real numbers such that*

$$\lambda_{n+1} \leq (1 - \alpha_n)\lambda_n + \alpha_n\beta_n + \gamma_n.$$

Then $\lim_{n\to\infty} \lambda_n = 0$.

2. Main results

Theorem 2.1. *Let H_1 and H_2 be two real Hilbert spaces. Let T be a strictly pseudocontractive mapping with coefficient $\kappa \in [0,1)$ on H_1 with fixed points. Let $A : H_1 \to H_2$ be a bounded linear operator and let A^* be the adjoint operator of A. Let $F : H_1 \to H_1$ be a $\mathscr{L}$-Lipschitz continuous and τ-strongly monotone mapping. Let M be a maximal monotone mapping on H_1 and let N be a maximal monotone mapping on H_2. Assume that $SFP(M,N) \cap Fix(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following process:*

$$\begin{cases} x_1 \in H_1, \\ y_n = Res_{r_n}^M(x_n + \gamma A^*(Res_{s_n}^N - I)Ax_n), \\ z_n = (1 - \beta_n)Ty_n + \beta_n y_n, \\ x_{n+1} = (1 - \gamma_n)x_n + \gamma_n(I - \mu\alpha_n F)z_n, \quad n \geq 1, \end{cases}$$

where $\{r_n\}$ and $\{s_n\}$ are two positive real number sequences, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$, μ and γ are two positive real numbers. Assume that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, $\{\beta_n\}$ is in

$[\kappa,\beta)$, where $\kappa \le \beta < 1$, such that $\lim_{n\to\infty}|\beta_{n+1}-\beta_n| = 0$, $\{\gamma_n\}$ is in $[\rho,\rho']$, where ρ and ρ' are two real numbers in $(0,1)$, such that $\lim_{n\to\infty}|\gamma_{n+1}-\gamma_n| = 0$, $\liminf_{n\to\infty} r_n > 0$, $\lim_{n\to\infty}|r_{n+1}-r_n| < \infty$, $\liminf_{n\to\infty} s_n > 0$, $\lim_{n\to\infty}|s_{n+1}-s_n| < \infty$, $\mu \in (0, \frac{2\tau}{\mathscr{L}^2})$, and $\gamma \in (0, \frac{1}{\|A\|^2})$. Then $\{x_n\}$ converge strongly to $\bar{x} \in SFP(M,N)\cap Fix(T)$, which is the unique solution of the following variational inequality

$$\langle \bar{x}-y, F\bar{x}\rangle \le 0, \quad \forall y \in SFP(M,N)\cap Fix(T). \tag{2.1}$$

Proof. Set

$$T_n = (1-\beta_n)T + \beta_n, \quad \forall n \ge 1.$$

Since $\{\beta_n\}$ is in $[\kappa,1)$, we obtain from Lemma 1.1 that T_n is a nonexpansive mapping and $Fix(T) = F(T_n)$ for each n. Note that $x \in SFP(M,N)\cap Fix(T)$ iff $x = Res_{r_n}^M x$, $Ax = Res_{s_n}^N Ax$, and $x = Tx$. Since both the resolvents are firmly nonexpansive, one has

$$\begin{aligned}
\|y_n - x\|^2 &= \|Res_{r_n}^M(x_n + \gamma A^*(Res_{s_n}^N - I)Ax_n) - Res_{r_n}^M x\|^2 \\
&\le \|(x_n - x) + \gamma A^*(Res_{s_n}^N - I)Ax_n\|^2 \\
&\le \|x_n - x\|^2 + 2\gamma\langle A^*(Res_{s_n}^N - I)Ax_n, x_n - x\rangle + \gamma^2\|A\|^2\|(Res_{s_n}^N - I)Ax_n\|^2 \\
&\le \|x_n - x\|^2 + 2\gamma\langle (Res_{s_n}^N - I)Ax_n, A(x_n - x) + (Res_{s_n}^N - I)Ax_n\rangle \\
&\quad - 2\gamma\langle (Res_{s_n}^N - I)Ax_n, (Res_{s_n}^N - I)Ax_n\rangle + \gamma^2\|A\|^2\|(Res_{s_n}^N - I)Ax_n\|^2.
\end{aligned}$$

Note that Ax is a fixed point of $Res_{s_n}^N$. Using (1.3), we see that

$$\langle A(x_n - x) + (Res_{s_n}^N - I)Ax_n, (Res_{s_n}^N - I)Ax_n\rangle \le \frac{1}{2}\|(Res_{s_n}^N - I)Ax_n\|^2.$$

It follows that

$$\|y_n - x\|^2 \le \|x_n - x_\|^2 - \gamma(1-\gamma\|A\|^2)\|(Res_{s_n}^N - I)Ax_n\|^2. \tag{2.2}$$

Since $\gamma \in (0, \frac{1}{\|A\|^2})$, we have $\|y_n - x\| \le \|x - x_n\|$. It follows from the nonexpansivity of T_n that

$$\|z_n - x\| = \|(1-\beta_n)Ty_n + \beta_n y_n - x\| = \|T_n y_n - T_n x\| \le \|y_n - x\| \le \|x - x_n\|.$$

Using Lemma 1.2, we have

$$\begin{aligned}
\|x_{n+1} - x\| &\le \gamma_n\|(I-\mu\alpha_n F)z_n - (I-\mu\alpha_n F)x - \mu\alpha_n Fx\| + (1-\gamma_n)\|x_n - x\| \\
&\le \gamma_n\|(I-\mu\alpha_n F)z_n - (I-\mu\alpha_n F)x\| + \mu\alpha_n\gamma_n\|Fx\| + (1-\gamma_n)\|x_n - x\| \\
&\le \gamma_n(1-\tau\alpha_n)\|z_n - x\| + \mu\alpha_n\gamma_n\|Fx\| + (1-\gamma_n)\|x_n - x\| \\
&\le (1-\gamma_n\tau\alpha_n)\|x_n - x\| + \alpha_n\mu\gamma_n\|Fx\| \\
&\le \max\{\|x_n - x\|, \frac{\mu}{\tau}\|Fx\|\}.
\end{aligned}$$

This implies that $\{x_n\}$ is a bounded vector sequence, so are $\{y_n\}$ and $\{z_n\}$. Using Lemma 1.3, we find that

$$\begin{aligned}
&\|(x_{n+1}+\gamma A^*(Res^N_{s_{n+1}}-I)Ax_{n+1})-(x_n+\gamma A^*(Res^N_{s_n}-I)Ax_n)\|\\
&=\|x_{n+1}-x_n+\gamma A^*\big(Ax_n-Ax_{n+1}+Res^N_{s_n}Ax_{n+1}-Res^N_{s_{n+1}}Ax_n\big)\|\\
&\le\|x_{n+1}-x_n-\gamma A^*(Ax_{n+1}-Ax_n)\|+\|\gamma A^*\big(Res^N_{s_n}Ax_n-Res^N_{s_{n+1}}Ax_{n+1}\big)\|\\
&\le(1-\gamma\|A\|^2)\|x_{n+1}-x_n\|+\gamma\|A\|\|Res^N_{s_n}Ax_n-Res^N_{s_n}\big(\frac{s_n}{s_{n+1}}Ax_{n+1}+(1-\frac{s_n}{s_{n+1}})Res^N_{s_{n+1}}Ax_{n+1}\big)\|\\
&\le(1-\gamma\|A\|^2)\|x_{n+1}-x_n\|+\gamma\|A\|\big(\frac{|s_{n+1}-s_n|}{s_{n+1}}\|Res^N_{s_{n+1}}Ax_{n+1}-Ax_{n+1}\|+\|Ax_{n+1}-Ax_n\|\big)\\
&\le\|x_n-x_{n+1}\|+\frac{|s_{n+1}-s_n|}{s_{n+1}}\gamma\|A\|\|Res^N_{s_{n+1}}Ax_{n+1}-Ax_{n+1}\|
\end{aligned}\tag{2.3}$$

and

$$\begin{aligned}
&\|Res^M_{r_{n+1}}(x_{n+1}+\gamma A^*(Res^N_{s_{n+1}}-I)Ax_{n+1})-Res^M_{r_n}(x_{n+1}+\gamma A^*(Res^N_{s_{n+1}}-I)Ax_{n+1})\|\\
&=\|Res^M_{r_n}\big(\frac{r_n}{r_{n+1}}(x_{n+1}+\gamma A^*(Res^N_{s_{n+1}}-I)Ax_{n+1})\\
&\quad+(1-\frac{r_n}{r_{n+1}})Res^M_{r_{n+1}}(x_{n+1}+\gamma A^*(Res^N_{s_{n+1}}-I)Ax_{n+1})\big)\\
&\quad-Res^M_{r_n}(x_{n+1}+\gamma A^*(Res^N_{s_{n+1}}-I)Ax_{n+1})\|\\
&\le\|\big(\frac{r_n}{r_{n+1}}(x_{n+1}+\gamma A^*(Res^N_{s_{n+1}}-I)Ax_{n+1})\\
&\quad+(1-\frac{r_n}{r_{n+1}})Res^M_{r_{n+1}}(x_{n+1}+\gamma A^*(Res^N_{s_{n+1}}-I)Ax_{n+1})\big)\\
&\quad-(x_{n+1}+\gamma A^*(Res^N_{s_{n+1}}-I)Ax_{n+1})\|\\
&\le\frac{|r_{n+1}-r_n|}{r_{n+1}}\|(x_{n+1}+\gamma A^*(Res^N_{s_{n+1}}-I)Ax_{n+1})\\
&\quad-Res^M_{r_{n+1}}(x_{n+1}+\gamma A^*(Res^N_{s_{n+1}}-I)Ax_{n+1})\big)\|.
\end{aligned}\tag{2.4}$$

On the other hand, one has

$$\begin{aligned}
&\|y_{n+1}-y_n\|\\
&\le\|Res^M_{r_{n+1}}(x_{n+1}+\gamma A^*(Res^N_{s_{n+1}}-I)Ax_{n+1})-Res^M_{r_n}(x_{n+1}+\gamma A^*(Res^N_{s_{n+1}}-I)Ax_{n+1})\|\\
&\quad+\|Res^M_{r_n}(x_{n+1}+\gamma A^*(Res^N_{s_{n+1}}-I)Ax_{n+1})-Res^M_{r_n}(x_n+\gamma A^*(Res^N_{s_n}-I)Ax_n)\|\\
&\le\|Res^M_{r_{n+1}}(x_{n+1}+\gamma A^*(Res^N_{s_{n+1}}-I)Ax_{n+1})-Res^M_{r_n}(x_{n+1}+\gamma A^*(Res^N_{s_{n+1}}-I)Ax_{n+1})\|\\
&\quad+\|(x_{n+1}+\gamma A^*(Res^N_{s_{n+1}}-I)Ax_{n+1})-(x_n+\gamma A^*(Res^N_{s_n}-I)Ax_n)\|.
\end{aligned}\tag{2.5}$$

Substituting (2.3) and (2.4) into (2.5) yields that

$$\|y_{n+1}-y_n\|\le\|x_n-x_{n+1}\|+M(\frac{|r_{n+1}-r_n|}{r_{n+1}}+\frac{|s_{n+1}-s_n|}{s_{n+1}}),\tag{2.6}$$

were M is an appropriate constant. Since T_n is nonexpansive, one obtains from (2.6) that

$$\begin{aligned}
\|z_{n+1}-z_n\|&\le\|T_{n+1}y_{n+1}-T_{n+1}y_n\|+\|T_{n+1}y_n-T_ny_n\|\\
&\le\|y_{n+1}-y_n\|+|\beta_{n+1}-\beta_n|\|Ty_n-y_n\|\\
&\le\|x_n-x_{n+1}\|+M(\frac{|r_{n+1}-r_n|}{r_{n+1}}+\frac{|s_{n+1}-s_n|}{s_{n+1}})+|\beta_{n+1}-\beta_n|\|Ty_n-y_n\|.
\end{aligned}\tag{2.7}$$

Setting $\lambda_n = (I-\mu\alpha_n F)z_n$, one has

$$\begin{aligned}
&\|\lambda_{n+1}-\lambda_n\| \\
&\le \|(I-\mu\alpha_{n+1}F)z_{n+1}-(I-\mu\alpha_{n+1}F)z_n\| + \|(I-\mu\alpha_{n+1}F)z_n-(I-\mu\alpha_n F)z_n\| \\
&\le (1-\tau\alpha_{n+1})\|z_{n+1}-z_n\| + \mu|\alpha_{n+1}-\alpha_n|\|Fz_n\| \\
&\le (1-\tau\alpha_{n+1})\|x_n-x_{n+1}\| + M(\frac{|r_{n+1}-r_n|}{r_{n+1}}+\frac{|s_{n+1}-s_n|}{s_{n+1}}) \\
&\quad + |\beta_{n+1}-\beta_n|\|Ty_n-y_n\| + \mu|\alpha_{n+1}-\alpha_n|\|Fz_n\|.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|\lambda_{n+1}-\lambda_n\| - \|x_n-x_{n+1}\| \le\ & |\beta_{n+1}-\beta_n|\|Ty_n-y_n\| + \mu|\alpha_{n+1}-\alpha_n|\|Fz_n\| \\
& + M(\frac{|r_{n+1}-r_n|}{r_{n+1}}+\frac{|s_{n+1}-s_n|}{s_{n+1}}).
\end{aligned}$$

Using the restrictions imposed on the control sequences, one finds that

$$\limsup_{n\to\infty}(\|\lambda_{n+1}-\lambda_n\| - \|x_n-x_{n+1}\|) \le 0.$$

From Lemma 1.4, one concludes that

$$\lim_{n\to\infty}\|x_n-\lambda_n\| = 0, \tag{2.8}$$

which implies that

$$\lim_{n\to\infty}\|x_n-z_n\| = 0, \tag{2.9}$$

Next, we show $x_n - y_n \to 0$ as $n\to\infty$. From (2.2), we have

$$\begin{aligned}
\|\lambda_n-x\|^2 &= \|(I-\mu\alpha_n F)z_n-(I-\mu\alpha_n F)x-\mu\alpha_n Fx\|^2 \\
&\le \|(I-\mu\alpha_n F)z_n-(I-\mu\alpha_n F)x\|^2 - 2\langle\mu\alpha_n Fx, \lambda_n-x\rangle \\
&\le (1-\tau\alpha_n)\|z_n-x\|^2 - 2\mu\alpha_n\langle Fx, \lambda_n-x\rangle \\
&\le (1-\tau\alpha_n)\|y_n-x\|^2 - 2\mu\alpha_n\langle Fx, \lambda_n-x\rangle \\
&\le (1-\tau\alpha_n)\|x_n-x\|^2 - (1-\tau\alpha_n)\gamma(1-\gamma\|A\|^2)\|(Res_{s_n}^N-I)Ax_n\|^2 \\
&\quad + 2\mu\alpha_n\|Fx\|\|\lambda_n-x\|.
\end{aligned}$$

Since squares of norms are convex, we have

$$\begin{aligned}
\|x_{n+1}-x\|^2 &\le (1-\gamma_n)\|x_n-x\|^2 + \gamma_n\|\lambda_n-x\|^2 \\
&\le (1-\gamma_n\tau\alpha_n)\|x_n-x\|^2 - \gamma_n(1-\tau\alpha_n)\gamma(1-\gamma\|A\|^2)\|(Res_{s_n}^N-I)Ax_n\|^2 \\
&\quad + 2\gamma_n\mu\alpha_n\|Fx\|\|\lambda_n-x\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\gamma_n(1-\tau\alpha_n)\gamma(1-\gamma\|A\|^2)\|(Res_{s_n}^N-I)Ax_n\|^2 \\
&\le \|x_n-x\|^2 - \|x_{n+1}-x\|^2 + 2\gamma_n\mu\alpha_n\|Fx\|\|\lambda_n-x\| \\
&\le (\|x_n-x\|+\|x_{n+1}-x\|)\|x_n-x_{n+1}\| + 2\gamma_n\mu\alpha_n\|Fx\|\|\lambda_n-x\|.
\end{aligned} \tag{2.10}$$

From (2.8), one has

$$\lim_{n\to\infty}\|x_n-x_{n+1}\| = 0, \tag{2.11}$$

which together with (2.10) yields that

$$\lim_{n\to\infty} \|Ax_n - Res_{s_n}^N Ax_n\| = 0. \tag{2.12}$$

Since $Res_{s_n}^N$ is firmly nonexpansive, we obtain from inequality (1.3) that

$$\begin{aligned}
&\|y_n - x\|^2 \\
&\leq \langle x_n + \gamma A^*(Res_{s_n}^N - I)Ax_n - x, y_n - x\rangle \\
&= \frac{1}{2}\{\|x_n + \gamma A^*(Res_{s_n}^N - I)Ax_n - x\|^2 + \|y_n - x\|^2 - \|x_n + \gamma A^*(Res_{s_n}^N - I)Ax_n - y_n\|^2\} \\
&= \frac{1}{2}\{\gamma^2\|A^*(Res_{r_n}^N - I)Ax_n\|^2 + 2\gamma\langle A^*(Res_{s_n}^N - I)Ax_n, x_n - x\rangle + \|x_n - x\|^2 \\
&\quad + \|y_n - x\|^2 - \|x_n + \gamma A^*(Res_{s_n}^N - I)Ax_n - y_n\|^2\} \\
&\leq \frac{1}{2}\{\gamma^2\|A\|^2\|(Res_{s_n}^N - I)Ax_n\|^2 + 2\gamma(\langle(Res_{s_n}^N - I)Ax_n, Res_{s_n}^N Ax_n - Ax\rangle - \|(Res_{s_n}^N - I)Ax_n\|^2) \\
&\quad + \|x_n - x\|^2 + \|y_n - x\|^2 - \|x_n - y_n\|^2 - 2\gamma\langle x_n - y_n, A^*(Res_{s_n}^N - I)Ax_n\rangle \\
&\quad - \|\gamma A^*(Res_{s_n}^N - I)Ax_n\|^2)\} \\
&\leq \frac{1}{2}\{\|x_n - x\|^2 + \|y_n - x\|^2 + 2\gamma\|A\|\|x_n - y_n\|\|(Res_{s_n}^N - I)Ax_n\| - \|x_n - y_n\|^2\}.
\end{aligned}$$

It follows that

$$\|y_n - x\|^2 \leq \|x_n - x\|^2 + 2\gamma\|A\|\|x_n - y_n\|\|(Res_{s_n}^N - I)Ax_n\| - \|x_n - y_n\|^2,$$

which implies that

$$\begin{aligned}
\|\lambda_n - x\|^2 &\leq \|(I - \mu\alpha_n F)z_n - (I - \mu\alpha_n F)x\|^2 - 2\langle\mu\alpha_n Fx, \lambda_n - x\rangle \\
&\leq (1 - \tau\alpha_n)\|z_n - x\|^2 - 2\mu\alpha_n\langle Fx, \lambda_n - x\rangle \\
&\leq (1 - \tau\alpha_n)\|x_n - x\|^2 + 2(1 - \tau\alpha_n)\gamma\|A\|\|x_n - y_n\|\|(Res_{s_n}^N - I)Ax_n\| \\
&\quad - (1 - \tau\alpha_n)\|x_n - y_n\|^2 - 2\mu\alpha_n\langle Fx, \lambda_n - x\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|x_{n+1} - x\|^2 &\leq (1 - \gamma_n)\|x_n - x\|^2 + \gamma_n\|\lambda_n - x\|^2 \\
&\leq (1 - \gamma_n)\|x_n - x\|^2 + \gamma_n(1 - \tau\alpha_n)\|x_n - x\|^2 \\
&\quad + 2\gamma_n(1 - \tau\alpha_n)\gamma\|A\|\|x_n - y_n\|\|(Res_{s_n}^N - I)Ax_n\| \\
&\quad - \gamma_n(1 - \tau\alpha_n)\|x_n - y_n\|^2 - 2\gamma_n\mu\alpha_n\langle Fx, \lambda_n - x\rangle \\
&\leq (1 - \gamma_n\tau\alpha_n)\|x_n - x\|^2 + 2\gamma_n(1 - \tau\alpha_n)\gamma\|A\|\|x_n - y_n\|\|(Res_{s_n}^N - I)Ax_n\| \\
&\quad - \gamma_n(1 - \tau\alpha_n)\|x_n - y_n\|^2 + 2\gamma_n\mu\alpha_n\|Fx\|\|\lambda_n - x\|.
\end{aligned}$$

This shows that

$$\begin{aligned}
&\gamma_n(1 - \tau\alpha_n)\|x_n - y_n\|^2 \\
&\leq (1 - \gamma_n\tau\alpha_n)\|x_n - x\|^2 + 2\gamma_n(1 - \tau\alpha_n)\gamma\|A\|\|x_n - y_n\|\|(Res_{s_n}^N - I)Ax_n\| \\
&\quad - \|x_{n+1} - x\|^2 + 2\gamma_n\mu\alpha_n\|Fx\|\|\lambda_n - x\| \\
&\leq (\|x_n - x\| + \|x_{n+1} - x\|)\|x_n - x_{n+1}\| + 2\gamma_n\gamma\|A\|\|x_n - y_n\|\|(Res_{s_n}^N - I)Ax_n\| \\
&\quad + 2\gamma_n\mu\alpha_n\|Fx\|\|\lambda_n - x\|.
\end{aligned}$$

So, we have $x_n - y_n \to 0$ as $n \to \infty$. Since $\{x_n\}$ is a bounded sequence, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x^*$ as $i \to \infty$. So, $y_{n_i} \rightharpoonup x^*$, $z_{n_i} \rightharpoonup x^*$ and $\lambda_{n_i} \rightharpoonup x^*$ as $k \to \infty$. Observe that

$$y_n = Res_{r_n}^M (x_n + \gamma A^*(Res_{s_n}^N - I)Ax_n),$$

which implies from the maximal monotonicity of M that

$$\langle y_{n_i} - u, \frac{x_{n_i} - y_{n_i} + \gamma A^*(Res_{s_{n_i}}^N Ax_{n_i} - Ax_{n_i})}{\gamma} - v\rangle \geq 0,$$

for $(u, v) \in M$. This shows that $0 \in M(x^*)$. Let s be some positive real number. From Lemma 1.3, one has

$$\begin{aligned} \|Res_s^N Ax_n - Res_{s_n}^N Ax_n\| &\leq \|Ax_n - \frac{s}{s_n} Ax_n - (1 - \frac{s}{s_n}) Res_{s_n}^N Ax_n\| \\ &\leq \frac{|s_n - s|}{s} \|Ax_n - Res_{s_n}^N Ax_n\|. \end{aligned} \tag{2.13}$$

It follows from (2.12) that

$$\lim_{n\to\infty} \|Res_{s_n}^N Ax_n - Res_s^N Ax_n\| = 0.$$

Taking into the fact that

$$\|Res_s^N Ax_n - Ax_n\| \leq \|Res_{s_n}^N Ax_n - Ax_n\| + \|Res_s^N Ax_n - Res_{s_n}^N Ax_n\|,$$

we conclude that $\lim_{n\to\infty} \|Ax_n - Res_s^N Ax_n\| = 0$. We also have

$$\lim_{i\to\infty} \|Ax_{n_i} - Res_s^N Ax_{n_i}\| = 0.$$

From Lemma 1.1, we find that $Ax^* \in Fix(Res_s^N)$, that is, $0 \in N(Ax^*)$.

Next, we prove that x^* is also a fixed point of T. From (2.1), we have

$$Ty_n - y_n = \frac{z_n - y_n}{1 - \beta_n}.$$

Since $\{\beta_n\}$ is bounded away from 1, we find that $\lim_{n\to\infty} \|Ty_n - y_n\| = 0$. Since $I - T$ is demiclosed, we obtain that $x^* \in Fix(T)$.

Since F is strongly monotone and Lipschitz continuous, we next use $\bar{x}$ to denote the unique solution of variational inequality (2.1). That is,

$$\limsup_{n\to\infty} \langle F(\bar{x}), \bar{x} - \lambda_n\rangle \leq 0.$$

We have

$$\begin{aligned} \|\lambda_n - \bar{x}\|^2 &= \|(I - \mu\alpha_n F)z_n - (I - \mu\alpha_n F)\bar{x} - \mu\alpha_n F\bar{x}\|^2 \\ &\leq \|(I - \mu\alpha_n F)z_n - (I - \mu\alpha_n F)\bar{x}\|^2 - 2\mu\alpha_n\langle F\bar{x}, \lambda_n - \bar{x}\rangle \\ &\leq (1 - \tau\alpha_n)^2\|z_n - \bar{x}\|^2 - 2\mu\alpha_n\langle F\bar{x}, \lambda_n - \bar{x}\rangle \\ &\leq (1 - \tau\alpha_n)^2\|y_n - \bar{x}\|^2 - 2\mu\alpha_n\langle F\bar{x}, \lambda_n - \bar{x}\rangle \\ &\leq (1 - \tau\alpha_n)^2\|x_n - \bar{x}\|^2 + 2\mu\alpha_n\langle F\bar{x}, \bar{x} - \lambda_n\rangle, \end{aligned}$$

which implies that

$$\begin{aligned} \|x_{n+1} - \bar{x}\|^2 &\leq (1 - \gamma_n)\|x_n - \bar{x}\|^2 + \gamma_n\|\lambda_n - \bar{x}\|^2 \\ &\leq (1 - \gamma_n)\|x_n - \bar{x}\|^2 + \gamma_n(1 - \tau\alpha_n)^2\|x_n - \bar{x}\|^2 + 2\gamma_n\mu\alpha_n\langle F\bar{x}, \bar{x} - \lambda_n\rangle \\ &\leq (1 - \xi_n)\|x_n - \bar{x}\|^2 + \xi_n \frac{\tau^2\alpha_n\|x_n - \bar{x}\|^2 + 2\mu\langle F\bar{x}, \bar{x} - \lambda_n\rangle}{2\tau}, \end{aligned}$$

where $\xi_n = 2\tau\gamma_n\alpha_n$. Observe $\xi_n \to 0$, $\sum_{n=1}^{\infty}\xi_n = \infty$ and

$$\limsup_{n\to\infty} \frac{\tau^2\alpha_n\|x_n-\bar{x}\|^2 + 2\mu\langle F\bar{x}, \bar{x}-\lambda_n\rangle}{2\tau} \leq 0.$$

Using Lemma 1.5, we obtain that $\|x_n - \bar{x}\| \to 0$. This completes the proof. □

From Theorem 2.1, the following result involving nonexpansive mappings are not hard to derived easily.

Corollary 2.1. *Let H_1 and H_2 be two real Hilbert spaces. Let T be a nonexpansive mapping on H_1 with fixed points. Let $A : H_1 \to H_2$ be a bounded linear operator and let A^* be the adjoint operator of A. Let $F : H_1 \to H_1$ be a $\mathscr{L}$-Lipschitz continuous and τ-strongly monotone mapping. Let M be a maximal monotone mapping on H_1 and let N be a maximal monotone mapping on H_2. Assume that $SFP(M,N)\cap Fix(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following process:*

$$\begin{cases} x_1 \in H_1, \\ z_n = TRes_{r_n}^M(x_n + \gamma A^*(Res_{s_n}^N - I)Ax_n), \\ x_{n+1} = (1-\gamma_n)x_n + \gamma_n(I - \mu\alpha_n F)z_n, \quad n \geq 1, \end{cases}$$

where $\{r_n\}$ and $\{s_n\}$ are two positive real number sequences, $\{\alpha_n\}$, and $\{\gamma_n\}$ are sequences in $(0,1)$, μ and γ are two positive real numbers, and $\{r_n\}$ is a positive sequence. Assume that $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\{\gamma_n\}$ is in $[\rho,\rho']$, where ρ and ρ' are two real numbers in $(0,1)$, such that $\lim_{n\to\infty}|\gamma_{n+1} - \gamma_n| = 0$, $\liminf_{n\to\infty} r_n > 0$, $\lim_{n\to\infty}|r_{n+1}-r_n| < \infty$, $\liminf_{n\to\infty} s_n > 0$, $\lim_{n\to\infty}|s_{n+1}-s_n| < \infty$, $\mu \in (0, \frac{2\tau}{\mathscr{L}^2})$, and $\gamma \in (0, \frac{1}{\|A\|^2})$. Then $\{x_n\}$ converge strongly to $\bar{x} \in SFP(M,N)\cap Fix(T)$, which is the unique solution of the following variational inequality

$$\langle \bar{x} - y, F\bar{x}\rangle \leq 0, \quad \forall y \in SFP(M,N)\cap Fix(T).$$

Proof. Every nonexpansive mapping is a strict pseudocontraction with coefficient 0. Setting $\beta_n = 0$, we have the desired conclusion immediately. □

From Theorem 2.1, we also have the following result on split inclusion problem (1.1)-(1.2).

Corollary 2.2. *Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator and let A^* be the adjoint operator of A. Let $F : H_1 \to H_1$ be a $\mathscr{L}$-Lipschitz continuous and τ-strongly monotone mapping. Let M be a maximal monotone mapping on H_1 and let N be a maximal monotone mapping on H_2. Assume that $SFP(M,N) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by the following process:*

$$\begin{cases} x_1 \in H_1, \\ z_n = Res_{r_n}^M(x_n + \gamma A^*(Res_{s_n}^N - I)Ax_n), \\ x_{n+1} = (1-\gamma_n)x_n + \gamma_n(I - \mu\alpha_n F)z_n, \quad n \geq 1, \end{cases}$$

where $\{r_n\}$ and $\{s_n\}$ are two positive real number sequences, $\{\alpha_n\}$, and $\{\gamma_n\}$ are sequences in $(0,1)$, μ and γ are two positive real numbers, and $\{r_n\}$ is a positive sequence. Assume that $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\{\gamma_n\}$ is in $[\rho,\rho']$, where ρ and ρ' are two real numbers in $(0,1)$, such that $\lim_{n\to\infty}|\gamma_{n+1} - \gamma_n| = 0$, $\liminf_{n\to\infty} r_n > 0$, $\lim_{n\to\infty}|r_{n+1}-r_n| < \infty$, $\liminf_{n\to\infty} s_n > 0$, $\lim_{n\to\infty}|s_{n+1}-s_n| < \infty$, $\mu \in (0, \frac{2\tau}{\mathscr{L}^2})$, and $\gamma \in (0, \frac{1}{\|A\|^2})$. Then $\{x_n\}$ converge strongly to $\bar{x} \in SFP(M,N)$, which is the unique solution of the following variational inequality $\langle \bar{x} - y, F\bar{x}\rangle \leq 0$, $\forall y \in SFP(M,N)$.

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