CONVERGENCE OF SOLUTIONS TO LEXICOGRAPHIC EQUILIBRIUM PROBLEMS

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Abstract. This article deals with lexicographic equilibrium problems on Banach spaces. We first study the existence of solutions for such problems. Then, we investigate the Painlevé-Kuratowski convergence of the solution sets for a family of perturbed problems in a such way that they are perturbed by sequences constrained sets and objective functions converging. Several illustrative examples are given which clarify the essentialness of imposed assumptions. As an application, we discuss various results on the Painlevé-Kuratowski convergence for lexicographic variational inequalities.

Keywords. Lexicographic order; Equilibrium problem; Variational inequality; Painlevé-Kuratowski convergence; Continuous convergence.

1. INTRODUCTION

The vector equilibrium problem contains many problems as special cases, including vector variational inequalities, vector optimization problems, vector complementarity problems, vector Nash equilibrium problems, etc. Because of the general form of this problem, in fact it was investigated earlier under other terminologies (see, e.g., [10, 14, 26]). Most works on vector variational inequalities and vector equilibrium problems are based on orders induced by convex closed cones, i.e., they used various extensions of the Pareto order. However, it is known from the theory of vector optimization that the set of Pareto-optimal points is usually too large, so that one needs certain additional rules to reduce it. One of the possible approaches is to utilize the lexicographic order, which was investigated in connection with its applications in optimization and decision making theory. Konnov [18] studied vector equilibrium problems using the lexicographic order and showed that several classes of inverse lexicographic optimization problems can be reduced to lexicographic vector equilibrium problems. In [8], Bianchi, Konnov and Pini established several existence results for lexicographic variational inequalities by using equivalence

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properties between various kinds of lexicographic variational inequalities and sequential ones. Bianchi, Konnov and Pini [9] discussed lexicographic equilibrium problems on a topological Hausdorff vector space and their relationships with some other vector equilibrium problems. They established existence results for the tangled lexicographic problem via the study of a related sequential problem. Anh, Duy and Khanh [3] established the sufficient conditions for the upper semicontinuity, closedness, and continuity of solution maps for a parametric lexicographic equilibrium problem. The topic of well-posedness for problems involving lexicographic cones has received attention by many researchers recently; see, e.g., [2, 4]. Very recently, Anh and Duy [5] proposed a penalty function method for solving the lexicographic equilibrium problem. They use regularized gap function to solve such penalized problem and establish an error bound result for the penalized problem.

Set convergence, which is based on the limits of sets, was introduced by Painlevé (between 1902 and 1905) and popularized by Kuratowski in his book [19] in 1968, and thus often called the Painlevé-Kuratowski convergence. To date, it has been viewed as a major tool for dealing with approximations of variational problems in optimization, systems of equations and related objects. This concept plays an important role in the stability theory of vector optimization and its applications. It gives us the relationships between cluster points of approximating solutions set and exact solutions to the original problem, and hence it is the bridge among stability, well-posedness and solution method topics. Since problems in terms of asymptotically solving sequences are different from such parametric problems with the parameters perturbed in spaces of parameters, investigation of Painlevé-Kuratowski convergence of approximate solution sets is significant. Therefore, this topic has attracted attention of many researchers all over the world. There are many publications concentrating on this topic for problems related to optimization, such as vector optimization problems [11, 12, 20, 25, 27, 29], vector variational inequalities [23, 24], and recently, this topic has been also discussed on vector equilibrium problems; see, e.g., [13, 21, 22]. It is worth noting that since the lexicographic cone is neither closed nor open, the results obtained in [13, 21, 22] cannot be applied to lexicographic equilibrium problems as the ordered cones in these papers are required to be closed. Moreover, to our knowledge, there is no work devoted to the Painlevé-Kuratowski convergence for lexicographic equilibrium problems.

Motivated and inspired by research works mentioned above, the aim of this article is to study the existence of solutions to lexicographic equilibrium problem and the Painlevé-Kuratowski convergence of solutions for a family of perturbed problems. Using the continuous convergence of bifunction sequences and the Painlevé-Kuratowski convergence of set sequences, we obtain the Painlevé-Kuratowski convergence of the approximate solution sets for perturbed lexicographic equilibrium problems to the solution set for the original problem. In addition, we also consider the special case of lexicographic variational inequalities as an application.

This paper is organized as follows. In Section 2, some preliminary notions and results are recalled. Section 3 is devoted to the study of existence of solutions for lexicographic equilibrium problems. The Painlevé-Kuratowski convergence of solution sets for perturbed lexicographic equilibrium problems to corresponding solution set of the original problem is established in Section 4. In the last section, Section 5, several results on the Painlevé-Kuratowski convergence for lexicographic variational inequalities are derived from the obtained results as an application.

2. PRELIMINARIES

Throughout this article, unless specified otherwise, let X be a Banach space with its dual space X^*. The norm and the dual pair between X and X^* are denoted respectively by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$. We recall that given a sequence of subsets $\{K_n\}$ in X, the Painlevé-Kuratowski upper and lower limits are defined, respectively, as

$$\limsup_{n\to\infty} K_n := \left\{x \in X \mid x = \lim_{k\to\infty} x_{n_k}, x_{n_k} \in K_{n_k}, n_k \text{ a selection of the integers}\right\};$$

$$\liminf_{n\to\infty} K_n := \left\{x \in X \mid x = \lim_{n\to\infty} x_n, x_n \in K_n \text{ for } n \text{ sufficiently large}\right\}.$$

In the literature, the condition $\limsup_{n\to\infty} K_n \subset K \subset \liminf_{n\to\infty} K_n$ is referred to as the convergence of K_n to K in the sense of *Painlevé-Kuratowski*, denoted by $K_n \xrightarrow{\text{P.K.}} K$. The relations $K \subset \liminf_n K_n$ and $\limsup_n K_n \subset K$ are referred to as the *lower part of the convergence* and *upper part of the convergence*, respectively.

Lemma 2.1. *Let K be a compact subset of X, and K_n $(n = 1,2,\ldots)$ be nonempty subsets of X such that $\limsup_{n\to\infty} K_n \subset K$. Then, for every sequence $\{x_n\}$ with $x_n \in K_n$ there exists a subsequence $\{x_{n_k}\}$ converging to some point x in K.*

Proof. Suppose to the contrary that for every $x \in K$ there exist a neighborhood V_x of x and $N \in \mathbb{N}$ such that $x_n \notin V_x$ for all $n \geq N$. Because the family $\{V_x : x \in K\}$ is an open cover of the compact set K, there is a finite subcover $V = \bigcup_{i=1}^m V_{x_i}$ containing K. Consequently, there exists an $n_0 \in \mathbb{N}$ such that $x_n \notin V$ for all $n \geq n_0$, which contradicts $\limsup_{n\to\infty} K_n \subset K$. $\square$

We also recall the notion of continuous convergence for a sequence of extended real-valued functions.

Definition 2.1. [28] A sequence of functions $\{f_n\}$ where $f_n : X \to \mathbb{R} \cup \{\pm\infty\}$, is said to converge to a function $f : X \to \mathbb{R} \cup \{\pm\infty\}$ in the sense of *continuous convergence* if and only if $f_n(x_n) \to f(x)$ for every sequence $\{x_n\}$ converging to x.

This convergence is denoted by $f_n \xrightarrow{c} f$.

Definition 2.2. [15, Definition 2.5.1] Let X and Y be Banach spaces and $\bar{x} \in X$. A set-valued mapping $F : X \rightrightarrows Y$ is said to be

(a) *upper semicontinuous* (usc) at $\bar{x}$ if for any open subset U of Y with $F(\bar{x}) \subset U$, there is a neighborhood N of $\bar{x}$ such that $F(N) \subset U$.
(b) *lower semicontinuous* (lsc) at $\bar{x}$ if for any open subset U of Y with $F(\bar{x}) \cap U \neq \emptyset$, there is a neighborhood N of $\bar{x}$ such that $F(x) \cap U \neq \emptyset$ for all $x \in N$.
(c) *continuous* at $\bar{x}$ if it is both usc and lsc at $\bar{x}$.

We say that a mapping has a certain property on a set A if so does it at every point of A.

Definition 2.3. [1, 17] Let K be a nonempty convex subset of X. A function $f : X \to \mathbb{R}$ is said to be

(a) *upper semicontinuous* at $x \in X$ if for every sequence $\{x_n\}$ in X converging to x, one has

$$f(x) \geq \limsup_{n\to\infty} f(x_n).$$

(b) *lower semicontinuous* at $x \in X$ if $-f$ is upper semicontinuous at x.

(c) *quasiconvex* on K if for each pair of points $x, y \in K$ and for all $t \in [0, 1]$, one has

$$f(tx + (1 - t)y) \leq \max\{f(x), f(y)\}.$$

(d) *strictly quasiconvex* on K if for each pair of points $x, y \in K, x \neq y$ and $t \in]0, 1[$, one has

$$f(tx + (1 - t)y) < \max\{f(x), f(y)\}.$$

(e) *semistrictly quasiconvex* on K if for each pair of points $x, y \in K, x \neq y$ such that $f(x) \neq f(y)$ and $t \in]0, 1[$, one has

$$f(tx + (1 - t)y) < \max\{f(x), f(y)\}.$$

(f) *explicitly quasiconvex* on K if it is quasiconvex and semistrictly quasiconvex on K.

(g) *quasiconcave* (respectively, *strictly quasiconcave*, *semistrictly quasiconcave*, *explicitly quasiconcave*) on K if $-f$ is quasiconvex (respectively, strictly quasiconvex, semistrictly quasiconvex, explicitly quasiconvex) on K.

It is worth noting that there is no relationship between the class of quasiconvex functions and that of semistrictly quasiconvex ones. However, a function f is explicitly quasiconvex whenever it is lower semicontinuous and semistrictly quasiconvex.

Definition 2.4. [9] Let K be a nonempty subset of X and $\varphi : K \times K \to \mathbb{R}$ be a bifunction.

(a) φ is said to be *pseudomonotone* on K if for each pair of points $x, y \in K$, the following implication holds

$$[\varphi(x, y) \geq 0] \Longrightarrow [\varphi(y, x) \leq 0].$$

(b) φ is said to be *properly quasimonotone* on K if for every finite set $A \subset K$ and for every x in the convex hull of A, it holds that $\max_{y \in A} \varphi(x, y) \geq 0$.

(c) φ is said to be *pseudo-symmetric* on K if for each pair of points $x, y \in K$, the following implication holds:

$$[\varphi(x, y) = 0] \Longrightarrow [\varphi(y, x) = 0].$$

(d) φ is said to be *pseudomonotone*$_*$ on K if it is pseudomonotone on K and for each pair of points $x, y \in K$, if $\varphi(x, y) = \varphi(y, x) = 0$ then for every $z \in K$, there exists a positive real number k depended on x, y, z such that

$$\varphi(x, z) = k\varphi(y, z).$$

(e) φ possesses the *triangle inequality* if

$$\varphi(x, y) \leq \varphi(x, z) + \varphi(z, y), \quad \forall x, y, z \in K.$$

(f) φ is said to be a *weak pseudo-distance* on K if for every triple $x, y, z \in K$, there exist positive real numbers k, l depended on x, y, z such that the following implication is satisfied

$$[\varphi(x, y) \geq 0] \Longrightarrow [k\varphi(x, z) + l\varphi(z, y) \geq 0].$$

It is clear that the class of weak pseudo-distance functions strictly contains that of triangle inequality functions. Now we propose a modified version of the weak pseudo-distance notion.

Definition 2.5. Let K be a nonempty subset of X. A bifunction $\varphi : K \times K \to \mathbb{R}$ is said to be a *transfer weak pseudo-distance* if $\varphi(x,y) = 0$ for each pair of points $x, y \in K$ then the following implication holds

$$[\varphi(x,z) \geq 0, \forall z \in K] \Longrightarrow [\varphi(y,z) \geq 0, \forall z \in K].$$

We discuss some sufficient conditions for the transfer weak pseudo-distance property.

Proposition 2.1. *Suppose that $\varphi : K \times K \to \mathbb{R}$ satisfies one of the following conditions:*

(i) φ *is pseudo-symmetric and pseudomonotone$_*$.*

(ii) φ *is a weak pseudo-distance.*

Then, φ is a transfer weak pseudo-distance.

Proof. The transfer weak pseudo-distance of φ follows immediately from (i). Suppose that φ satisfies condition (ii). Let $x, y \in K$ be such that $\varphi(x,y) = 0$. Then, for each $z \in K$, $\varphi(x,z) \geq 0$, there exist $k, l > 0$ satisfying

$$k\varphi(x,y) + l\varphi(y,z) \geq 0.$$

Consequently, $\varphi(y,z) \geq 0$, which implies that φ is a transfer weak pseudo-distance. The proof is complete. □

We now recall the concept of lexicographic cone in finite dimensional space and models of equilibrium problems with the order induced by such cone. The lexicographic cone of $\mathbb{R}^n$, denoted C_{lex}, is the collection of zero and all vectors in $\mathbb{R}^n$ whose the first nonzero coordinate is a positive number, i.e.,

$$C_{\text{lex}} := \{0\} \cup \{x \in \mathbb{R}^n \mid x_1 = \cdots = x_j = 0, x_{j+1} > 0, \text{ for some } j, 0 \leq j < n\}.$$

This cone is convex and pointed, and induces the total order as follow:

$$x \geq_L y \Leftrightarrow x - y \in C_{\text{lex}}.$$

We also observe that it is neither closed nor open. Indeed, when comparing with the cone $C_1 := \{x \in \mathbb{R}^n \mid x_1 \geq 0\}$, we see that $\text{int} C_1 \subsetneq C_{\text{lex}} \subsetneq C_1$, while

$$\text{int} C_{\text{lex}} = \text{int} C_1 \text{ and } \text{cl} C_{\text{lex}} = C_1.$$

Let $f := (f^1, f^2) : X \times X \to \mathbb{R}^2$ be a vector-valued bifunction satisfying $f(x,x) = 0$ for all $x \in X$, and K be a nonempty closed subset of X. Let us now consider the following lexicographic vector equilibrium problem.

(LEP) Find $\bar{x} \in K$ such that

$$f(\bar{x}, y) \geq_L 0, \quad y \in K.$$

The set of the solutions to the problem (LEP) is denoted by S_{LEP}. We consider the lexicographic equilibrium problem under perturbations by a sequence of problems which are expressed in terms of perturbing the constraint set and the objective function. Then, (LEP) is embedded into the following family.

(LEP)$_n$ Find $x_n \in K_n$ such that

$$f_n(x_n, y) \geq_L 0, \quad \forall y \in K_n,$$

where $f_n : X \times X \to \mathbb{R}^2$ and K_n are nonempty subsets of X.

By S_{LEP}^n we denote the solution sets of the such problems. It is worth noticing that the problem (LEP) can be stated in the following equivalent way:

(LEP) Find $\bar{x} \in K$ such that

$$\begin{cases} f^1(\bar{x}, y) \geq 0, & \forall y \in K, \\ f^2(\bar{x}, z) \geq 0, & \forall z \in Z(\bar{x}), \end{cases}$$

where $Z : X \rightrightarrows K, Z(x) := \{y \in K \mid f^1(x, y) = 0\}$. The problem (LEP) can clearly be rewritten as: Find $\bar{x} \in S_{\mathrm{EP}_1}$ such that

$$f^2(\bar{x}, z) \geq 0, \quad \forall z \in Z(\bar{x}),$$

where $S_{\mathrm{EP}_1} := \{\bar{x} \in K \mid f^1(\bar{x}, y) \geq 0, \forall y \in K\}$.

The family of perturbed lexicographic vector equilibrium problems $(\mathrm{LEP})_n$, $n \in \mathbb{N}$, is restated similarly by

$(\mathrm{LEP})_n$ Find $x_n \in K_n$ such that

$$\begin{cases} f_n^1(x_n, y) \geq 0, & \forall y \in K_n \\ f_n^2(x_n, z) \geq 0, & \forall z \in Z_n(x_n), \end{cases}$$

where $Z_n : X \rightrightarrows K_n, Z_n(x_n) := \{y_n \in K_n \mid f_n^1(x_n, y_n) = 0\}$.

3. EXISTENCE CONDITIONS

In this section, we establish the existence of solutions to the lexicographic equilibrium problem. Unless otherwise specified, we assume throughout this article that K is a nonempty-convex-compact subset of X. The compactness assumption used here is only for the sake of simplifying the presentation. In fact, this assumption can be replaced by the various coercivity conditions, for more details we refer the reader to [1, 7, 10].

We first recall some well-known results about the Kakutani–Fan–Glicksberg fixed point theorem and the Ky Fan's minimax inequality theorem.

Proposition 3.1. [16] *Let A be a nonempty compact convex subset of a locally convex Hausdorff space X. Let $\Phi : A \rightrightarrows A$ be an upper semicontinuous mapping with nonempty closed and convex values. Then, Φ has a fixed point in A.*

Proposition 3.2. [14] *Let $\theta : K \times K \to \mathbb{R}$ be an equilibrium bifunction, i.e., $\theta(x, x) = 0$ for every $x \in K$. If $\theta(\cdot, y)$ is upper semicontinuous for each $y \in K$, and $\theta(x, \cdot)$ is quasiconvex for each $x \in K$, then there exists a point $\bar{x} \in K$ such that*

$$\theta(\bar{x}, y) \geq 0, \quad \forall y \in K.$$

It is obvious that under conditions assumed in Proposition 3.2, the solutions set of the scalar equilibrium $\mathrm{EP}(K, \theta)$ is compact. To obtain some additional properties of the solutions set to equilibrium problems, we would like to recall the following results; see, for instance, [1, 6].

Proposition 3.3. *Under assumptions of Proposition 3.2 and suppose further that:*

(i) *θ is pseudomonotone;*
(ii) *$\theta(x, \cdot)$ is explicitly quasiconvex for each $x \in K$.*

Then, the solutions set of $\mathrm{EP}(K, \theta)$ is nonempty, compact and convex.

Proposition 3.4. *Suppose that*

(i) *θ is properly quasimonotone;*

(ii) $\theta(\cdot, y)$ *is upper semicontinuous for each* $y \in K$;

(iii) $\theta(\cdot, y)$ *is quasiconcave for each* $y \in K$.

Then, the solutions set of $\mathrm{EP}(K, \theta)$ *is nonempty, compact and convex.*

Inspired by the Kakutani–Fan–Glicksberg fixed point theorem, we propose the following existence result for (LEP).

Theorem 3.1. *Assume that the following assumptions hold:*

  (i) $f^1(\cdot, y)$ *and* f^2 *are upper semicontinuous for each* $y \in K$;

 (ii) $f^1(x, \cdot)$ *and* $f^2(x, \cdot)$ *are explicitly quasiconvex for each* $x \in K$;

(iii) f^1 *and* f^2 *are pseudomonotone*;

(iv) Z *is lsc on* S_{EP_1}.

Then, the set S_{LEP} *is nonempty and compact.*

Proof. In light of Proposition 3.1, we see that S_{EP_1} is nonempty, compact and convex. For each $x \in S_{\mathrm{EP}_1}$, it is easy to see that $Z(x)$ is convex. Let's consider the set-valued mapping $\Phi : S_{\mathrm{EP}_1} \rightrightarrows S_{\mathrm{EP}_1}$ defined as

$$\Phi(x) := \{u \in S_{\mathrm{EP}_1} \mid f^2(u, y) \geq 0, \forall y \in Z(x)\}.$$

We first check the convexity of $\Phi(x)$. Let $u_1, u_2 \in \Phi(x)$. Then, for all $y \in Z(x)$, we have

$$f^2(u_1, y) \geq 0 \qquad \text{and} \qquad f^2(u_2, y) \geq 0.$$

Since f^2 is pseudomonotone,

$$f^2(y, u_1) \leq 0 \qquad \text{and} \qquad f^2(y, u_2) \leq 0.$$

This fact together with the quasiconvexity of $f^2(y, \cdot)$ gives

$$f^2(y, tu_1 + (1-t)u_2) \leq \max\{f^2(y, u_1), f^2(y, u_2)\} \leq 0.$$

Consequently, $f^2(tu_1 + (1-t)u_2, y) \geq 0$, which implies that $tu_1 + (1-t)u_2 \in \Phi(x)$. Hence, $\Phi(x)$ is convex. We next claim that $\Phi(x)$ is closed. Let $\{u_n\} \subset \Phi(x)$ be a sequence converging to w. Then, $f^2(u_n, y) \geq 0$ for all $y \in Z(x)$. From the upper semicontinuous of $f^2(\cdot, y)$ at u, we have

$$f^2(u, y) \geq 0.$$

Thus, $u \in \Phi(x)$, i.e., $\Phi(x)$ is closed. Finally, we prove that Φ is upper semicontinuous on S_{EP_1}. Suppose to the contrary that Φ is not usc at some point $x_0 \in S_{\mathrm{EP}_1}$. Then, there exists a neighborhood U of $\Phi(x_0)$ such that there are x_n converging to x_0 and $u_n \in \Phi(x_n)$ with $u_n \notin U$ for all n. Since S_{EP_1} is compact, u_n (take a subsequence if necessary) converge to some point $u_0 \in S_{\mathrm{EP}_1}$. For each $y_0 \in Z(x_0)$, the lower semicontinuity of Z ensures the existence of $y_n \in Z(x_n)$ with $y_n \to y_0$. Since $u_n \in \Phi(x_n)$, we have $f^2(u_n, y_n) \geq 0$. It follows from the upper semicontinuous of f^2 at (u_0, y_0) that $f^2(u_0, y_0) \geq 0$, i.e., $w_0 \in \Phi(x_0)$, which is a contradiction as $u_n \notin U$ for all n. Hence, Φ is usc on S_{EP_1}. By Kakutani-Fan-Glicksberg fixed point theorem, there exists $x \in S_{\mathrm{EP}_1}$ such that $\Phi(x) = x$. Therefore, S_{LEP} is nonempty. The compactness of the set of solutions is obvious. $\qquad \square$

Remark 3.1. In [9], Bianchi, Konnov and Pini proposed an approach the existence of solutions of (LEP) via the analysis of the solutions of sequential equilibrium problems. Theorem 3.1 is a direct investigation for the existence of solutions of (LEP). In this way, we have to utilize the lower semicontinuity of the

set-valued mapping Z. The following result provides sufficient conditions for the mentioned property of Z.

Proposition 3.5. *Assume that S_{EP_1} is a nonempty compact set and*

(i) *$f^1(\cdot, y)$ is strictly quasiconcave and lower semicontinuous for each $y \in K$;*

(ii) *f^1 is pseudomonotone and is a transfer weak pseudo-distance on $K \times K$.*

Then, Z is lsc on S_{EP_1}.

Proof. For each $\bar{x} \in S_{\mathrm{EP}_1}$, let $\bar{z}$ be an arbitrary point in $Z(\bar{x})$. It is clear that $f^1(\bar{x}, \bar{z}) = 0$ and $f^1(\bar{x}, y) \geq 0$ for all $y \in K$. Because f^1 is a transfer weak pseudo-distance on $K \times K$, we have $f^1(\bar{z}, y) \geq 0$, which implies that $\bar{z} \in S_{\mathrm{EP}_1}$. It follows from the nonemptiness and compactness of S_{EP_1} that there exists a sequence of points $z_n \in S_{\mathrm{EP}_1}$ converging to $\bar{z}$. We claim that $z_n \in Z(x_n)$ for every $x_n \in S_{\mathrm{EP}_1}$. Indeed, since $z_n \in S_{\mathrm{EP}_1}$, $f^1(z_n, x_n) \geq 0$. Combining this and the pseudomonotonicity of f^1 gives us $f^1(x_n, z_n) \leq 0$, which yields that $f_1(x_n, z_n) = 0$, as $x_n \in S_{\mathrm{EP}_1}$. Therefore, $z_n \in Z(x_n)$, this brings the proof to its end. $\square$

The following result is an alternative approach for the existence solutions of (LEP).

Theorem 3.2. *Assume that*

(i) *$f^1(\cdot, y)$ is quasiconcave for each $y \in K$;*

(ii) *f^1 is a transfer weak pseudo-distance on $K \times K$;*

(iii) *f^1 and f^2 are properly quasimonotone on $K \times K$;*

(iv) *$f^1(\cdot, y)$ and $f^2(\cdot, y)$ are upper semicontinuous for each $y \in K$.*

Then, the set S_{LEP} is nonempty.

Proof. It follows immediately from Proposition 3.4 that S_{EP_1} is nonempty, compact and convex. By setting

$$T := \{x \in S_{\mathrm{EP}_1} \mid f^2(x, y) \geq 0, \forall y \in S_{\mathrm{EP}_1}\}$$

and applying Proposition 3.4 again we obtain that T is nonempty. For every $x \in S_{\mathrm{EP}_1}$, let z be an arbitrary point in $Z(x)$. Because f^1 is a transfer weak pseudo-distance and $f^1(x, z) = 0$, it yields from the fact $f^1(x, y) \geq 0$ for all $y \in K$ that $f^1(z, y) \geq 0$. Thus, $z \in S_{\mathrm{EP}_1}$, i.e., $Z(x) \subset S_{\mathrm{EP}_1}$. Consequently, $T \subset S_{\mathrm{LEP}}$. The proof is complete. $\square$

Remark 3.2. Theorem 3.2 is strictly improve Theorem 2 in [9]. Namely, the weak pseudo-distance assumption on f^1 is relaxed by the transfer weak pseudo-distance.

4. CONVERGENCE CONDITIONS

In this section, we mainly study the Painlevé-Kuratowski convergence of the solution sets of a family of perturbed lexicographic vector equilibrium problems by perturbing both the constraint set and the objective function. We investigate this convergence by using the Painlevé-Kuratowski convergence of the sequence of the perturbed constraint sets and the continuous convergence of the sequence of the perturbed objective functions. In this regard, we firstly establish the following result.

Lemma 4.1. *Assume that*

(i) *$K_n \xrightarrow{\mathrm{P.K.}} K$ and $f_n^1 \xrightarrow{\mathrm{c}} f^1$;*

(ii) *The Fréchet derivative, $\nabla_2 f^1$, of f^1 with respect to the second argument exists and $\nabla_2 f^1(x,\cdot)$ is surjective on $X \setminus \{x\}$ for each $x \in X$.*

Then, for each $x_n \to x$ and $z \in Z(x) \setminus \{x\}$, there exist $z_n \in Z_n(x_n)$ such that $z_n \to z$.

Proof. For each $x_n \to \bar{x}$ and $\bar{z} \in Z(\bar{x}) \setminus \{\bar{x}\}$, we claim that for each neighborhood V of $\bar{z}$, $V \subset X$, there exist a neighborhood U of $\bar{x}$ and a function $s_n : U \to V$ such that $s_n(x) \in Z_n(x)$ for all $x \in U$. Let $m = \left\| \nabla_2 f^1(\bar{x},\bar{z})^{-1} \right\|$ and α be a positive real number such that $B_\alpha(\bar{z}) \subset V$. From the Fréchet differentiability of f^1, one can choose a real number $\beta \in (0,\alpha]$ such that

$$\left| f^1(\bar{x},z) - f^1(\bar{x},\bar{z}) - \langle \nabla_2 f^1(\bar{x},\bar{z}), z - \bar{z} \rangle \right| \le \frac{1}{2m} \|z - \bar{z}\|, \forall z \in B_\beta(\bar{z}),$$

and consequently,

$$\left| f^1(\bar{x},z) - \langle \nabla_2 f^1(\bar{x},\bar{z}), z - \bar{z} \rangle \right| \le \frac{\beta}{2m}, \quad \forall z \in B_\beta(\bar{z}).$$

Since $f_n^1 \xrightarrow{c} f^1$, there exists a positive real number $\gamma \le \beta$ such that for all $x \in B_\gamma(\bar{x}), z \in B_\beta(\bar{z})$, one has

$$\left| f_n^1(x,z) - f^1(\bar{x},z) \right| \le \frac{\beta}{2m}.$$

For each $x \in B_\gamma(\bar{x})$, we construct the function $\xi_x^{(n)} : B_\beta(\bar{z}) \to X$ satisfying

$$\xi_x^{(n)}(z) = \nabla_2 f^1(\bar{x},\bar{z})^{-1} \left(\langle \nabla_2 f^1(\bar{x},\bar{z}), z \rangle - f_n^1(x,z) \right).$$

Obviously, $\xi_x^{(n)}$ is continuous in $B_\beta(\bar{z})$. Further, for any $z \in B_\beta(\bar{z})$, one has

$$\begin{aligned}
\left\| \xi_x^{(n)}(z) - \bar{z} \right\| &= \left\| \nabla_2 f^1(\bar{x},\bar{z})^{-1} \left(\langle \nabla_2 f^1(\bar{x},\bar{z}), z \rangle - f_n^1(x,z) \right) - \bar{z} \right\| \\
&= \left\| \nabla_2 f^1(\bar{x},\bar{z})^{-1} \right\| \left| \langle \nabla_2 f^1(\bar{x},\bar{z}), z - \bar{z} \rangle - f_n^1(x,z) \right| \\
&\le m \left(\left| \langle \nabla_2 f^1(\bar{x},\bar{z}), z - \bar{z} \rangle - f^1(\bar{x},z) \right| + \left| f^1(\bar{x},z) - f_n^1(x,z) \right| \right) \\
&\le m \left(\frac{\beta}{2m} + \frac{\beta}{2m} \right) = \beta.
\end{aligned}$$

Hence, thanks to the Schauder's fixed-point theorem, for all $x \in B_\gamma(\bar{x})$, there exists a point $s_n(x) \in B_\beta(\bar{z}) \subset V$ such that $\xi_{(x)}^{(n)}(s_n(x)) = s_n(x)$. Thus,

$$s_n(x) = \nabla_2 f^1(\bar{x},\bar{z})^{-1} \left(\langle \nabla_2 f^1(\bar{x},\bar{z}), s_n(x) \rangle - f_n^1(x,s_n(x)) \right),$$

which is equivalent to $f_n^1(x,s_n(x)) = 0$, i.e. $s_n(x) \in Z_n(x)$.   □   □

We next focus on the upper part of the convergence of the solution sets S_{LEP}^n.

Theorem 4.1. *Assume that*

(i) $K_n \xrightarrow{\text{P.K.}} K$ *and* $f_n^i \xrightarrow{c} f^i$*, for* $i = 1,2$*;*

(ii) $f^1(x,\cdot)$ *and* $f^2(x,\cdot)$ *are explicitly quasiconvex for each* $x \in K$*;*

(iii) f^1 *and* f^2 *are pseudomonotone;*

(iv) *The Fréchet derivative, $\nabla_2 f^1$, of f^1 with respect to the second argument exists and $\nabla_2 f^1(x,\cdot)$ is surjective on $X \setminus \{x\}$.*

Then, the solution set of (LEP) *is nonempty and* $\limsup_{n \to \infty} S_{\text{LEP}}^n \subset S_{\text{LEP}}$.

Proof. The nonemptiness of the solution set S_{LEP} follows from Theorem 3.1. We show that $\limsup_{n\to\infty} S_{\text{LEP}}^n$ $\subset S_{\text{LEP}}$. Let x_0 be an arbitrary point of $\limsup_{n\to\infty} S_{\text{LEP}}^n$. Then, there exists a subsequence $\{x_{n_k}\}$ in $S_{\text{LEP}}^{n_k}$ such that $x_{n_k} \to x_0$. Since $K_{n_k} \xrightarrow{\text{P.K.}} K$, we have $x_0 \in K$. Because $x_{n_k} \in S_{\text{LEP}}^{n_k}$,

$$\begin{cases} f_{n_k}^1(x_{n_k}, y) \geq 0, & \forall y \in K_{n_k}, \\ f_{n_k}^2(x_{n_k}, z) \geq 0, & \forall z \in Z_{n_k}(x_{n_k}). \end{cases} \tag{4.1}$$
$$\tag{4.2}$$

For each $y \in K$, as $K_{n_k} \xrightarrow{\text{P.K.}} K$, there are $y_{n_k} \in K_{n_k}$ such that $y_{n_k} \to y$. It follows from (4.1) and the continuous convergence of f^1 that $f^1(x_0, y) \geq 0$. If $x_0 \notin S_{\text{LEP}}$, then there is $z_0 \in Z(x_0)$ such that

$$f^2(x_0, z_0) < 0. \tag{4.3}$$

Applying Lemma 3.5, there exist $z_{n_k} \in Z_{n_k}(x_{n_k})$ such that $z_{n_k} \to z_0$. Combining this with (4.2) and taking into account the continuous convergence of f^2, we arrive at $f^2(x_0, z_0) \geq 0$, which contradicts (4.3). Therefore, $x_0 \in S_{\text{LEP}}$, this brings the proof to the end. $\qquad\square$

The essentialness of all assumptions is now explained by the following counterexamples.

Example 4.1. (The Painlevé-Kuratowski convergence of constraint sets is essential) Let $X = \mathbb{R}, K = [-1, 1], K_n = [-2 - n^{-1}, 2 + n^{-1}]$. Let $f, f_n : X \times X \to \mathbb{R}^2$ be the functions defined, respectively, by

$$f(x, y) = (y - x, y - x) \text{ and } f_n(x, y) = \left(\left(1 + \frac{1}{n}\right)(y - x), \left(1 + \frac{1}{n}\right)(y - x)\right).$$

Obviously, assumption (ii) and the lower part of convergence of K_n are satisfied while the upper part of convergence is not. We have $S_{\text{LEP}} = \{-1\}$ and $S_{\text{LEP}}^n = \{-2 - n^{-1}\}$; however, $\{-2\} = \limsup_{n\to\infty} S_{\text{LEP}}^n \not\subset S_{\text{LEP}} = \{-1\}$.

Example 4.2. (The continuous convergence of objective function is essential) Let $X = \mathbb{R}^2, K = [0, 2]^2$ and $f(x, y) = (f^1(x, y), f^2(x, y))$ defined by

$$f^1(x, y) = y_1 - x_1, \quad f^2(x, y) = \begin{cases} x_2 - y_2, & \text{if } x_2 \leq 1, \\ 0, & \text{if } x_2 > 1. \end{cases}$$

It is clear that assumption (ii) is satisfied. Direct computations give us $S_{\text{LEP}} = \{(0, x_2) \mid 1 < x_2 \leq 2\}$. Let $K_n = [0, 2]^2$, and $f_n(x, y) = (f_n^1(x, y), f_n^2(x, y))$, where $f_n^1(x, y) = n^{-1}(x_2 - y_2)^2 + y_1 - x_1$, and

$$f_n^2(x, y) = \begin{cases} (x_2 - y_2) \exp\left(\frac{x_1 - y_1}{n}\right), & \text{if } x_2 \leq 1, \\ (x_1 - y_1)\left(\exp\left(\frac{x_2 - y_2}{n}\right) - 1\right), & \text{if } x_2 > 1. \end{cases}$$

Let $x_n = (0, (n+1)n^{-1})$. It is not hard to verify that $\{x_n\} \subset S_{\text{LEP}}^n$. But $x_n \to (0, 1) \notin S_{\text{LEP}}$. Hence, $\limsup_{n\to\infty} S_{\text{LEP}}^n \not\subset S_{\text{LEP}}$. The reason is that f_n^2 does not converge continuously to f^2. Indeed, taking $x_n = (1, (n+1)n^{-1}), y_n = (1, 2)$, we have $x_n \to x = (1, 1), y_n \to y = (1, 2)$ and $f_n^2(x_n, y_n) = 0$, but $f_2(x, y) = -1 < 0$.

Example 4.3. (The surjectivity of $\nabla_2 f^1$ cannot be dispensed) Let $X = \mathbb{R}, K = [0, 1], K_n = [-n^{-1}, 1 + n^{-1}]$. Consider the functions $f, f_n : X \times X \to \mathbb{R}^2$ defined as $f(x, y) = (x(x - y)^2, \exp(y)(x - y))$, and

$$f_n(x, y) = \left(\left(x + \frac{1}{n}\right)(x - y)^2, \left(1 + \frac{y}{n}\right)^n (x - y)\right).$$

Obviously, assumption (i) is satisfied. Direct computations give us $S_{\text{LEP}} = (0,1]$ and $S_{\text{LEP}}^n = (-n^{-1}, 1 + n^{-1}]$. Hence, $\limsup_{n \to \infty} S_{\text{LEP}}^n \not\subset S_{\text{LEP}}$. This is due to the violation of assumption (ii) as $\nabla_2 f^1(0,y) = 0$, for all $y \in X$.

Now we can state the main result about the convergence in the sense of Painlevé-Kuratowski for the solutions of the perturbed problems.

Theorem 4.2. *Impose all assumptions of Theorem 4.1 and assume further that*

(iii) *K is convex;*

(iv) *$f^1(\cdot, y)$ is strictly quasiconcave on K, for all $y \in K$.*

Then, S_{LEP}^n converge to S_{LEP} in the sense of Painlevé-Kuratowski.

Proof. We need only to prove that $S_{\text{LEP}} \subset \liminf S_{\text{LEP}}^n$. Suppose to the contrary that, there exists $x_0 \in S_{\text{LEP}}$ such that $x_0 \notin \liminf S_{\text{LEP}}^n$. Then, there is a neighborhood U_0 of the origin in X such that

$$S_{\text{LEP}}^n \cap (x_0 + U_0) = \emptyset \text{ for } n \text{ sufficiently large} . \tag{4.4}$$

We discuss the following two cases:

Case 1. $S_{\text{LEP}} = \{x_0\}$. Let $\{x_n\}$ be an arbitrary sequence in S_{LEP}^n. Since $K_n \xrightarrow{\text{P.K.}} K$, one can assume that $x_n \to \bar{x}_0$ for some $\bar{x}_0 \in K$. The same arguments given in the proof of Theorem 4.1 show that $\bar{x}_0 \in S_{\text{LEP}}$. Noting the fact that S_{LEP} is a singleton, we have $x_0 = \bar{x}_0$, and so $x_n \to \bar{x}_0 = x_0$. Thus, $x_n \in x_0 + U_0$ for n large enough. This together with $x_n \in S_{\text{LEP}}^n$ imply that $S_{\text{LEP}}^n \cap (x_0 + U_0) \neq \emptyset$, which contradicts (4.4).

Case 2. There exists $\bar{x} \in S_{\text{LEP}}$ satisfying $\bar{x} \neq x_0$. Then, for any $y \in K$, one has $f^1(x_0, y) \geq 0$ and $f^1(\bar{x}, y) \geq 0$. By the strict quasiconcavity of $f^1(\cdot, y)$ on K, one has

$$f^1(t\bar{x} + (1-t)x_0, y) > \min\{f^1(\bar{x}, y), f^1(x_0, y)\} \geq 0, \forall t \in (0,1). \tag{4.5}$$

Putting $x(t) := t\bar{x} + (1-t)x_0$, we have $x(t) \in K$. It is worth noting that for the chosen U_0, there exist a neighborhood U_1 of the origin in X and $t_0 \in (0,1)$ satisfying $U_1 + U_1 \subset U_0$ and $x(t_0) \in x_0 + U_1$. Hence, $x(t_0) + U_1 \subset x_0 + U_0$. Since $x(t_0) \in K$ and $K_n \xrightarrow{\text{P.K.}} K$, there are $\bar{x}_n(t_0) \in K_n$ with $\bar{x}_n(t_0) \to x(t_0)$. Thus, $\bar{x}_n(t_0) \in x(t_0) + U_1 \subset x_0 + U_0$, for n sufficiently large. Combining this with (4.4), one has $\bar{x}_n(t_0) \notin S_{\text{LEP}}^n$, for n large enough. Hence, there exist $\bar{y}_n \in K_n$ such that

$$f_n^1(\bar{x}_n(t_0), \bar{y}_n) < 0. \tag{4.6}$$

According to Lemma 2.1, we can assume that $\{\bar{y}_n\}$ (taking a subsequence if necessary) converges to some $\bar{y} \in K$. It follows from (4.6) and the continuous convergence of f_n^1 that $f^1(x(t_0), \bar{y}) \leq 0$, which contradicts (4.5). This brings the proof to the end. $\qquad\square$

We give an example to illustrate that the strict quasiconcavity assumption on f^1 is essential.

Example 4.4. Let $X = \mathbb{R}, K = [0,1]$ and $K_n = [\exp(n^{-1}) - 1, \exp(2n^{-1})]$. Consider the functions $f, f_n : X \times X \to \mathbb{R}^2$ defined as

$$f(x,y) = ((x-y)^2, (x-y)\exp(xy)),$$

and

$$f_n(x,y) = \left((x-y)(x-y-\frac{1}{n}), \left(1+\frac{xy}{n}\right)^n (x-y)\right).$$

It is obvious that assumptions (i)-(iii), except (iv), are satisfied. Direct computations give us $S_{\text{LEP}} = [0,1]$ and $S_{\text{LEP}}^n = \{\exp(n^{-1}) - 1\}$. Hence, S_{LEP}^n does not converge to S_{LEP} in the sense of Painlevé-Kuratowski.

5. APPLICATIONS TO LEXICOGRAPHIC VARIATIONAL INEQUALITIES

Let X, X^* and K be as in the preceding sections, and $A^i : X \to X^*$, for $i = 1, 2$. We consider the following lexicographic variational inequality.

(LVI) Find $\bar{x} \in K$ such that

$$(\langle A^1(\bar{x}), y - \bar{x} \rangle, \langle A^2(\bar{x}), y - \bar{x} \rangle) \geq_L 0, \quad \forall y \in K.$$

The family of perturbed lexicographic variational inequalities $(LVI)_n, n \in \mathbb{N}$ is given as

$(LVI)_n$ Find $\bar{x}_n \in K_n$ such that

$$(\langle A_n^1(\bar{x}_n), y - \bar{x} \rangle, \langle A_n^2(\bar{x}_n), y - \bar{x} \rangle) \geq_L 0, \quad \forall y \in K_n,$$

where $K_n \subset X$ and $A_n^i : X \to X^*$.

The following results are derived directly from the corresponding ones in Section 4 by setting $f^i(x, y) = \langle A^i(x), y - x \rangle$ for $i = 1, 2$.

Corollary 5.1. *For (LVI), assume that the following conditions hold:*

(i) $K_n \xrightarrow{P.K.} K$ and $A_n^i \xrightarrow{c} A^i$ for $i = 1, 2$;

(ii) A^1 is surjective.

Then, $\limsup_{n \to \infty} S_{LVI}^n \subset S_{LVI}$.

Corollary 5.2. *For (LVI), impose the assumptions of Corollary 5.1 and the additional conditions:*

(iii) K is convex;

(iv) the function $x \mapsto \langle A^1(x), y - x \rangle$ is strictly quasiconcave on K, for all $y \in K$.

Then, S_{LVI}^n *converge to* S_{LVI} *in the sense of Painlevé-Kuratowski.*

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