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APPROXIMATION OF COMMON SOLUTIONS OF A SPLIT INCLUSION PROBLEM AND A FIXED-POINT PROBLEM

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Abstract. The purpose of this paper is to introduce a composite viscosity iterative algorithm for finding a common solution of a split variational inclusion problem and a fixed point of a family of pseudocontractive mappings. Strong convergence of the proposed iterative algorithm is obtained under some mild assumptions in the framework of Hilbert spaces.
Keywords. Split variational inclusion problem; Fixed-point problem; Pseudocontractive mapping; Composite viscosity iterative method.

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1. INTRODUCTION-PRELIMINARIES

Let H_1 and H_2 be two real Hilbert spaces with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, while H refers to as any of these spaces. Let C be a nonempty closed convex subset of H_1, and let Q be a nonempty closed convex subset of H_2. In 1994, Censor and Elfving [7] introduced the following well known split feasibility problem: find $x \in H_1$ such that

$$x \in C, \quad Ax \in Q,$$

where $A : H_1 \to H_2$ is a bounded linear operator. The split feasibility problem has attracted many authors' attention due to its application in signal processing and image reconstruction, with particular progress in intensity-modulated radiation therapy [1, 8, 9, 20] and many other applied fields.

In 2002, Byrne [1] introduced the following so-called CQ algorithm.

$$x_0 \in H_1, x_{n+1} = P_C\big(I - \gamma A^*(I - P_Q)A\big)x_n, \quad n \geq 1,$$

where A^* is the adjoint operator of A, $\gamma > 0$ is a real constant in $(0, \frac{2}{r})$ (r is the spectral radius of the self-adjoint operator A^*A), P_C is the metric projection from H_1 onto C, P_Q is the metric projection from H_2 onto Q and I is the identity mapping on H_1. He proved that the sequence $\{x_n\}$ generated in the CQ algorithm converges weakly to some solution of the split feasibility problem. Recently, many authors intensively investigated the split feasibility problem based on different regularization methods in Hilbert spaces or Banach spaces; see [2, 3, 4, 5, 6, 18, 21] and the references therein.

Let $B : C \to H$ be a nonlinear mapping. The classical variational inequality (VI) is to find $x^* \in C$ such that

$$\langle Bx^*, x - x^*\rangle \geq 0, \quad \forall x \in C. \tag{1.1}$$

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We denote by $\mathrm{VI}(C,B)$ the solution set of VI (1.1). As a very effective and powerful tool, VI (1.1) has been applied to study a wide range of problems arising in differential equations, mechanics, contact problems in elasticity, optimization and control problems, management science, operations research, and general equilibrium problems in economics and transportation.

A set-valued mapping $M: H \to 2^H$ is said to be monotone if, for all $x,y \in H$, $f \in Mx$ and $g \in My$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $M: H \to 2^H$ is maximal if the graph $\mathrm{Gph}(M)$ of M is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping M is maximal if and only if for $(x,f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$, $\forall (y,g) \in \mathrm{Gph}(M)$ implies $f \in Mx$. Let $M: H \to 2^H$ be a multi-valued maximal monotone mapping. For any positive number λ and identity operator I on H, the single-valued mapping $J_\lambda^M : H \to H$ defined by

$$J_\lambda^M(x) := (I+\lambda M)^{-1}(x), \quad \forall x \in H,$$

is called the resolvent operator associated with M. It is known that the resolvent operator J_λ^M is firmly nonexpansive, in particular, nonexpansive.

Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings. We consider the following split variational inclusion problem (SVIP): find $x^* \in H_1$ such that

$$0 \in B_1(x^*), \tag{1.2}$$

and

$$y^* = Ax^* \in H_2 \quad \text{solves} \quad 0 \in B_2(y^*). \tag{1.3}$$

In this paper, $\mathrm{Sol}(B_1,B_2)$ stands for the solution set of SVIP (1.2)-(1.3).

It is obvious that SVIP (1.2)-(1.3) is equivalent to the problem of finding $x^* \in H_1$ with $x^* = J_\lambda^{B_1}(x^*)$ such that

$$y^* = Ax^* \in H_2 \quad \text{and} \quad y^* = J_\lambda^{B_2}(y^*),$$

for some $\lambda > 0$. Recently, SVIP (1.2)-(1.3) has received much attention and extensively investigated based on fixed-point algorithms due to its importance in convex optimization problems; see [2, 11, 14, 16] and the references therein.

Recently, Byrne *et al.* [2] studied the convergence of the following iterative method for solving SVIP (1.2)-(1.3):

$$x_1 \in H, \quad x_{n+1} = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n) \quad \forall n \geq 1,\ \exists \lambda > 0.$$

Weak and strong convergence theorems were established in the framework of Hilbert spaces.

Let $T: C \to C$ be a nonlinear mapping. Denote by $\mathrm{Fix}(T)$ the fixed-point set of T, i.e., $\mathrm{Fix}(T) = \{x \in C : x = Tx\}$. Recall that T is said to be κ-Lipschitzian if there exists a constant $\kappa > 0$ such that $\|Tx - Ty\| \leq \kappa\|x-y\|$, $\forall x,y \in C$. In particular, if $\kappa = 1$, then T is said to be nonexpansive. If $\kappa < 1$, then T is said to be contractive.

For any $x \in H$, there exists a unique nearest point on the nonempty closed convex subset C denoted by $P_C x$ such that $\|x - P_C x\| \leq \|x-y\|$, $\forall y \in C$. The mapping P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping from H onto C. The metric projection P_C can be characterized by $P_C x \in C$ and

$$\langle x-y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x,y \in H.$$

For all $x \in H$ and $y \in C$, $P_C x$ is characterized by $\langle x - P_C x, y - P_C x\rangle \leq 0$. It is easy to see that the above inequality is equivalent to the following inequality:

$$\|x-y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H_1, y \in C.$$

It is not hard to find that every nonexpansive mapping $T : H \to H$ satisfies the following inequality

$$\langle (I-T)x - (I-T)y, Ty - Tx\rangle \leq \frac{1}{2}\|(I-T)x - (I-T)y\|^2, \quad \forall (x,y) \in H \times H. \tag{1.4}$$

A mapping $T : H \to H$ is said to be averaged mapping if it can be written as an average of the identity I and a nonexpansive mapping, that is,

$$T = (1-\alpha)I + \alpha S,$$

where $\alpha \in (0,1)$ and $S : H \to H$ is a nonexpansive mapping. We note that averaged mappings are nonexpansive. Further, firmly nonexpansive mappings (in particular, projections on nonempty closed and convex subsets and resolvent operators of maximal monotone operators).

The composite of finitely many averaged mappings is averaged. The following is a remarkable property of averaged mappings.

$$\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1T_2\cdots T_{N-1}T_N) = \mathrm{Fix}(T_2T_3\cdots T_NT_1) = \cdots = \mathrm{Fix}(T_NT_1\cdots T_{N-2}T_{N-1}).$$

In particular, as $N = 2$, we have $\mathrm{Fix}(T_1T_2) = \mathrm{Fix}(T_2T_1) = \mathrm{Fix}(T_1)\bigcap \mathrm{Fix}(T_2)$.

In 2014, Kazmi and Rizvi [14] studied SVIP (1.2)-(1.3) and fixed points of a nonexpansive mapping S via the following iterative algorithm

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Su_n, \quad \forall n \geq 0, \end{cases} \tag{1.5}$$

where f is a contractive mapping in H_1, $\gamma \in (0, \frac{1}{L})$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$. They proved that both $\{u_n\}$ and $\{x_n\}$ converge strongly to a point $z \in \mathrm{Sol}(B_1,B_2)$, which is the unique solution to the variational inequality $\langle (I-f)z, z-p\rangle \leq 0$, $\forall p \in \mathrm{Sol}(B_1,B_2)$.

Recall that a mapping T is said to be pseudocontractive if

$$\langle Tx - Ty, x-y\rangle \leq \|x-y\|^2, \quad \forall x,y \in C.$$

T is said to be strongly pseudocontractive if there exists a constant $\beta \in (0,1)$ such that

$$\langle Tx - Ty, x-y\rangle \leq \beta\|x-y\|^2, \quad \forall x,y \in C.$$

Let $\{S_n\}_{n=1}^{\infty}$ be a sequence of continuous pseudocontractive self mappings on C. Then $\{S_n\}_{n=1}^{\infty}$ is said to be a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on C if there exists a constant $\ell > 0$ such that each S_n is ℓ-Lipschitz continuous. Recall that a mapping $F : H \to H$ is said to be a strongly positive bounded linear operator if there exists a constant $\bar{\gamma} > 0$ such that $\langle Fx, x\rangle \geq \bar{\gamma}\|x\|^2$, $\forall x \in H$.

In this paper, we introduce a composite viscosity iterative algorithm for finding a common solution of a split variational inclusion problem and a fixed point of a family of pseudocontractive mappings. Strong convergence of the proposed iterative algorithm is obtained under some mild assumptions in the framework of Hilbert spaces. To prove our main results, we also need the following tools.

Lemma 1.1 ([10]). *Let C be a nonempty closed convex subset of a Banach space X and let $T : C \to C$ be a continuous and strong pseudocontraction mapping. Then, T has a unique fixed point in C.*

Lemma 1.2 ([22]). *Let H be a Hilbert space. Let λ be a number in $(0,1]$ and let $T : H \to H$ be a nonexpansive mapping. Define a mapping $T^\lambda : H \to H$ by $T^\lambda x := Tx - \lambda\mu F(Tx)$, $\forall x \in H$, where $F : H \to H$ is a κ-Lipschitzian and η-strongly monotone operator. Then T^λ is a contraction provided $0 < \mu < \frac{2\eta}{\kappa^2}$, that is, $\|T^\lambda x - T^\lambda y\| \leq (1-\lambda\tau)\|x-y\|$, $\forall x,y \in H$, where $\tau = 1 - \sqrt{1-\mu(2\eta-\mu\kappa^2)} \in (0,1]$.*

Lemma 1.3 ([2]). *Let H_1 and H_2 be two real Hilbert spaces. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be multi-valued maximal monotone mappings. Let A be a bounded linear operator from H_1 to H_2 and let A^* be its adjoint operator. Let L be the spectral radius of the A^*A. For any given $\lambda > 0$, let $G : H \to H$ be a mapping defined by $G := J_\lambda^{B_1}(I + \gamma A^*(J_\lambda^{B_2} - I)A)$, where $\gamma \in (0, \frac{1}{L})$. Then G is a nonexpansive mapping. If $\mathrm{Sol}(B_1,B_2) \neq \emptyset$, then $\mathrm{Sol}(B_1,B_2) = \mathrm{Fix}(G)$.*

Lemma 1.4 ([19]). *Let $\{x_n\}$ and $\{z_n\}$ be bounded sequences in a Banach space X and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Suppose $x_{n+1} = \beta_n x_n + (1-\beta_n)z_n$ for all integers $n \geq 1$ and*

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then, $\lim_{n\to\infty} \|z_n - x_n\| = 0$.

Lemma 1.5 ([12]). *Let C be a nonempty closed convex subset of a Banach space X. Let $S_1, S_2, \dots$ be a sequence of mappings of C into itself. Suppose that $\sum_{n=1}^\infty \sup\{\|S_n x - S_{n+1}x\| : x \in C\} < \infty$. Then for each $y \in C$, $\{S_n y\}$ converges strongly to some point of C. Moreover, let S be a mapping of C into itself defined by $Sy = \lim_{n\to\infty} S_n y$ for all $y \in C$. Then $\lim_{n\to\infty} \sup\{\|Sx - S_n x\| : x \in C\} = 0$.*

Lemma 1.6 ([17]). *Assume that S is a nonexpansive self-mapping on a nonempty closed convex subset C of a Hilbert space H. If S has a fixed point, then $I - S$ is demiclosed at zero, i.e., if $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ and the sequence $\{(I-S)x_n\}$ converges strongly to zero, then $(I-S)x = 0$, where I is the identity mapping of H_1.*

Lemma 1.7 ([23]). *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the conditions $a_{n+1} \leq (1-\lambda_n)a_n + \lambda_n\gamma_n$, $\forall n \geq 1$, where $\{\lambda_n\}$ and $\{\gamma_n\}$ are sequences of real numbers such that $\{\lambda_n\} \subset [0,1]$ and $\sum_{n=1}^\infty \lambda_n = \infty$, $\limsup_{n\to\infty} \gamma_n \leq 0$. Then $\lim_{n\to\infty} a_n = 0$.*

Lemma 1.8 ([13]). *Let C be a nonempty closed convex subset of a real Hilbert space H_1, and let $B : C \to H_1$ be a monotone and hemicontinuous mapping. Then the following hold:*

(i) $\mathrm{VI}(C,B) = \{x^* \in C : \langle By, y - x^*\rangle \geq 0, \forall y \in C\}$;

(ii) $\mathrm{VI}(C,B) = \mathrm{Fix}(P_C(I - \lambda B))$ *for all* $\lambda > 0$;

(iii) $\mathrm{VI}(C,B)$ *consists of one point, if B is strongly monotone and Lipschitz continuous.*

2. Main Results

We are now in a position to state and prove the main result in this paper.

Theorem 2.1. *Let H_1 and H_2 be real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator and let A^* be the adjoint of A. Let L be the spectral radius of the operator A^*A. Suppose that $B_1 : H_1 \to 2^{H_1}$*

and $B_2 : H_2 \to 2^{H_2}$ are maximal monotone mappings. Let $G : H_1 \to H_1$ be a mapping defined as $G := J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)$, where $\lambda > 0$, $\gamma \in (0, \frac{1}{L})$. Let $f : H_1 \to H_1$ be a nonexpansive mapping and let $F : H_1 \to H_1$ be κ-Lipschitzian and η-strongly monotone with constants $\kappa, \eta > 0$ such that $0 < \delta < \tau := 1-\sqrt{1-\mu(2\eta-\mu\kappa^2)} \in (0,1]$, $0 < \mu < \frac{2\eta}{\kappa^2}$. Assume that $\{S_n\}_{n=1}^\infty$ is a countable family of ℓ-uniformly Lipschitzian pseudocontractive self-mappings on H_1 such that $\Omega := (\bigcap_{n=1}^\infty \mathrm{Fix}(S_n)) \cap \mathrm{Sol}(B_1,B_2) \neq \emptyset$. For an arbitrary $x_1 \in H_1$, let $\{x_n\}$ and $\{u_n\}$ be sequences generated by

$$\begin{cases} u_n = \gamma_n x_n + (1-\gamma_n)S_n u_n, \\ x_{n+1} = \alpha_n \delta f(x_n) + \beta_n x_n + [(1-\beta_n)I - \alpha_n \mu F]G u_n, \quad \forall n \geq 1, \end{cases} \tag{2.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $(0,1)$ satisfying the following conditions:

(1) $\{\alpha_n+\beta_n\} \subset (0,1]$ and $\{\beta_n\}_{n=1}^\infty \subset [a,b]$ for some $a,b \in (0,1)$;

(2) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$;

(3) $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$ and $\lim_{n\to\infty} |\gamma_{n+1}-\gamma_n| = 0$.

Assume that $\sum_{n=1}^\infty \sup_{x\in K} \|S_n x - S_{n+1}x\| < \infty$ for any bounded subset K of H_1. Let S be a mapping of H_1 into itself defined by $Sx = \lim_{n\to\infty} S_n x$ for all $x \in H_1$ with $\mathrm{Fix}(S) = \bigcap_{n=1}^\infty \mathrm{Fix}(S_n)$. Then $\{x_n\}$ and $\{u_n\}$ converge strongly to a point $z \in \Omega$, which is the unique solution to the variational inequality

$$\langle (\mu F - \delta f)z, z-p\rangle \leq 0, \quad \forall p \in \Omega.$$

Proof. Taking into account $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$, we may assume, without loss of generality, that $\{\gamma_n\} \subset [c,d] \subset (0,1)$ for some $c,d \in (0,1)$. Define a mapping F_n by $F_n x = \gamma_n x_n + (1-\gamma_n)S_n x$, $\forall x \in H_1$. Since each $S_n : H_1 \to H_1$ is a continuous pseudocontraction mapping, we deduce that

$$\langle F_n x - F_n y, x-y\rangle \leq (1-\gamma_n)\|x-y\|^2, \quad \forall x,y \in H_1.$$

This shows that F_n is a continuous strong pseudocontraction. From Lemma 1.1, we know that, for each $n \geq 1$, $u_n = \gamma_n x_n + (1-\gamma_n)S_n u_n$ in (2.1) is well defined.

We now claim that $\{x_n\}$, $\{y_n\}$, $\{u_n\}$, $\{S_n u_n\}$, $\{f(x_n)\}$ and $\{F(y_n)\}$ are bounded. Indeed, take an element $p \in \Omega$ arbitrarily. Then $p = J_\lambda^{B_1} p$, $Ap = J_\lambda^{B_2}(Ap)$ and $S_n p = p$ for all $n \geq 1$. Since each $S_n : H_1 \to H_1$ is a pseudocontraction mapping, it follows that

$$\begin{aligned} \|u_n - p\|^2 &= \langle u_n - p, u_n - p\rangle \\ &= \gamma_n\langle x_n - p, u_n - p\rangle + (1-\gamma_n)\langle S_n u_n - p, u_n - p\rangle \\ &\leq \gamma_n \|x_n - p\|\|u_n - p\| + (1-\gamma_n)\|u_n - p\|^2, \end{aligned}$$

which yields $\|u_n - p\| \leq \|x_n - p\|$. Putting

$$y_n = J_\lambda^{B_1}(u_n + \gamma A^*(J_\lambda^{B_2} - I)Au_n),$$

we obtain

$$\begin{aligned} \|y_n - p\|^2 &= \|J_\lambda^{B_1}(u_n + \gamma A^*(J_\lambda^{B_2} - I)Au_n) - J_\lambda^{B_1} p\|^2 \\ &\leq \|u_n + \gamma A^*(J_\lambda^{B_2} - I)Au_n - p\|^2 \\ &= \|u_n - p\|^2 + \gamma^2\|A^*(J_\lambda^{B_2} - I)Au_n\|^2 + 2\gamma\langle u_n - p, A^*(J_\lambda^{B_2} - I)Au_n\rangle. \end{aligned}$$

Thus,

$$\|y_n - p\|^2 \leq \|x_n - p\|^2 + \gamma^2\langle (J_\lambda^{B_2} - I)Au_n, AA^*(J_\lambda^{B_2} - I)Au_n\rangle + 2\gamma\langle u_n - p, A^*(J_\lambda^{B_2} - I)Au_n\rangle.$$

Note that

$$\gamma^2\langle (J_\lambda^{B_2}-I)Au_n, AA^*(J_\lambda^{B_2}-I)Au_n\rangle \le L\gamma^2\|(J_\lambda^{B_2}-I)Au_n\|^2.$$

From (1.4), we have

$$\begin{aligned}
&2\gamma\langle u_n-p, A^*(J_\lambda^{B_2}-I)Au_n\rangle\\
&=2\gamma\langle A(u_n-p)+(J_\lambda^{B_2}-I)Au_n-(J_\lambda^{B_2}-I)Au_n,(J_\lambda^{B_2}-I)Au_n\rangle\\
&=2\gamma\{\langle Ap-J_\lambda^{B_2}Au_n, Au_n-J_\lambda^{B_2}Au_n\rangle-\|(J_\lambda^{B_2}-I)Au_n\|^2\}\\
&\le 2\gamma\{\frac{1}{2}\|(J_\lambda^{B_2}-I)Au_n\|^2-\|(J_\lambda^{B_2}-I)Au_n\|^2\}\\
&=-\gamma\|(J_\lambda^{B_2}-I)Au_n\|^2.
\end{aligned} \tag{2.2}$$

It follows that

$$\|y_n-p\|^2\le\|x_n-p\|^2+\gamma(L\gamma-1)\|(J_\lambda^{B_2}-I)Au_n\|^2. \tag{2.3}$$

From $\gamma\in(0,\frac{1}{L})$, we arrive at $\|y_n-p\|\le\|x_n-p\|$. Since $f:H_1\to H_1$ is nonexpansive, we conclude that

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\alpha_n(\delta f(x_n)-\mu Fp)+\beta_n(x_n-p)+[(1-\beta_n)I-\alpha_n\mu F]y_n-[(1-\beta_n)I-\alpha_n\mu F]p\|\\
&\le \alpha_n\delta\|f(x_n)-f(p)\|+\alpha_n\|\delta f(p)-\mu Fp\|+\beta_n\|x_n-p\|\\
&\quad +(1-\beta_n)\|[I-\tfrac{\alpha_n}{1-\beta_n}\mu F]y_n-[I-\tfrac{\alpha_n}{1-\beta_n}\mu F]p\|\\
&\le \alpha_n\delta\|x_n-p\|+\alpha_n\|\delta f(p)-\mu Fp\|+\beta_n\|x_n-p\|+(1-\beta_n)(1-\tfrac{\alpha_n}{1-\beta_n}\tau)\|y_n-p\|\\
&\le \alpha_n\delta\|x_n-p\|+\alpha_n\|\delta f(p)-\mu Fp\|+\beta_n\|x_n-p\|+(1-\beta_n-\alpha_n\tau)\|x_n-p\|\\
&= [1-\alpha_n(\tau-\delta)]\|x_n-p\|+\alpha_n(\tau-\delta)\cdot\tfrac{\|\delta f(p)-\mu Fp\|}{\tau-\delta}\\
&\le \max\{\|x_n-p\|,\tfrac{\|\delta f(p)-\mu Fp\|}{\tau-\delta}\}.
\end{aligned}$$

By induction, we have

$$\|x_n-p\|\le\max\{\|x_1-p\|,\frac{\|\delta f(p)-\mu Fp\|}{\tau-\delta}\},\quad \forall n\ge 1.$$

It immediately follows that $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{u_n\}$, $\{f(x_n)\}$ and $\{F(y_n)\}$. Taking into account that $\{S_n\}$ is ℓ-uniformly Lipschitzian on C, we know that

$$\|S_nu_n\|\le\|S_nu_n-p\|+\|p\|\le\ell\|u_n-p\|+\|p\|,$$

which implies that $\{S_nu_n\}$ is also bounded. Setting $x_{n+1}=\beta_nx_n+(1-\beta_n)v_n$, we see that

$$\begin{aligned}
v_n &= \tfrac{1}{1-\beta_n}\{\alpha_n\delta f(x_n)+\beta_nx_n+[(1-\beta_n)I-\alpha_n\mu F]y_n\}-\tfrac{\beta_n}{1-\beta_n}x_n\\
&= \tfrac{1}{1-\beta_n}\{\alpha_n\delta f(x_n)+\beta_nx_n+y_n-\beta_ny_n-\alpha_n\mu Fy_n\}-\tfrac{\beta_n}{1-\beta_n}x_n\\
&= \tfrac{1}{1-\beta_n}\{\alpha_n(\delta f(x_n)-\mu Fy_n)+(1-\beta_n)y_n\}\\
&= \tfrac{\alpha_n}{1-\beta_n}(\delta f(x_n)-\mu Fy_n)+y_n.
\end{aligned}$$

Hence,

$$\|v_{n+1}-v_n\|\le\|y_{n+1}-y_n\|+\frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\delta f(x_{n+1})-\mu Fy_{n+1}\|+\frac{\alpha_n}{1-\beta_n}\|\delta f(x_n)-\mu Fy_n\|.$$

By using Lemma 1.3, we know that $G:=J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)$ is nonexpansive. So

$$\|y_{n+1}-y_n\|=\|Gu_{n+1}-Gu_n\|\le\|u_{n+1}-u_n\|.$$

On the other hand, simple calculations show that

$$\begin{aligned}\|u_{n+1}-u_n\|^2 &= \gamma_{n+1}\langle x_{n+1}-x_n, u_{n+1}-u_n\rangle + (1-\gamma_{n+1})[\langle S_{n+1}u_{n+1}-S_nu_{n+1}, u_{n+1}-u_n\rangle \\ &\quad + \langle S_nu_{n+1}-S_nu_n, u_{n+1}-u_n\rangle] + (\gamma_{n+1}-\gamma_n)\langle x_n - S_nu_n, u_{n+1}-u_n\rangle \\ &\le \gamma_{n+1}\|x_{n+1}-x_n\|\|u_{n+1}-u_n\| + (1-\gamma_{n+1})[\|S_{n+1}u_{n+1}-S_nu_{n+1}\|\|u_{n+1}-u_n\| \\ &\quad + \|u_{n+1}-u_n\|^2] + |\gamma_{n+1}-\gamma_n|\|x_n-S_nu_n\|\|u_{n+1}-u_n\|,\end{aligned}$$

that is,

$$\begin{aligned}\|u_{n+1}-u_n\| &\le \gamma_{n+1}\|x_{n+1}-x_n\| + (1-\gamma_{n+1})[\|S_{n+1}u_{n+1}-S_nu_{n+1}\| \\ &\quad + \|u_{n+1}-u_n\|] + |\gamma_{n+1}-\gamma_n|\|x_n-S_nu_n\|,\end{aligned}$$

which immediately leads to

$$\begin{aligned}\|u_{n+1}-u_n\| &\le \|x_{n+1}-x_n\| + \tfrac{1-\gamma_{n+1}}{\gamma_{n+1}}\|S_{n+1}u_{n+1}-S_nu_{n+1}\| + |\gamma_{n+1}-\gamma_n|\tfrac{\|x_n-S_nu_n\|}{\gamma_{n+1}} \\ &\le \|x_{n+1}-x_n\| + \tfrac{1}{c}\|S_{n+1}u_{n+1}-S_nu_{n+1}\| + |\gamma_{n+1}-\gamma_n|\tfrac{\|x_n-S_nu_n\|}{c}.\end{aligned}$$

Putting $K := \{u_n : n \ge 1\}$, we know that K is a bounded subset of H_1. By the assumption, we get $\sum_{n=1}^{\infty}\sup_{x\in K}\|S_{n+1}x-S_nx\| < \infty$. In view of

$$\|S_{n+1}u_{n+1}-S_nu_{n+1}\| \le \sup_{x\in K}\|S_{n+1}x-S_nx\|,$$

we have

$$\sum_{n=1}^{\infty}\|S_{n+1}u_{n+1}-S_nu_{n+1}\| < \infty.$$

Therefore,

$$\begin{aligned}\|v_{n+1}-v_n\| &\le \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}\|\delta f(x_{n+1})-\mu Fy_{n+1}\| + \tfrac{\alpha_n}{1-\beta_n}\|\delta f(x_n)-\mu Fy_n\| + \|y_{n+1}-y_n\| \\ &\le \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}\|\delta f(x_{n+1})-\mu Fy_{n+1}\| + \tfrac{\alpha_n}{1-\beta_n}\|\delta f(x_n)-\mu Fy_n\| + \|u_{n+1}-u_n\| \\ &\le \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}\|\delta f(x_{n+1})-\mu Fy_{n+1}\| + \tfrac{\alpha_n}{1-\beta_n}\|\delta f(x_n)-\mu Fy_n\| + \|x_{n+1}-x_n\| \\ &\quad + \tfrac{1}{c}\|S_{n+1}u_{n+1}-S_nu_{n+1}\| + |\gamma_{n+1}-\gamma_n|\tfrac{\|x_n-S_nu_n\|}{c},\end{aligned}$$

which immediately yields

$$\begin{aligned}\|v_{n+1}-v_n\| - \|x_{n+1}-x_n\| &\le \tfrac{\alpha_{n+1}}{1-\beta_{n+1}}\|\delta f(x_{n+1})-\mu Fy_{n+1}\| + \tfrac{\alpha_n}{1-\beta_n}\|\delta f(x_n)-\mu Fy_n\| \\ &\quad + \tfrac{1}{c}\|S_{n+1}u_{n+1}-S_nu_{n+1}\| + |\gamma_{n+1}-\gamma_n|\tfrac{\|x_n-S_nu_n\|}{c}.\end{aligned}$$

It follows that $\limsup_{n\to\infty}(\|v_{n+1}-v_n\| - \|x_{n+1}-x_n\|) \le 0$. An application of Lemma 1.4 yields that

$$\lim_{n\to\infty}\|v_n-x_n\| = 0. \tag{2.4}$$

Since

$$\|x_{n+1}-x_n\| = \|\gamma_nx_n + (1-\gamma_n)v_n - [\gamma_nx_n + (1-\gamma_n)x_n]\| = (1-\gamma_n)\|v_n-x_n\|,$$

we conclude from (2.4) that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0, \tag{2.5}$$

so are

$$\lim_{n\to\infty}\|u_{n+1}-u_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|y_{n+1}-y_n\| = 0. \tag{2.6}$$

Set $f_n = \delta f(x_n) - \mu F y_n$ for all $n \geq 1$. For any $p \in \Omega$, we observe that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n \delta f(x_n) + \beta_n x_n + (1-\beta_n) y_n - \alpha_n \mu F y_n - p\|^2 \\
&= \|\alpha_n f_n + \beta_n x_n + (1-\beta_n) y_n - \beta_n p - (1-\beta_n) p\|^2 \\
&\leq \|\beta_n (x_n - p) + (1-\beta_n)(y_n - p)\|^2 + 2\langle \alpha_n f_n, x_{n+1} - p\rangle \\
&\leq \beta_n \|x_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 - \beta_n(1-\beta_n)\|x_n - y_n\|^2 \\
&\quad + 2\alpha_n \|f_n\| \|x_{n+1} - p\| \\
&\leq \beta_n \|x_n - p\|^2 + (1-\beta_n)\|y_n - p\|^2 + 2\alpha_n M^2,
\end{aligned} \tag{2.7}$$

where $M = \max\{\sup_{n\geq 1} \|f_n\|, \sup_{n\geq 1} \|x_n - p\|\}$. Substituting (2.3) into (2.7), we obtain that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \beta_n \|x_n - p\|^2 + (1-\beta_n)[\|u_n - p\|^2 + \gamma(L\gamma - 1)\|(J_\lambda^{B_2} - I)Au_n\|^2] + 2\alpha_n M^2 \\
&\leq \|x_n - p\|^2 - \gamma(1-\beta_n)(1 - L\gamma)\|(J_\lambda^{B_2} - I)Au_n\|^2 + 2\alpha_n M^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\gamma(1-\beta_n)(1-L\gamma)\|(J_\lambda^{B_2} - I)Au_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n M^2 \\
&\leq (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + 2\alpha_n M^2,
\end{aligned}$$

From conditions (1), (2), and (2.5), we get

$$\lim_{n\to\infty} \|(J_\lambda^{B_2} - I)Au_n\| = 0. \tag{2.8}$$

Since $J_\lambda^{B_1}$ is firmly nonexpansive mapping, we conclude from inequality (2.2) that

$$\begin{aligned}
\|y_n - p\|^2 &\leq \langle y_n - p, u_n + \gamma A^*(J_\lambda^{B_2} - I)Au_n - p\rangle \\
&= \tfrac{1}{2}\{\|y_n - p\|^2 + \|u_n + \gamma A^*(J_\lambda^{B_2} - I)Au_n - p\|^2 - \|y_n - p \\
&\quad - [u_n + \gamma A^*(J_\lambda^{B_2} - I)Au_n - p]\|^2\} \\
&\leq \tfrac{1}{2}\{\|y_n - p\|^2 + \|u_n - p\|^2 - \gamma\|(J_\lambda^{B_2} - I)Au_n\|^2 + \gamma^2\|A^*(J_\lambda^{B_2} - I)Au_n\|^2 \\
&\quad - \|y_n - u_n - \gamma A^*(J_\lambda^{B_2} - I)Au_n\|^2\} \\
&= \tfrac{1}{2}\{\|y_n - p\|^2 + \|u_n - p\|^2 - \gamma\|(J_\lambda^{B_2} - I)Au_n\|^2 + \gamma^2\|A^*(J_\lambda^{B_2} - I)Au_n\|^2 \\
&\quad - \|y_n - u_n\|^2 - \gamma^2\|A^*(J_\lambda^{B_2} - I)Au_n\|^2 + 2\gamma\langle y_n - u_n, A^*(J_\lambda^{B_2} - I)Au_n\rangle\} \\
&\leq \tfrac{1}{2}\{\|y_n - p\|^2 + \|u_n - p\|^2 - \|y_n - u_n\|^2 + 2\gamma\langle y_n - u_n, A^*(J_\lambda^{B_2} - I)Au_n\rangle\} \\
&\leq \tfrac{1}{2}\{\|y_n - p\|^2 + \|u_n - p\|^2 - \|y_n - u_n\|^2 + 2\gamma\|A(y_n - u_n)\|\|(J_\lambda^{B_2} - I)Au_n\|\}.
\end{aligned}$$

Hence, we obtain

$$\|y_n - p\|^2 \leq \|u_n - p\|^2 - \|y_n - u_n\|^2 + 2\gamma\|A(y_n - u_n)\|\|(J_\lambda^{B_2} - I)Au_n\|. \tag{2.9}$$

Substituting (2.9) into (2.7), one concludes that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \beta_n\|x_n - p\|^2 + (1-\beta_n)[\|u_n - p\|^2 - \|y_n - u_n\|^2 \\
&\quad + 2\gamma\|A(y_n - u_n)\|\|(J_\lambda^{B_2} - I)Au_n\|] + 2\alpha_n M^2 \\
&\leq \|x_n - p\|^2 - (1-\beta_n)\|y_n - u_n\|^2 \\
&\quad + 2\gamma(1-\beta_n)\|A(y_n - u_n)\|\|(J_\lambda^{B_2} - I)Au_n\| + 2\alpha_n M^2.
\end{aligned}$$

So,

$$\begin{aligned}
(1-\beta_n)\|y_n - u_n\|^2 &\leq (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| \\
&\quad + 2\gamma(1-\beta_n)\|A(y_n - u_n)\|\|(J_\lambda^{B_2} - I)Au_n\| + 2\alpha_n M^2.
\end{aligned}$$

From conditions (1), (2), (2.5), and (2.8), we obtain that $\lim_{n\to\infty}\|y_n-u_n\|=0$. Since each S_n is a pseudocontractive mapping, we have

$$\begin{aligned}\|u_n-p\|^2 &= \langle\gamma_n(x_n-p)+(1-\gamma_n)(S_nu_n-p),u_n-p\rangle\\ &= \gamma_n\langle x_n-p,u_n-p\rangle+(1-\gamma_n)\langle S_nu_n-p,u_n-p\rangle\\ &\le \gamma_n\langle x_n-p,u_n-p\rangle+(1-\gamma_n)\|u_n-p\|^2,\end{aligned}$$

which immediately leads to

$$\begin{aligned}\|u_n-p\|^2 &\le \langle x_n-p,u_n-p\rangle\\ &= \tfrac{1}{2}[\|x_n-p\|^2+\|u_n-p\|^2-\|x_n-u_n\|^2].\end{aligned}$$

Hence,

$$\|y_n-p\|^2\le\|u_n-p\|^2\le\|x_n-p\|^2-\|x_n-u_n\|^2,$$

which, together with (2.7), yields

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|y_n-p\|^2-\beta_n(1-\beta_n)\|x_n-y_n\|^2+2\alpha_nM^2\\ &\le \beta_n\|x_n-p\|^2+(1-\beta_n)[\|x_n-p\|^2-\|x_n-u_n\|^2]-\beta_n(1-\beta_n)\|x_n-y_n\|^2+2\alpha_nM^2\\ &= \|x_n-p\|^2-(1-\beta_n)\|x_n-u_n\|^2-\beta_n(1-\beta_n)\|x_n-y_n\|^2+2\alpha_nM^2.\end{aligned}$$

This implies that

$$\begin{aligned}&(1-\beta_n)\|x_n-u_n\|^2+\beta_n(1-\beta_n)\|x_n-y_n\|^2\\ &\le \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\alpha_nM^2\\ &\le (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+2\alpha_nM^2.\end{aligned}$$

From conditions (1), (2), and (2.5), we get

$$\lim_{n\to\infty}\|x_n-u_n\|=0 \quad\text{and}\quad \lim_{n\to\infty}\|x_n-y_n\|=0. \tag{2.10}$$

Noticing that $\|u_n-x_n\|=(1-\gamma_n)\|S_nu_n-x_n\|\ge(1-d)\|S_nu_n-x_n\|$,

$$\|x_n-S_nx_n\|\le\|x_n-S_nu_n\|+\|S_nu_n-S_nx_n\|\le\|x_n-S_nu_n\|+\ell\|u_n-x_n\|,$$

and

$$\|x_n-Gx_n\|\le\|x_n-u_n\|+\|u_n-y_n\|+\|Gu_n-Gx_n\|\le2\|x_n-u_n\|+\|u_n-y_n\|,$$

we deduce from (2.10) that

$$\lim_{n\to\infty}\|x_n-S_nu_n\|=0,\quad \lim_{n\to\infty}\|x_n-S_nx_n\|=0 \quad\text{and}\quad \lim_{n\to\infty}\|x_n-Gx_n\|=0. \tag{2.11}$$

Next, we claim that $\|x_n-\overline{S}x_n\|\to0$ as $n\to\infty$, where $\overline{S}:=(2I-S)^{-1}$. First, let us show that $S:H_1\to H_1$ is pseudocontractive and ℓ-Lipschitzian such that $\lim_{n\to\infty}\|Sx_n-x_n\|=0$, where $Sx=\lim_{n\to\infty}S_nx$, $\forall x\in H_1$. Observe that, for all $x,y\in H_1$, $\lim_{n\to\infty}\|S_nx-Sx\|=0$ and $\lim_{n\to\infty}\|S_ny-Sy\|=0$. Since each S_n is pseudocontractive, we get

$$\langle Sx-Sy,x-y\rangle=\lim_{n\to\infty}\langle S_nx-S_ny,x-y\rangle\le\|x-y\|^2.$$

This means that S is pseudocontractive. Because $\{S_n\}_{n=1}^{\infty}$ is ℓ-uniformly Lipschitzian on H_1, we have

$$\|Sx-Sy\|=\lim_{n\to\infty}\|S_nx-S_ny\|\le\ell\|x-y\|,\quad \forall x,y\in H_1.$$

This means that S is ℓ-Lipschitzian. Taking into account the boundedness of $\{x_n\}$ and putting $K = \overline{\text{conv}}\{x_n : n \geq 1\}$ (the closed convex hull of the set $\{x_n : n \geq 1\}$), we have $\sum_{n=1}^{\infty} \sup_{x \in K} \|S_n x - S_{n+1} x\| < \infty$. Hence, by Lemma 1.5, we get $\lim_{n\to\infty} \sup_{x\in K} \|S_n x - Sx\| = 0$, which immediately yields

$$\lim_{n\to\infty} \|S_n x_n - Sx_n\| = 0. \tag{2.12}$$

Combining (2.11) with (2.12) we have $\|x_n - Sx_n\| \leq \|x_n - S_n x_n\| + \|S_n x_n - Sx_n\| \to 0$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{2.13}$$

Define $\overline{S} := (2I - S)^{-1}$. Then $\overline{S} : H_1 \to H_1$ is nonexpansive with $\text{Fix}(\overline{S}) = \text{Fix}(S) = \bigcap_{n=1}^{\infty} \text{Fix}(S_n)$ and $\lim_{n\to\infty} \|x_n - \overline{S} x_n\| = 0$. Indeed, put $\overline{S} := (2I - S)^{-1}$, where I is the identity mapping of H_1. Then it is known that $\overline{S}$ is nonexpansive and $\text{Fix}(\overline{S}) = \text{Fix}(S) = \bigcap_{n=1}^{\infty} \text{Fix}(S_n)$ as a consequence of [15, Theorem 6]. From (2.13), it follows that

$$\begin{aligned}
\|x_n - \overline{S} x_n\| &= \|\overline{S}\,\overline{S}^{-1} x_n - \overline{S} x_n\| \\
&\leq \|\overline{S}^{-1} x_n - x_n\| \\
&= \|(2I - S)x_n - x_n\| = \|x_n - Sx_n\| \to 0 \quad (n \to \infty).
\end{aligned}$$

That is,

$$\lim_{n\to\infty} \|x_n - \overline{S} x_n\| = 0. \tag{2.14}$$

Next, we claim that $\limsup_{n\to\infty} \langle (\delta f - \mu F)z, x_n - z\rangle \leq 0$, where $z = P_\Omega(z - \mu Fz + \delta f(z))$. For any $x, y \in H_1$, we conclude from Lemma 1.2 that

$$\begin{aligned}
&\|P_\Omega(I - \mu F + \delta f)x - P_\Omega(I - \mu F + \delta f)y\| \\
&\leq \delta \|f(x) - f(y)\| + \|(I - \mu F)x - (I - \mu F)y\| \\
&\leq \delta \|x - y\| + (1 - \tau)\|x - y\| \\
&= [1 - (\tau - \delta)]\|x - y\|,
\end{aligned}$$

which implies that $P_\Omega(I - \mu F + \delta f)$ is a contractive mapping. Banach's Contraction Mapping Principle tells us that $P_\Omega(I - \mu F + \delta f)$ has a unique fixed point, say $z \in H_1$, that is, $z = P_\Omega(z - \mu Fz + \delta f(z))$. Since $\{x_n\}$ is a bounded sequence in H_1, without loss of generality, we may choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle (\delta f - \mu F)z, x_n - z\rangle = \lim_{i\to\infty} \langle (\delta f - \mu F)z, x_{n_i} - z\rangle. \tag{2.15}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w. Without loss of generality, we may assume that $x_{n_i} \rightharpoonup w$. Thus, in terms of Lemma 1.6, (2.14) and the nonexpansivity of $\overline{S}$, we obtain that $w \in \text{Fix}(\overline{S}) = \text{Fix}(S) = \bigcap_{n=1}^{\infty} \text{Fix}(S_n)$. In addition, according to Lemma 1.6, (2.11) and the nonexpansivity of G, we have that $w \in \text{Fix}(G)$ (due to Lemma 1.3). Consequently, we get $w \in \Omega$. Since $z = P_\Omega(z - \mu Fz + \delta f(z))$ and $w \in \Omega$, we deduce that

$$\begin{aligned}
\limsup_{n\to\infty} \langle (\delta f - \mu F)z, x_n - z\rangle &= \lim_{i\to\infty} \langle (\delta f - \mu F)z, x_{n_i} - z\rangle \\
&= \langle (\delta f - \mu F)z, w - z\rangle \\
&= \langle (z - \mu Fz + \delta f(z)) - z, w - z\rangle \\
&\leq 0.
\end{aligned} \tag{2.16}$$

Finally, we claim that $x_n \to z$ and $u_n \to z$ as $n \to \infty$. From Lemma 1.3, we have

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \alpha_n\langle(\delta f-\mu F)z, x_{n+1}-z\rangle + \beta_n\langle x_n - z, x_{n+1}-z\rangle \\ &\quad +\langle[(1-\beta_n)I-\alpha_n\mu F]y_n - [(1-\beta_n)I-\alpha_n\mu F]z, x_{n+1}-z\rangle \\ &\le \alpha_n\langle(\delta f-\mu F)z, x_{n+1}-z\rangle + \beta_n\langle x_n - z, x_{n+1}-z\rangle \\ &\quad +(1-\beta_n-\alpha_n\tau)\|y_n-z\|\|x_{n+1}-z\| \\ &\le \alpha_n\langle(\delta f-\mu F)z, x_{n+1}-z\rangle + \tfrac{1}{2}\beta_n(\|x_n-z\|^2+\|x_{n+1}-z\|^2) \\ &\quad +(1-\beta_n-\alpha_n\tau)\|x_n-z\|\|x_{n+1}-z\| \\ &\le \alpha_n\langle(\delta f-\mu F)z, x_{n+1}-z\rangle + \tfrac{1}{2}(1-\alpha_n\tau)(\|x_n-z\|^2+\|x_{n+1}-z\|^2).\end{aligned}$$

This immediately implies that

$$\begin{aligned}2\|x_{n+1}-z\|^2 &\le 2\alpha_n\langle(\delta f-\mu F)z, x_{n+1}-z\rangle + (1-\alpha_n\tau)(\|x_n-z\|^2+\|x_{n+1}-z\|^2) \\ &= 2\alpha_n\langle(\delta f-\mu F)z, x_{n+1}-z\rangle + (1-\alpha_n\tau)\|x_n-z\|^2+(1-\alpha_n\tau)\|x_{n+1}-z\|^2 \\ &\le 2\alpha_n\langle(\delta f-\mu F)z, x_{n+1}-z\rangle + (1-\alpha_n\tau)\|x_n-z\|^2+\|x_{n+1}-z\|^2,\end{aligned}$$

that is,

$$\|x_{n+1}-z\|^2 \le (1-\alpha_n\tau)\|x_n-z\|^2 + 2\alpha_n\langle(\delta f-\mu F)z, x_{n+1}-z\rangle.$$

From condition (2), (3.32), and Lemma 1.7, we see that $\lim_{n\to\infty}\|x_n-z\|=0$. This completes the proof. □

Remark 2.1. Comparing Theorem 2.1 with [18, Theorem 3.1], we have the following points. (1) Our algorithm, which is based on the Mann implicit iteration method, viscosity approximation method, and hybrid steepest-descent method, is more more subtle than algorithm 3.2 in [18, Theorem 3.1] because our algorithm involves the predictor-corrector for finding a common fixed point of a countable family of ℓ-uniformly Lipschitzian pseudocontractive mappings $\{S_i\}_{i=1}^{\infty}$, that is, the implicit iterative step $u_n = \gamma_n x_n + (1-\gamma_n)S_n u_n$ is the predictor one for finding their common fixed points. SVIP (1.2)-(1.3) with a HVI constraint for a countable family of nonexpansive mappings in Theorem [18, Theorem 3.1] is extended to a countable family of ℓ-uniformly Lipschitzian pseudocontractive mappings, which is an essential difference between two HVI constraints.

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